International Journal of Partial Differential Equations and Applications, 2014, Vol. 2, No. 6, 96-104
Available online at http://pubs.sciepub.com/ijpdea/2/6/1

DOI:10.12691/ijpdea-2-6-1

Study of MHD Nanofluid Flow over a Horizontal Stretching Plate by Analytical Methods

A. Vahabzadeh[1,*], M. Fakour[1], D. D. Ganji[2]

[1]Department of Mechanical Engineering, Sari Branch, Islamic Azad University, Sari, Iran
[2]Department of Mechanical Engineering, Mazandaran University, Babol, Iran
*Corresponding author: vahabzadeh_a@yahoo.com

Received November 24, 2014; Revised December 03, 2014; Accepted December 11, 2014

Abstract The nonlinear two-dimensional forced-convection boundary-layer magneto hydrodynamic (MHD) incompressible flow of nanofluid over a horizontal stretching flat plate with variable magnetic field including the viscous dissipation effect is solved using the Variational iteration method (VIM), homotopy perturbation method (HPM) and Adomian decomposition method (ADM). In the paper, our results of the VIM, HPM and ADM are compared with the numerical method (Runge-Kutta fourth-rate). Also the influence of physical factors such as m, Eckert number (Ec) and the percentage of nanoparticles (ϕ) on the velocity and temperature profiles have been investigated. The comparisons of the results show that the HPM has the capability of solving the nonlinear boundary layer MHD flow of nanofluid with sufficient accuracy. The results show with increasing of the parameter m, velocity decreases but for temperature the reverse trend is observed and also with increasing of nanoparticles volume, the temperature value decreases and the velocity value increases. Also with increasing of Eckert number, the temperature value increases.

Keywords: *magneto hydrodynamic (MHD) flow of nanofluid, Adomian decomposition method (ADM), Variational iteration method (VIM), Homotopy perturbation method (HPM)*

Cite This Article: A. Vahabzadeh, M. Fakour, and D. D. Ganji, "Study of MHD Nanofluid Flow over a Horizontal Stretching Plate by Analytical Methods." *International Journal of Partial Differential Equations and Applications*, vol. 2, no. 6 (2014): 96-104. doi: 10.12691/ijpdea-2-6-1.

1. Introduction

The forced convection heat transfer over a permeable stretching plate has relevance in applications such as solar receivers exposed to wind currents, electronic devices cooled by fans, nuclear reactors cooled during emergency shutdown, heat exchanges placed in a low-velocity environment, extrusion processes, cooling of reactors, glass fiber production and crystal growing [1,2,3]. During the last decade, nanofluid heat transfer problems have been given considerable attention by researchers. Most of the nonlinear differential equations do not have an analytical solution. However, so far there have been many researchers that attempted to solve the nonlinear differential equations by using numeric methods. Using the numeric methods, a tremendous amount of CPU time as well as huge memory is required. Semi analytical methods which are more suitable than the numerical methods are applied for the solution of nonlinear nonhomogeneous partial differential equations [4-10]. Comparing with other methods, the Semi analytical methods have the advantage of simplicity when applying to solve complicated nonlinear problems. The HPM, ADM, and VIM methods are used to solve the nonhomogeneous variable coefficient partial differential equations with accurate approximation. Consequently, to extend the validity of the solution to a broader range, one needs to handle huge amount of computational effort. The most powerful Semi analytical method to the solution of nonhomogeneous variable coefficient partial differential equations is the homotopy perturbation method (HPM).

He [11-15] developed the homotopy perturbation method for solving linear, nonlinear, and initial and boundary value problems by combining the standard homotopy and the perturbation methods. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods and has been modified later by some scientists to obtain more accurate results, rapid convergence, and to reduce the amount of computation [16,17,18,19].

Recently, some of researchers have solved many problems in different fields of engineering. Singh et al. [20] solved space-time fractional solidification in a finite slab with HPM. Ajadi and Zuilino [21] applied HPM to reaction-diffusion equations with source term. They concluded that rapid convergence is obtained to the exact solution by HPM. Slota [22] applied the HPM to Stefan solidification heat equation problem, and his results show that HPM is a capable method for solving the problems under consideration.

The basic motivation of this paper is to solve a two-dimensional forced-convection boundary-layer MHD problem formed by a magneto hydrodynamic (MHD) incompressible nanofluid flow in the presence of variable magnetic field over a horizontal flat plate including the viscous dissipation term using the VIM, HPM and ADM. The two-dimensional forced-convection boundary-layer MHD problem is also exact solutions, and the results of simulation are compared with the results obtained by solving the problem using the VIM, HPM and ADM. In the present problem, a nano incompressible fluid in the presence of a variable magnetic field and the viscous dissipation effect over a horizontal stretching flat plate are considered. The results are compared with the previous results of exact simulation. To our knowledge, there have been no results reported so far for the boundary layer flow of nanofluid, using the VIM, HPM and ADM methods, including the MHD with variable magnetic field, and viscous dissipation effect.

2. Decribe Analytical Methods

2.1. Homotopy-perturbation Method

Consider the following function:

$$A(u) - f(r) = 0 \tag{1}$$

with the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0 \tag{2}$$

Where $A(u)$ is defined as follows:

$$A(u) = L(u) + N(u) \tag{3}$$

homotopy-perturbation structure is shown as:

$$H(v,p) = L(v) - L(u_0) + p.L(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{4}$$

or

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0 \tag{5}$$

where

$$v(r,p) : \Omega \times [0,1] \to R \tag{6}$$

Obviously, considering Eqs. (4) and (5), we have:

$$H(v,0) = L(v) - L(u_0) = 0\,,\ H(v,1) = A(v) - f(r) = 0 \tag{7}$$

Where $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. The process of the changes in p from zero to unity is that of $v(r, p)$ changing from u_0 to u_r. We consider v as:

$$v = v_0 + p.v_1 + p^2.v_2 \tag{8}$$

and the best approximation is:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{9}$$

The above convergence is discussed in [23,24].

2.2. Variational Iteration Method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Fu = g(t) \tag{10}$$

where L is a linear operator, F is a nonlinear operator and $g(t)$ is a heterogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + F\tilde{u}_n(\tau) - g(\tau)\right).d\tau \tag{11}$$

Where λ is a general Lagrangian multiplier [25,26] which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation [25,26], i.e., $\delta\tilde{u}_n = 0$.

2.3. Adomian Decomposition Method

To clarify the basic ideas of ADM, we consider the following differential equation:

$$Lu + Ru + Nu = g \tag{12}$$

Where L is the highest order derivative which is assumed to be easily invertible, R the linear differential operator of less order than L, Nu presents the nonlinear terms and g is the source term. Applying the invers operator L^{-1} to the both sides of Eq. (12), and using the given conditions we obtain:

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{13}$$

where the function $f(x)$ represents the terms arising from integration the source term $g(x)$, using given conditions. For nonlinear differential equations, the nonlinear operator $Nu = F(u)$ is represented by an infinite series of the so-called Adomian polynomials.

$$F(u) = \sum_{m=0}^{\infty} A_m \tag{14}$$

The polynomials A_m are generated for all kind of nonlinearity so that A_0 depends only on u_0, A_1 depends on u_0 and u_1, and so on. The Adomian method defines the solution $u(x)$ by the series:

$$u_0 = u_1 + u_2 + u_3 + \cdots \tag{15}$$

In the case of $F(u)$, the infinite series is a Taylor expansion about u_0. In other words

$$F(u) = F(u_0) + F'(u_0)(u-u_0) + F''(u_0)\frac{(u-u_0)^2}{2!} + F'''(u_0)\frac{(u-u_0)^3}{3!} + \ldots \tag{16}$$

By rewriting Eq. (15) as $u_0 = u_1 + u_2 + u_3 + \cdots$, substituting it into Eq. (16) and then equating two expressions for F(u) found in Eq. (16) and Eq. (14) defines formulas for the Adomian polynomials:

$$F(u) = A_1 + A_2 + A_3 + ... = F(u_0) + F'(u_0)(u_1 + u_2 + ...) + F''(u_0)\frac{(u_1 + u_2 + ...)^2}{2!} + ... \quad (17)$$

By equating terms in Eq. (17), the first few Adomian's polynomials A_0, A_1, A_2, A_3 and A_4 are given:

$$A_0 = F(u_0) \quad (18)$$

$$A_1 = u_1 F'(u_0) \quad (19)$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) \quad (20)$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0)$$
$$\vdots \quad (21)$$

Now that the A_k are known, Eq. (15) can be substituted in Eq. (14) to specify the terms in the expansion for the solution of Eq. (16).

3. Mathematical Formulation

The governing two-dimensional forced-convection boundary-layer flow over a horizontal stretching flat plate including the viscous dissipation term is written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (22)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho_{nf}}\left(\mu_{nf}\frac{\partial^2 u}{\partial y^2} - \sigma B(x)^2 u\right), \quad (23)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf} \cdot \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{(\rho C_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2, \quad (24)$$

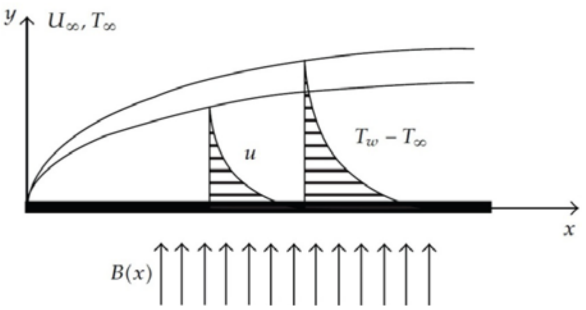

Figure 1. Schematic of the physical model and coordinate system

Eq (22) describes the continuity equation, where *u* and *v* are the velocity components in the *x* and *y* directions, respectively, (see Figure 1). Eq (23) describes the two dimensional momentum equation in the presence of a variable magnetic field, where *u* and *v* are the *x* and *y* components of velocity, respectively, μ_{nf} and ρ_{nf} are the dynamic viscosity and the density of the nanofluid, respectively, σ is the electrical conductivity, and *B(x)* is the variable magnetic field acting in the perpendicular direction to the horizontal flat plate. Eq (24) describes the two-dimensional energy equation including the viscous dissipation term, where, *u*, *v*, and *T* are the *x* and *y* components of velocity and temperature, respectively, α_{nf} is the thermal diffusivity, and $(\rho C_p)_{nf}$ is the heat capacitance of the nanofluid.

The boundary conditions are defined as

$$u = u_w = bx^m, \; v = 0, \; T = T_w, \; at \; y = 0$$
$$u \to 0, \; T \to T_\infty, \; as \; y \to \infty \quad (25)$$

where u_w is the *x*-component of velocity on the horizontal flat plate, *b* and *m* are constants, and T_w and T_∞ are the plate and ambient temperatures, respectively. The nanofluid properties such as the density, ρ_{nf}, the dynamic viscosity, μ_{nf}, the heat capacitance, $(\rho C_p)_{nf}$, and the thermal conductivity, k_{nf}, are defined in terms of fluid and nanoparticles properties as in [27],

$$\rho_{nf} = \rho_f(1-\phi) + \rho_s\phi,$$
$$(\rho C_p)_{nf} = (\rho C_p)_f(1-\phi) + (\rho C_p)_s\phi,$$
$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{0.25}}$$
$$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)},$$
$$\alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}}, \quad (26)$$

where ρ_f is the density of fluid, ρ_s is the density of nanoparticles, φ is defined as the volume fraction of the nanoparticles, μ_f is the dynamic viscosity of fluid, $(\rho C_p)_f$ is the thermal capacitance of fluid, $(\rho C_p)_s$ is the thermal capacitance of nanoparticles, and k_f, k_s are the thermal conductivities of fluid and nanoparticles, respectively.

The variable magnetic field is defined as [28,29]

$$B(x) = B_0\sqrt{x^{m+1}} \quad (27)$$

where B_0 and *m* are constant.

The following dimensionless similarity variable is used to transform the governing equations into the ordinary differential equations

$$\eta = \frac{y}{x}\sqrt{\mathrm{Re}_x},$$
$$\mathrm{Re}_x = \frac{\rho_f u_w(x)}{\mu_f}x \quad (28)$$

The dimensionless stream function and dimensionless temperature are defined as

$$f(\eta) = \frac{\psi(x,y)(\mathrm{Re}_x)^{\frac{1}{2}}}{u_w(x)},$$
$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \quad (29)$$

where the stream function $\psi(x,y)$ is defined as

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x} \tag{30}$$

By applying the similarity transformation parameters, the momentum Eq (23) and the energy Eq (24) can be rewritten as

$$f''' + \left((1-\phi) + \phi(\frac{\rho_s}{\rho_f})\right)(1-\phi)^{2.5}(\frac{m+1}{2})^2 f\, f''$$
$$-\left((1-\phi) + \phi(\frac{\rho_s}{\rho_f})\right)(1-\phi)^{2.5}(m) f'^2 - \left[(1-\phi)^{2.5} Mn\right] f' = 0,$$
$$\theta'' + \left((1-\phi) + \phi(\frac{\rho_s}{\rho_f})\right)\Pr f\,\theta' + \frac{Ec\Pr}{(1-\phi)^{2.5}} f''^2 = 0. \tag{31}$$

Therefore, the transformed boundary conditions are

$$f'(0) = 1,\; f(0) = 0,\; f'(\infty) = 0,\; \theta(0) = 1,\; \theta(\infty) = 0. \tag{32}$$

The dimensionless parameters of *Mn*, *Pr*, *Ec*, and Re_x are the magnetic parameter, Prandtl, Eckert, and Reynolds numbers, respectively. They are defined as

$$Mn = \frac{\sigma . B_0^2}{\rho_f . b},\; \Pr = \frac{(\rho C_p)_f}{k_{ef}} \upsilon_f,\; Ec = \frac{u_w(x)^2}{C_p \Delta T},\; \mathrm{Re}_x = \frac{\rho_f u_w(x)}{\mu_f} x. \tag{33}$$

Eq (31) is rewritten as

$$f''' + A f f'' - B f'^2 - C f' = 0 \tag{34}$$

$$\theta'' + D f \theta' + E f''^2 = 0 \tag{35}$$

where coefficients, *A*, *B*, *C*, *D*, and *E* are written as

$$A = \left((1-\phi) + \phi(\frac{\rho_s}{\rho_f})\right)(1-\phi)^{2.5}(\frac{m+1}{2})^2,$$
$$B = \left((1-\phi) + \phi(\frac{\rho_s}{\rho_f})\right)(1-\phi)^{2.5}(m),\; C = \left[(1-\phi)^{2.5} Mn\right], \tag{36}$$
$$D = \left((1-\phi) + \phi(\frac{\rho_s}{\rho_f})\right)\Pr,\; E = \frac{Ec\Pr}{(1-\phi)^{2.5}}.$$

4. Analytical Methods Applied to the Problem

4.1. The HPM Applied to the Problem

A homotopy perturbation method can be constructed as follows:

$$H(f,p) = (1-p)(f''') + p\left(f''' + A f f'' - B f'^2 - C f'\right) \tag{37}$$

$$H(f,p) = (1-p)(\theta'') + p\left(\theta'' + D f \theta' + E f''^2\right) \tag{38}$$

One can now try to obtain a solution of Eqs. (37, 38) in the form of:

$$v(f) = v_0(\eta) + p.v_1(\eta) + \dots \tag{39}$$

$$v(\theta) = v_0(\eta) + p.v_1(\eta) + \dots \tag{40}$$

where $v_i(\eta)$, i = 0, 1, 2,... are functions yet to be determined. According to Eqs. (37,38) the initial approximation to satisfy initial condition is:

$$f_0(\eta) = -0.1\eta^2 + \eta \tag{41}$$

$$\theta_0(\eta) = -0.2\eta + 1 \tag{42}$$

Substituting Eqs. (39, 40) into Eqs. (37, 38) yields:

$$\frac{d^3}{d\eta^3} f_1(\eta) + 0.0045689\eta^2 - 0.0227916\eta - 0.114487 = 0, \tag{43}$$

$$\frac{d^2}{d\eta^2}\theta_1(\eta) + 0.009456\eta^2 - 0.09456\eta + 0.04332384 = 0 \tag{44}$$

The solutions of Eqs. (43, 44) may be written as follow:

$$f_1(\eta) = -0.0000761483\eta^5 + 0.00094965\eta^4 + 0.0190812\eta^3 - 0.1668\eta^2 \tag{45}$$

$$\theta_1(\eta) = 0.000788\eta^4 + 0.01576\eta^3 - 0.02166192\eta^2 - 0.1871904\eta \tag{46}$$

In the same manner, the rest of components were obtained by using the Maple package. According to the HPM, we can conclude:

$$f(\eta) = -0.0000761483\eta^5 + 0.00094965\eta^4 + 0.0190812\eta^3 - 0.2668\eta^2 + \eta \tag{47}$$

$$\theta(\eta) = 0.000788\eta^4 + 0.01576\eta^3 - 0.02166192\eta^2 - 0.3871904\eta + 1 \tag{48}$$

4.2. The VIM Applied to the Problem

In order to solve Eqs. (34, 35) with boundary conditions (33) using VIM, we construct a correction functional, as follows:

$$f_{n+1} = f_n + \int_0^\eta \lambda.\left\{f_n''' + A f_n f_n'' - B f_n'^2 - C f_n'\right\} d\tau \tag{49}$$

$$\theta_{n+1} = \theta_n + \int_0^\eta \lambda.\left\{\theta_n'' + D f_n \theta_n' + E f_n''^2\right\} d\tau \tag{50}$$

Its stationary conditions can be obtained as follows:

$$\lambda'''(\tau) - C\lambda'(\tau) = 0 \rightarrow L\{\lambda'''(\tau)\} - C\, L\{\lambda'(\tau)\} = -1$$
$$\rightarrow L\{\lambda(\tau)\}\Big|_{\tau=\eta} = -\frac{1}{s^3 - C.s} \tag{51}$$

$$\lambda''(\tau) = 0 \rightarrow L\{\lambda''(\tau)\} = 1$$
$$\rightarrow L\{\lambda(\tau)\}\Big|_{\tau=\eta} = \frac{1}{s^2} \tag{52}$$

The Lagrangians multiplier can therefore be identified as:

$$\lambda_f = \frac{1 - \cos h(\sqrt{C}\,(\tau - t))}{c} \tag{53}$$

$$\lambda_\theta = \tau - t \tag{54}$$

As results, we obtain the following iteration formula:

$$f_{n+1} = f_n + \int_0^\eta \frac{1-\cosh\left(\sqrt{C}\,(\tau - t)\right)}{c} \left\{f_n''' + A f_n f_n'' - B f_n'^2 - C f_n'\right\} d\tau \tag{55}$$

$$\theta_{n+1} = \theta_n + \int_0^\eta (\tau - t).\left\{\theta_n'' + D f_n \theta_n' + E f_n''^2\right\} d\tau \tag{56}$$

Now we start with an arbitrary initial approximation that satisfies the initial conditions:

$$f_0(\eta) = -0.1\eta^2 + \eta \tag{57}$$

$$\theta_0(\eta) = -0.2\eta + 1 \tag{58}$$

Using the above Variational formula (49, 50), we have:

$$f_1 = f_0 + \int_0^\eta \frac{1-\cosh(\sqrt{C}(\tau - t))}{c}. \{ f_0''' + Af_0 f_0'' - Bf_0'^2 - Cf_0' \}.d\tau \tag{59}$$

$$\theta_1 = \theta_0 + \int_0^\eta (\tau - t).\{\theta_0'' + D f_0 \theta_0' + E f_0''^2\}.d\tau \tag{60}$$

Substituting Eq. (57, 58) in to Eq. (59, 60) and after simplification, we have:

$$f(\eta) = -0.1995379\eta^2 + 0.697155\eta + 0.89503 sinh(0.33838)\eta + 1.738852 cosh(0.338365)\eta + 0.013302\eta^3 - 1.738852 \tag{61}$$

$$\theta(\eta) = -0.2\eta + 1 - 0.000788\eta^4 + 0.01576\eta^3 - 0.021662\eta^2 \tag{62}$$

and so on. In the same manner the rest of the components of the iteration formula can be obtained.

4.3. The ADM Applied to the Problem

In order to apply ADM to nonlinear equation in fluids problem, we rewrite Eqs (34, 35) in the following operator form:

$$L_{\eta\eta\eta} f''' = -A f f'' + B f'^2 + C f' \tag{63}$$

$$L_{\eta\eta}\theta'' = -D f \theta' - E f''^2 \tag{64}$$

where the notation:

$$L_{\eta\eta\eta} = \frac{\partial^3}{\partial\eta^3} \ , \ L_{\eta\eta} = \frac{\partial^2}{\partial\eta^2} \tag{65}$$

is the linear operator. By using the inverse operator, we can write Eqs (63, 64) in the following form:

$$f(\eta) = L^{-1}{}_{\eta\eta\eta}[-A f f'' + B f'^2 + C f'] \tag{66}$$

$$\theta(\eta) = L^{-1}{}_{\eta\eta}[-D f \theta' - E f''^2] \tag{67}$$

where the inverse operator is defined by:

$$L^{-1}{}_{\eta\eta\eta}() = \int_0^\eta\int_0^\eta\int_0^\eta ().d\eta.d\eta.d\eta \quad , \qquad L^{-1}{}_{\eta\eta}() = \int_0^\eta\int_0^\eta .d\eta.d\eta \tag{68}$$

where

$$N_1(f) = A f f'' \ , \ N_2(f) = B f'^2 \ , \ N_3(f,\theta) = D f \theta' \ , \ N_4(f) = E f''^2 \tag{69}$$

The nonlinear operators $N_1(f), N_2(f), N_3(f,\theta), N_4(f)$ are defined by the following infinite series:

$$N_i(f) = \sum_{n=0}^{\infty} A_{i,n} \ , \ i = 1,2 \qquad N_i(\theta) = \sum_{n=0}^{\infty} A_{i,n} \ , \ i = 1,2 \tag{70}$$

where A_{in} is called Adomian polynomials and defined by:

$$A_{i,n} = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_i\left[\sum_{k=0}^{n} \lambda^k f_k\right]\right]_{\lambda=0} \tag{71}$$

Hence we obtain the components series solution by the following recursive relations:

$$f_{n+1}(\eta) = L^{-1}{}_{\eta\eta\eta}[-A f_n f_n'' + B f_n'^2 + C f_n'] \tag{72}$$

$$\theta_{n+1}(\eta) = L^{-1}{}_{\eta\eta}[-D f_n \theta_n' - E f_n''^2] \tag{73}$$

Where $n \geq 0$. Adomian's polynomials formula, Eq. (71), is easy to set computer cod to get as many polynomials as we need in the calculation. We can give the first few Adomian's polynomials of the $A_{i,n}$ as:

$$\begin{aligned} A_{1,0} &= -0.2\eta + 0.02\eta^2, \\ A_{1,1} &= -0.00038\eta^4 + 0.00001523\eta^3 \\ &+ \left(\eta - 0.1\eta^2\right)\left(0.0228\eta^2 - 0.001523\eta^3\right) \end{aligned} \tag{74}$$

$$\begin{aligned} A_{2,0} &= (1-0.2\eta)^2, \\ A_{2,1} &= 2(1-0.2\eta)(0.0076148\eta^3 - 0.00038\eta^4) \end{aligned} \tag{75}$$

and so on, the rest of the polynomials can be constructed in a similar manner. Using the recursive relation, Eqs (72, 73) and Adomian's polynomials formula, Eq. (71), with the initial conditions, Eq. (32), gives:

$$\begin{aligned} f_0(\eta) &= -0.1\eta^2 + \eta, \\ f_1(\eta) &= 0.001903\eta^4 - 0.0000761\eta^5, \\ f_2(\eta) &= -1.139.10^{(-7)}\eta^8 + 0.00000455\eta^7 - 0.0000434\eta^6 \\ &\vdots \end{aligned} \tag{76}$$

$$\begin{aligned} \theta_0(\eta) &= -0.2\eta + 1, \\ \theta_1(\eta) &= 0.0157\eta^3 - 0.00078\eta^4 - 0.0216\eta^2, \\ \theta_2(\eta) &= -0.371969.10^{-5}\eta^7 + 0.00013\eta^6 - 0.001253\eta^5 \\ &+ 0.00253\eta^4 \\ &\vdots \end{aligned} \tag{77}$$

where

$$\begin{aligned} f(\eta) &= \eta - 0.1\eta^2 + 0.0019\eta^4 - 0.000076\eta^5 \\ &- 1.139.10^{(-7)}\eta^8 + 0.455610^{(-5)}\eta^7 \\ &- 0.0000434\eta^6 \end{aligned} \tag{78}$$

$$\begin{aligned} \theta(\eta) &= 1 - 0.2\eta + 0.01576\eta^3 + 0.001743\eta^4 \\ &- 0.02166\eta^2 - 0.371969.10^{-5}\eta^7 \\ &+ 0.00013\eta^6 - 0.0012531\eta^5 \end{aligned} \tag{79}$$

and so on. In the same manner the rest of the components of the iteration formula can be obtained.

Table 1. Thermophysical properties of water and nanoparticles

Physical properties	Fluid (water)	Nanoparticles Al_2O_3
$\rho\ (kg.m^{-3})$	997.1	3970
$C_p\ (J.kg^{-1}.k^{-1})$	4179	765
$k\ (W.m^{-1}.k^{-1})$	0.613	40

These functions, f and θ, are calculated for the case where, $Ec = 0.1$, $m = 0$, $\Pr = 6.2$, $\phi = 0.02$, and $Mn = 0.2$. The physical properties of the fluid, water, and the nanoparticles, aluminum oxide Al_2O_3, are given in Table 1.

5. Results and Discussion

Comparison between the results of VIM, HPM and ADM methods is shown in (Figure 2). It can be seen that the HPM method of this analytic methods is closer to numeric method, also Table 2, Table 3 display the numerical magnitude of velocity and temperature profiles. Based on results of these tables discrepancy between the obtained result of HPM and exact solution is less than discrepancy the result of VIM and ADM with exact solution. This is obvious that HPM method is appropriate method for this problem. With observation of results we are starting to clear study about HPM with more iteration to approach to better answer.

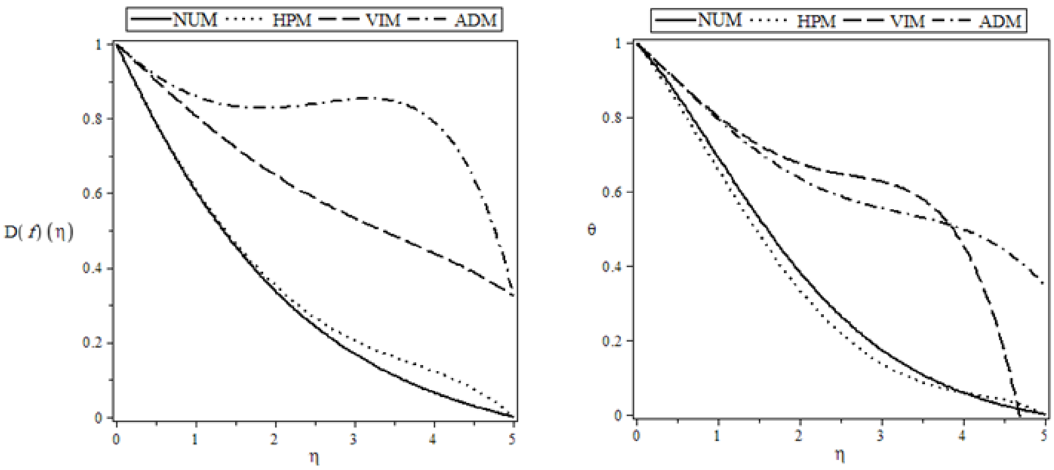

Figure 2. Comparison of dimensionless velocity & temperature profiles versus the numeric method with the results obtained by HPM, VIM & ADM at $Ec = 0.1$, $m = 0$, $\Pr = 6.2$, $\phi = 0.02$, and $Mn = 0.2$.

Table 2. The results of VIM, HPM and ADM methods and their errors for f'

η	VIM	HPM	ADM	NUM	ErrorofVIM	ErrorofHPM	ErrorofADM
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000	0.0000000000
0.5	0.9293511010	0.8287998889	0.9009203948	0.8271859472	0.1021651538	0.0016139417	0.0737344476
1.0	0.8590727760	0.6109629230	0.8070041369	0.6064555886	0.2526171874	0.0045073344	0.2005485483
1.5	0.8350721170	0.4923703573	0.7221387807	0.4850785546	0.3499935624	0.0072918027	0.2370602261
2.0	0.8277486770	0.3523358286	0.6484013232	0.3382516133	0.4894970637	0.0140842153	0.3101497099
2.5	0.8359920900	0.2819034827	0.5858570676	0.2607671382	0.5752249518	0.0211363445	0.3250899294
3.0	0.8418096430	0.2050411896	0.5326096353	0.1693569467	0.6724526963	0.0356842429	0.3632526886
3.5	0.8495700100	0.1681899808	0.4850662501	0.1217573510	0.7278126590	0.0464326298	0.3633088991
4.0	0.7903484000	0.1218114196	0.4383824167	0.0655737576	0.7247746424	0.0562376620	0.3728086590
4.5	0.6808884400	0.0859558650	0.3870501126	0.0359369129	0.6449515271	0.0500189521	0.3511131990
5.0	0.3287604000	0.0000000000	0.3255936159	0.0000000000	0.3287604000	0.0000000000	0.3255936159

Table 3. The results of VIM, HPM and ADM methods and their errors for θ

η	VIM	HPM	ADM	Exact	ErrorofVIM	ErrorofHPM	ErrorofADM
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000	0.0000000000
0.5	0.9180632270	0.8747693514	0.8966263479	0.8904989294	0.0275642976	0.0157295780	0.0061274185
1.0	0.8013727981	0.6594092739	0.7947151613	0.6932176135	0.1081551846	0.0338083396	0.1014975478
1.5	0.7381211854	0.5183756214	0.7051818573	0.5605876443	0.1775335411	0.0422120229	0.1445942130
2.0	0.6743369275	0.3356504193	0.6350883868	0.3838204506	0.2905164769	0.0481700313	0.2512679362
2.5	0.6509362135	0.2399835564	0.5861174467	0.2870140325	0.3639221810	0.0470304761	0.2991034142
3.0	0.6258464405	0.1377714227	0.5540719324	0.1745945390	0.4512519015	0.0368231163	0.3794773934
3.5	0.5931152555	0.0948164923	0.5292531693	0.1191092162	0.4740060393	0.0242927239	0.4101439531
4.0	0.4548371812	0.0588248147	0.4975714550	0.0589675267	0.3958696545	0.0001427120	0.4386039283
4.5	0.2454386305	0.0432670702	0.4422424590	0.0305855598	0.2148530707	0.0126815104	0.4116568992
5.0	- 0.388276024	0.0000000000	0.3459230120	0.0000000000	0.3882760240	0.0000000000	0.3459230120

Figure 3 shows the comparison of dimensionless velocity profiles versus the numeric method with the results obtained by the HPM at VIM & ADM at $Ec = 0.1$, $m = 0$, $\Pr = 6.2$, $\phi = 0.02$ and $Mn = 0.2$. The results obtained from the HPM are reported for four different sums of terms, S=2,4,8, and 12, in the HPM series solution. It is obvious from Figure 3 that as the number of sums of terms in the HPM series solution increases, the results approach towards the profile obtained from the exact solution. The mean discrepancies between the results of velocity obtained from the HPM for S=12 and the results obtained from the exact solution are at most 1%.

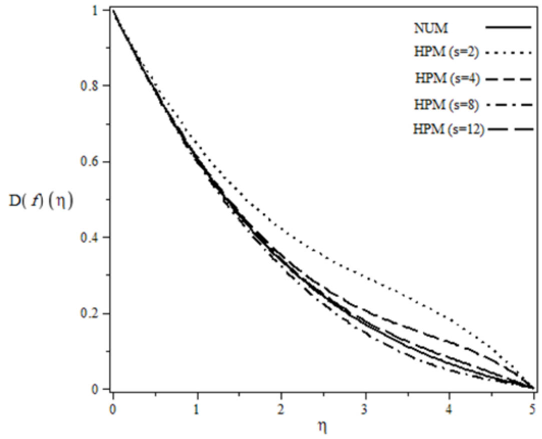

Figure 3. Comparison of dimensionless velocity profiles versus the numeric method with the results obtained by HPM at VIM & ADM at $Ec = 0.1$, $m = 0$, $\Pr = 6.2$, $\phi = 0.02$, and $Mn = 0.2$.

Figure 4 shows the comparison of dimensionless temperature profiles versus the numeric method with the results obtained by the HPM at VIM & ADM at $Ec = 0.1$, $m = 0$, $\Pr = 6.2$, $\phi = 0.02$ and $Mn = 0.2$. The results obtained from the HPM are reported for four different sums of terms, S=2,4,8, and 12, in the HPM series solution. It is obvious from Figure 4 that as the number of sums of terms in the HPM series solution increases, the results approach towards the profile obtained from the exact solution. The mean discrepancies between the results of velocity obtained from the HPM for S=12 and the results obtained from the exact solution are at most 3%.

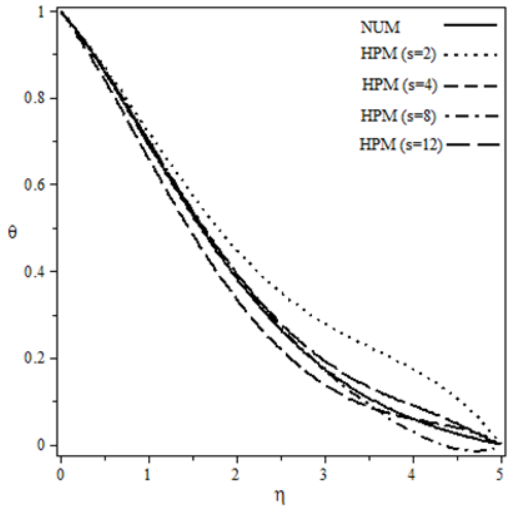

Figure 4. Comparison of dimensionless temperature profiles versus the numeric method with the results obtained by HPM at VIM & ADM at $Ec = 0.1$, $m = 0$, $\Pr = 6.2$, $\phi = 0.02$, and $Mn = 0.2$

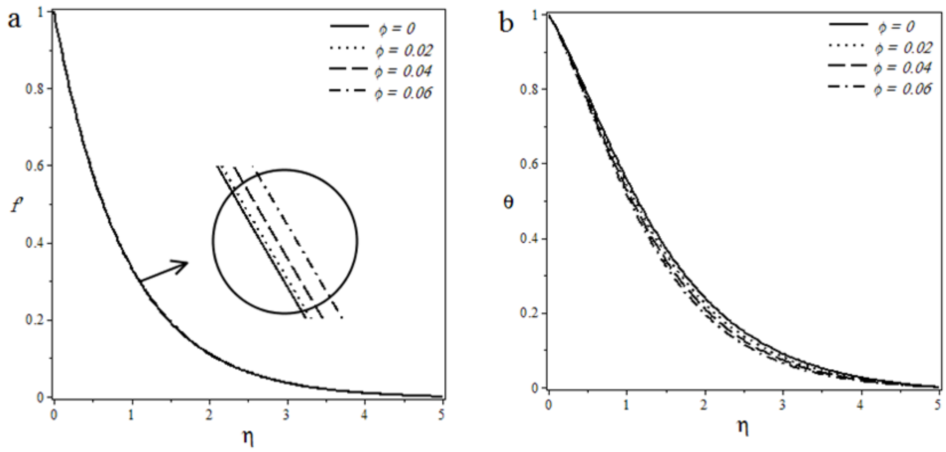

Figure 5. The influence of volume fraction of the nanofluid (ϕ) on (a) velocity, (b) temperature

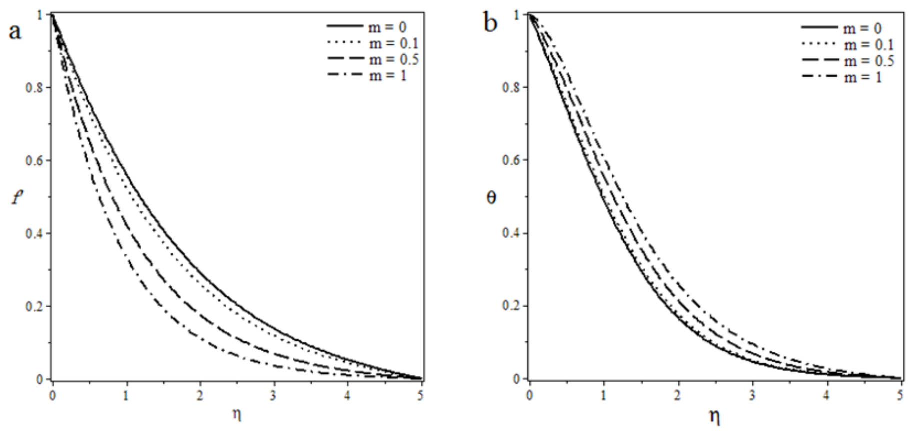

Figure 6. The effect of different value of *m* on (a) velocity, (b) temperature

The fluctuation of temperature and velocity with different values of nanoparticles volume is shown in Figure 5. Considering this figure it is obvious that with increasing of nanoparticles volume, the temperature value decreases and with increasing of nanoparticles volume, the velocity value increases. Figure 6 indicates the influence of *m* on the temperature and velocity profiles. According to this figure it is obtained that the values of velocity decrease with increasing of parameter *m*, but for temperature the reverse trend is observed and with increasing of *m* the value of temperature profile increases. Finally Figure 7 show the effect of Eckert number on temperature profile. Considering this figure it is obvious that with increasing of Ec, the temperature value increases.

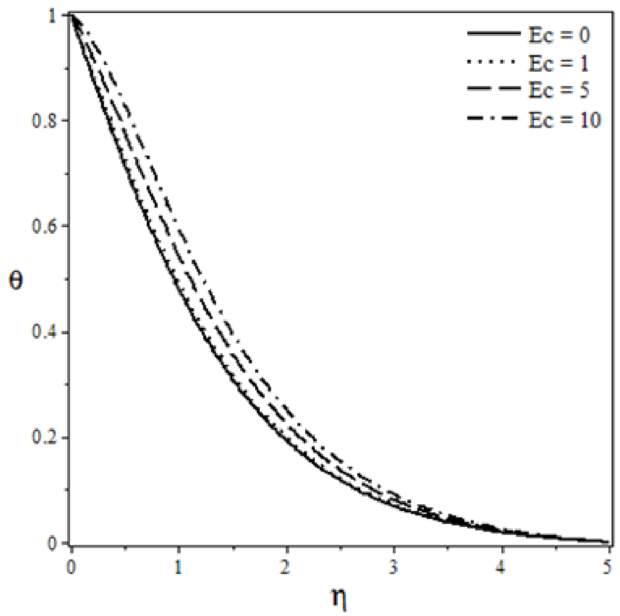

Figure 7. The effect of Eckert number on temperature profile

6. Conclusion

In this work, the nonlinear two-dimensional forced-convection boundary-layer magneto hydrodynamic (MHD) incompressible flow of nanofluid over a horizontal stretching flat plate with variable magnetic field including the viscous dissipation effect is solved using the homotopy perturbation method (HPM), Variational iteration method (VIM) and Adomian decomposition method (ADM). The results are justified and compared with the results obtained from the numeric method. Our results obtained from the HPM, when the number of sums of terms in the HPM series solution increases, showed a monotonic convergence towards the results using the exact solution. The results obtained from the HPM show at most less than 5% mean deviations when compared with the results obtained from the exact solution. For the nonlinear MHD problem, this is encouraging because these results are only achieved by including at most S=12 number of sums of terms in the HPM series solution. Also the influence of physical factors such as *m*, Eckert number (*Ec*) and the percentage of nanoparticles (ϕ) on the velocity and temperature profiles have been investigated. The results show with increasing of the parameter *m*, velocity decreases but for temperature the reverse trend is observed and also with increasing of nanoparticles volume, the temperature value decreases and the velocity value increases. Also with increasing of Eckert number, the temperature value increases.

References

[1] Fisher E. G., Extrusion of Plastics, Wiley, New York, 1976.

[2] Altan, T., Gegel, S. O. H., Metal Forming Fundamentals and Applications, American Society of Metals, Metals Park, OH, 1979.

[3] Tadmor, Z., Klein, I., Engineering Principles of Plasticating Extrusion, Polymer Science and Engineering Series, Van Nostrand Reinhold, New York, 1970.

[4] J. H. He, Non-perturbative methods for strongly nonlinear problems [Ph.D. thesis], de-Verlagim Internet GmbH, Berlin, Germany, 2006.

[5] J.-H. He, “Some asymptotic methods for strongly nonlinear equations,” International Journal of Modern Physics B, vol. 20, no. 10, pp. 1141-1199, 2006.

[6] J.-H. He, “Homotopy perturbation method for solving boundary value problems,” Physics Letters A, vol. 350, no. 1-2, pp. 87-88, 2006.

[7] J. H. He, “Application of homotopy perturbation method to nonlinear wave equations,” Chaos, Solutions& Fractals, vol. 26, no. 3, pp. 695-700, 2005.

[8] J.-H. He, “Approximate analytical solution for seepage flow with fractional derivatives in porous media,” Computer Methods in Applied Mechanics and Engineering, vol. 167, no. 1-2, pp. 57-68, 1998.

[9] J. H. He, “Approximate solution of nonlinear differential equations with convolution product nonlinearities,” Computer Methods in Applied Mechanics and Engineering, vol. 167, no. 1-2, pp. 69-73, 1998.

[10] J. H. He, “Variational iteration method-a kind of non-linear analytical technique: some examples,” International Journal of Non-Linear Mechanics, vol. 34, no. 4, pp. 699-708, 1999.

[11] J.-H.He, “Homotopy perturbation technique,” Computer Methods in Applied Mechanics and Engineering, vol. 178, no. 3-4, pp. 257-262, 1999.

[12] J.-H. He, “A coupling method of a homotopy technique and a perturbation technique for non-linear problems,” International Journal of Non-Linear Mechanics, vol. 35, no. 1, pp. 37-43, 2000.

[13] J.-H. He and X.-H.Wu, “Construction of solitary solution and compacton-like solution by variational iteration method,” Chaos, Solitons & Fractals, vol. 29, no. 1, pp. 108-113, 2006.

[14] J.-H. He, “Periodic solutions and bifurcations of delay-differential equations,” Physics Letters A, vol. 347, no. 4-6, pp. 228-230, 2005.

[15] J. H. He, “Limit cycle and bifurcation of nonlinear problems,” Chaos, Solitons & Fractals, vol. 26, no. 3, pp. 827-833, 2005.

[16] A. Majidian, M. Fakour, A. Vahabzadeh, Analytical investigation of the Laminar Viscous Flow in a Semi-Porous Channel in the Presence of a Uniform Magnetic Field, International Journal of Partial Differential Equations and Applications, Vol. 2, No. 4, pp. 79-85, 2014.

[17] A. Vahabzadeh, M. Fakour, D.D. Ganji, I.Rahimi Petroudi, Analytical accuracy of the one dimensional heat transfer in geometry with logarithmic various surfaces, Cent. Eur. J. Eng., vol. 4, pp. 341-355, 2014.

[18] M. Fakour, A. Vahabzadeh, D.D. Ganji, Scrutiny of mixed convection flow of a nanofluid in a vertical channel, International journal of Case Studies in Thermal Engineering, (2014).

[19] D. D. Ganji, M. Fakour, A. Vahabzadeh, S.H.H. Kachapi, Accuracy of VIM, HPM and ADM in Solving Nonlinear Equations for the Steady Three-Dimensional Flow of a Walter’s B Fluid in Vertical Channel, Walailak Journal of Science and Technology, Vol. 11, No 7, pp. 593-609, 2014.

[20] J. Singh, P. K. Gupta, K. N. Rai, and CIMS-DST, “Homotopy perturbation method to space-time fractional solidification in a finite slab,” Applied Mathematical Modeling. Simulation and Computation for Engineering and Environmental Systems, vol. 35, no. 4, pp. 1937-1945, 2011.

[21] S. O. Ajadi and M. Zuilino, “Approximate analytical solutions of reaction-diffusion equations with exponential source term: homotopy perturbation method (HPM),” Applied Mathematics Letters, vol. 24, no. 10, pp. 1634-1639, 2011.

[22] D. Slota, “The application of the homotopy perturbation method to one-phase inverse Stefan problem,” International Communications in Heat and Mass Transfer, vol. 37, no. 6, pp. 587-592, 2010.

[23] M M Rashidi, G Domairry and S Dinarvand. Approximate solutions for the Burger and regularized long wave equations by means of the homotopy analysis method Original Research Article. Communications in Nonlinear Science and Numerical Simulation, 14, pp. 708-717, 2009.

[24] AA Soliman. A numerical simulation and explicit solutions of KdV-Burgers’ and Lax’s seventh-order KdV equations. Chaos, Solitons Fractals, vol. 29, pp. 294-302, 2006.

[25] S Momani and S Abuasad. Application of He's variational iteration method to Helmholtz equation. Chaos, Solitons Fractals, vol. 27, pp. 1119-1123, 2006.

[26] M. Fakour, D.D. Ganji, M. Abbasi, Scrutiny of underdeveloped nanofluid MHD flow and heat conduction in a channel with porous walls, International journal of Case Studies in Thermal Engineering, (2014).

[27] S. M. Aminossadati and B. Ghasemi, "Natural convection cooling of a localised heat source at the bottom of a nanofluid-filled enclosure," European Journal of Mechanics, B, vol. 28, no. 5, pp. 630-640, 2009.

[28] T. C. Chiam, "Hydromagnetic flow over a surface stretching with a power-law velocity," International Journal of Engineering Science, vol. 33, no. 3, pp. 429-435, 1995.

[29] S. P. A. Devi and M. Thiyagarajan, "Steady nonlinear hydromagnetic flow and heat transfer over a stretching surface of variable temperature," Heat and Mass Transfer, vol. 42, no. 8, pp. 671-677, 2006.