International Journal of Partial Differential Equations and Applications, 2014, Vol. 2, No. 5, 91-95
Available online at http://pubs.sciepub.com/ijpdea/2/5/2

DOI:10.12691/ijpdea-2-5-2

Matrix Fourier Transforms and Application

O. Yaremko, O. Nikitina, E. Zuravleva*

Oleg Yaremko,Olga Nikitina, Ekaterina Zuravleva Penza State University, Penza, Russia
*Corresponding author: yaremki@mail.ru

Received October 20, 2014; Revised November 12, 2014; Accepted November 18, 2014

Abstract In this work, we introduce in an explicit form a special types of matrix Fourier transforms on real axis and real semi- axis: matrix cos- transforms, matrix sin- transforms and matrix transforms with piecewise trigonometric kernels. The integral transforms of such kinds are used for solving analytically the problems of mathematical physics in homogeneous and piecewise homogeneous media. Analytical solution of vector heat conduction equation,vector wave equation and vector Poisson equation is obtained.

Keywords: *fourier matrix transforms, integral transform, heat conduction equation, wave equation, Poisson equation*

Cite This Article: O. Yaremko, O. Nikitina, and E. Zuravleva, "Matrix Fourier Transforms and Application." *International Journal of Partial Differential Equations and Applications*, vol. 2, no. 5 (2014): 91-95. doi: 10.12691/ijpdea-2-5-2.

1. Introduction

Matrix Fourier transforms with sine, cosine and piece-wise trigonometric kernels represent an important branch of mathematical analysis. It is based on the expansion of a function over a set of cosine or sine basis functions.

Integral transforms arise in a natural way through the principle of linear superposition in constructing integral representations of solutions to linear differential equations. The theory of integral Fourier transforms with piecewise trigonometric kernels in a scalar case was studied by Ufljand J.S. [1,2], Lenjuk M. P [3], Najda L.S. [4], Protsenko V. S [5]. The matrix version is adapted for the solution to the problems in piecewise homogeneous medium and have been developed by O.Yaremko in [6,8,9]. The necessary proofs by method of contour integration were conducted in [6] and [9]. It is clear this method is effective to obtain the exact solution of boundary-value problems for piecewise homogeneous media [7,10,11,13]. Special types of matrix Fourier transforms on real axis and semi-axis and their applications are analyzed in this article.

Fourier transforms with sine, cosine and piece-wise trigonometric kernels have shown their special applicability in description of consistent mathematical models. To show the versatility of these transforms constructed by authors we solve the vector problems of mathematical physics in homogeneous and piecewise homogeneous media. We find analytical solutions on real axis and real semi- axis of classical mathematical models:vector heat conduction equation,vector wave equation and vector Poisson equation.

Now we consider classical Sturm-Liouville problems of variety types in boundary conditions. Further we construct their matrix analogues in section 2,3.

The Sturm-Liouville problem with Dirichlet boundary condition

$$-A^2\frac{d^2}{dx^2}\phi(x,\lambda)=\lambda^2\phi(x,\lambda),\quad 0<x<\infty,$$
$$\phi(0,\lambda)=0,\left|\phi(x,\lambda)\right|<\infty,$$

provides the kernels for direct and inverse integral sin - transforms on the real semi-axis:

$$F_+[f](\lambda)\equiv\tilde f(\lambda)=\int_0^\infty\sin\left(A^{-1}\lambda\xi\right)A^{-1}f(\xi)\,d\xi,$$
$$f(x)=\frac{2}{\pi}\int_0^\infty\sin\left(A^{-1}\lambda x\right)\tilde f(\lambda)\,d\lambda.$$

The Sturm-Liouville problem with Neumann boundary condition

$$-A^2\frac{d^2}{dx^2}\phi(x,\lambda)=\lambda^2\phi(x,\lambda),\quad 0<x<\infty,$$
$$\phi'(0,\lambda)=0,\left|\phi(x,\lambda)\right|<\infty,$$

provides the kernels for direct and inverse integral cos - transforms on the real semi-axis:

$$F_+[f](\lambda)\equiv\tilde f(\lambda)=\int_0^\infty\cos\left(A^{-1}\lambda\xi\right)f(\xi)\,d\xi,$$
$$f(x)=\frac{2A^{-1}}{\pi}\int_0^\infty\cos\left(A^{-1}\lambda x\right)\tilde f(\lambda)\,d\lambda.$$

The Sturm-Liouville problem with Robin boundary condition

$$-A^2\frac{d^2\phi}{dx^2}=\lambda^2\phi(x,\lambda),x>0,$$
$$H\phi+\frac{d\phi}{dx}=0,x=0,$$
$$\left|\phi(x,\lambda)\right|<\infty,$$

where H - negative, provides the kernels for direct and inverse Fourier- type transforms on the real semi-axis:

$$F_+[f](\lambda) \equiv \tilde{f}(\lambda) = \int_0^\infty \begin{pmatrix} h \sin A^{-1}\lambda\xi \\ -\lambda \cos A^{-1}\lambda\xi \end{pmatrix} A^{-1} f(\xi)\, d\xi, \quad (1)$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \begin{pmatrix} \sin \lambda A^{-1}xh \\ -\lambda \cos \lambda A^{-1}x \end{pmatrix} \begin{pmatrix} h^2 \\ +E\lambda^2 \end{pmatrix}^{-1} \tilde{f}(\lambda) d\lambda, \quad (2)$$

$h = AH$.

Theorem 1. *1If $A = 1$ in (1), (2) then the inverse Fourier- type transform F^{-1} has the form*

$$F^{-1}[\tilde{f}](\lambda) = f(x) =$$
$$\frac{2}{\pi}\int_0^\infty \tilde{f}(\lambda)(h sin\lambda x - \lambda cos\lambda x)(h^2 + \lambda^2)^{-1} d\lambda$$

subject to the existence of the integral used in the definition.

Proof. Let function $f(x)$ takes the form

$$f(x) = \frac{2}{\pi}\int_0^\infty \tilde{f}(\lambda)\frac{h sin\lambda x - \lambda cos\lambda x}{h^2 + \lambda^2} d\lambda.$$

Then

$$hf(x) + f'(x) = \frac{2}{\pi}\int_0^\infty \tilde{f}(\lambda) sin\lambda x d\lambda.$$

Due to the inverse sin- transform, we get

$$\tilde{f}(\lambda) = \int_0^\infty sin\lambda x\left(hf(x) + f'(x)\right) d\lambda$$
$$= sin\lambda x f(x)|_0^\infty + \int_0^\infty \left(h sin\lambda x - \lambda cos\lambda x\right) f(x) dx$$
$$= \int_0^\infty \left(h sin\lambda x - \lambda cos\lambda x\right) f(x) dx.$$

Note. *2If $f(x)$ is a vector function and h is a square negative- definite matrix then theorem 1 is true.*

2. Matrix Fourier Transforms on Real Axis

In linear algebra, a symmetric $n \times n$ real matrix A is said to be positive definite if $z^T Az$ is positive for every non-zero column vector z of n real numbers. Here z^T denotes the transpose of z. Let $P\Lambda P^{-1}$ be an eigendecomposition of A, where P is a unitary real matrix whose rows comprise an orthonormal basis of eigenvectors of A, and Λ is a real diagonal matrix whose main diagonal contains the corresponding eigenvalues. Let $A = P\Lambda P^{-1}$, where

$$P = \|p_{ij}\|, P^{-1} = \|p_{ji}\|, \Lambda = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_N \end{pmatrix}.$$

Definition 1. 3*Let $f = f(x)$ be a vector-function*

$$f = \begin{pmatrix} f_1(x) \\ \dots \\ f_N(x) \end{pmatrix}$$

and $A = P\Lambda P^{-1}$ be the matrix give an account of above, then vector-function f of matrix Ax is defined by the following equality

$$f(Ax) = \|p_{ij}\| \begin{pmatrix} \sum_{j=1}^{N} p_{j1} f_j(x\lambda_1) \\ \dots \\ \sum_{j=1}^{N} p_{jN} f_j(x\lambda_N) \end{pmatrix}.$$

Theorem 2. *4The matrix -value Sturm-Liouville problem*

$$-A^2 \frac{d^2}{dx^2}\phi(x,\lambda) = \lambda^2\phi(x,\lambda), \quad -\infty < x < \infty,$$
$$|\phi(x,\lambda)| < \infty,$$

provides the direct and inverse matrix Fourier transforms on the real axis:

$$F[f](\lambda) \equiv \tilde{f}(\lambda) = \int_{-\infty}^{\infty} e^{-iA^{-1}\lambda\xi} A^{-1} f(\xi) d\xi,$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iA^{-1}\lambda x} \tilde{f}(\lambda) d\lambda.$$

Theorem 3. *5The matrix -value Sturm-Liouville problem on the composite axis*

$$-A_1^2 \frac{d^2\phi_1}{dx^2} = \lambda^2\phi_1(x,\lambda), x \in (-\infty, 0),$$
$$A_2^2 \frac{d^2\phi_2}{dx^2} = \lambda^2\phi_2(x,\lambda), x \in (0,\infty);$$
$$\phi_1 = \phi_2, \qquad \lambda_1\phi_1' = \lambda_2\phi_2', x = 0,$$
$$|\phi_i(x,\lambda)| < \infty$$

provides the direct F_1 and inverse F_1^{-1} matrix integral transforms on the composite real axis

$$I_{1+} = (-\infty, 0) \cup (0, \infty):$$
$$F_1[f](\lambda) \equiv \tilde{f}(\lambda) =$$
$$2(E+\chi)^{-1}\int_{-\infty}^{0} (\chi \cos q_1\xi - i \sin q_1\xi) A_1^{-1} f_1(\xi) d\xi$$
$$+2(E+\chi)^{-1}\int_0^\infty e^{-iq_2\xi} A_2^{-1} f_2(\xi) d\xi,$$
$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iq_1 x} \tilde{f}(\lambda) d\lambda,$$
$$f_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\cos q_2 x + i\chi \sin q_2 x) \tilde{f}(\lambda) d\lambda,$$

where

$$q_1 = A_1^{-1}\lambda_1, q_2 = A_2^{-1}\lambda_1, \chi = A_2\lambda_2^{-1}\lambda_1 A_1^{-1}.$$

All necessary proofs were conducted in [6,9,15] by method of contour integration.

3. Matrix Fourier Transforms on Real Semi-axis

Theorem 4. *6The matrix value Sturm-Liouville problem with Dirichlet boundary condition*

$$-A^2 \frac{d^2}{dx^2}\phi(x,\lambda) = \lambda^2\phi(x,\lambda), \quad 0 < x < \infty,$$
$$\phi(0,\lambda) = 0, |\phi(x,\lambda)| < \infty,$$

provides the direct and inverse matrix integral sin *- transforms on the real semi-axis:*

$$F_+[f](\lambda) \equiv \tilde{f}(\lambda) = \int_0^\infty \sin\left(A^{-1}\lambda\xi\right) f(\xi)\, d\xi,$$
$$f(x) = \frac{2A^{-1}}{\pi}\int_0^\infty \sin\left(A^{-1}\lambda x\right) \tilde{f}(\lambda)\, d\lambda.$$

Theorem 5. *7The matrix value Sturm-Liouville problem with Neumann boundary condition*

$$-A^2 \frac{d^2}{dx^2}\phi(x,\lambda) = \lambda^2\phi(x,\lambda), \quad 0 < x < \infty,$$
$$\phi'(0,\lambda) = 0, |\phi(x,\lambda)| < \infty,$$

provides the direct and inverse matrix integral cos *- transforms on the real semi-axis:*

$$F_+[f](\lambda) \equiv \tilde{f}(\lambda) = \int_0^\infty \cos\left(A^{-1}\lambda\xi\right) f(\xi)\, d\xi,$$
$$f(x) = \frac{2A^{-1}}{\pi}\int_0^\infty \cos\left(A^{-1}\lambda x\right) \tilde{f}(\lambda)\, d\lambda.$$

The kernels of integral transforms in theorems 4,5 may be calculated by formulasfrom [15].

Theorem 6. *The matrix -value Sturm-Liouville problem with Robin boundary condition*

$$-A^2 \frac{d^2\phi}{dx^2} = \lambda^2\phi(x,\lambda), x > 0,$$
$$H\phi + \frac{d\phi}{dx} = 0, x = 0,$$
$$|\phi(x,\lambda)| < \infty,$$

where H - square matrix with negative eigenvalues, provides the direct and inverse matrix Fourier type transforms on the real semi-axis:

$$F_+[f](\lambda) \equiv \tilde{f}(\lambda) = \int_0^\infty \begin{pmatrix} AH\sin A^{-1}\lambda\xi \\ -\lambda\cos A^{-1}\lambda\xi \end{pmatrix} A^{-1} f(\xi)\, d\xi,$$
$$f(x) = \frac{2}{\pi}\int_0^\infty \begin{pmatrix} \sin A^{-1}\lambda x \cdot AH \\ -\lambda\cos A^{-1}\lambda x \end{pmatrix} \left((AH)^2 + \lambda^2 \mathrm{E}\right)^{-1} \tilde{f}(\lambda)\, d\lambda.$$

Proof. Let

$$\tilde{f}(\lambda) = \int_0^\infty \begin{pmatrix} AH\sin A^{-1}\lambda\xi \\ -\lambda\cos A^{-1}\lambda\xi \end{pmatrix} A^{-1} f(\xi)\, d\xi,$$

then

$$\tilde{f}(\lambda) = \int_0^\infty (AH\sin\lambda\xi - \lambda\cos\lambda\xi) f(A\xi)\, d\xi,$$

Applyingtheorem 1, замечание 1исчитаяприэтом $h = AH$ whence we may write

$$f(Ax) = \frac{2}{\pi}\int_0^\infty \begin{pmatrix} \sin\lambda xAH \\ -\lambda\cos\lambda xE \end{pmatrix} ((AH)^2 + E\lambda^2)^{-1} \tilde{f}(\lambda)\, d\lambda.$$

Using the definition offunction $f(Ax)$, we obtain

$$f(x) = \frac{2}{\pi}\int_0^\infty \begin{pmatrix} \sin\lambda A^{-1}x \cdot AH \\ -\lambda\cos\lambda A^{-1}x \end{pmatrix} ((AH)^2 + E\lambda^2)^{-1} \tilde{f}(\lambda)\, d\lambda.$$

4. Vector Wave Equation

In this section we can solve Cauchy problem for vector wave equation

$$u_{tt} = A^2 u_{xx}, t > 0, x \in R,$$
$$u(0,x) = f(x), u'(0,x) = 0,$$
$$u = \begin{pmatrix} u_1 \\ \dots \\ u_N \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \dots \\ f_N \end{pmatrix}.$$

The solution of this Cauchy problem has the form

$$u = F^{-1}\left[e^{-\lambda^2 t}\tilde{f}(\lambda)\right] =$$
$$\frac{A^{-1}}{2\pi}\int_{-\infty}^{\infty} e^{iA^{-1}\lambda x}\left[\cos\lambda t\int_{-\infty}^{\infty} e^{-iA^{-1}\lambda\xi} f(\xi)d\xi\right] d\lambda.$$

Simplify this formula we get

$$u = \frac{1}{2}\frac{A^{-1}}{2\pi}\int_{-\infty}^{\infty} e^{iA^{-1}\lambda(x+At)}\left[\int_{-\infty}^{\infty} e^{-iA^{-1}\lambda\xi} f(\xi)d\xi\right] d\lambda$$
$$+\frac{1}{2}\frac{A^{-1}}{2\pi}\int_{-\infty}^{\infty} e^{iA^{-1}\lambda(x-At)}\left[\int_{-\infty}^{\infty} e^{-iA^{-1}\lambda\xi} f(\xi)d\xi\right] d\lambda$$
$$= \frac{f(Ex+At) + f(Ex-At)}{2}.$$

So vector version of d'Alembert formula [14] is obtained.

5. Vector Heat Equation

In this section we can solve Cauchy problem for vector heat equation

$$u_t = A^2 u_{xx}, t > 0, x \in R,$$
$$u(0,x) = f(x),$$
$$u = \begin{pmatrix} u_1 \\ \dots \\ u_N \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \dots \\ f_N \end{pmatrix}.$$

The solution of Cauchy problem has the form

$$u = F^{-1}\left[e^{-\lambda^2 t}\tilde{f}(\lambda)\right] =$$
$$\frac{A^{-1}}{2\pi}\int_{-\infty}^{\infty} e^{iA^{-1}\lambda x}\left[e^{-\lambda^2 t}\int_{-\infty}^{\infty} e^{-iA^{-1}\lambda\xi} f(\xi)d\xi\right] d\lambda =$$
$$= \frac{A^{-1}}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-A^{-2}\frac{(x-\xi)^2}{4t}} f(\xi)d\xi.$$

So vector version of Poisson's formula is obtained.

We consider Cauchy problem for vector heat equation on real composite axis

$$u_{i,t} = A_i^2 u_{i,xx}, t > 0, x \in R_1,$$
$$u_i(0,x) = f_i(x),$$
$$u = \begin{pmatrix} u_1 \\ \dots \\ u_N \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \dots \\ f_N \end{pmatrix},$$

with internal boundary conditions at the point $x = 0$

$$u_1 = u_2, \lambda_1 u_1' = \lambda_2 u_2'.$$

The solution of this problem has the form

$$u = F_1^{-1}\left[e^{-\lambda^2 t}\tilde{f}(\lambda)\right].$$

Then using theorem 3 we obtain

$$u_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda A_1^{-1}x} e^{-\lambda^2 t} F(\lambda)d\lambda,$$

$$u_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix} e^{i\lambda A_2^{-1}x}\frac{E+\chi}{2} \\ +e^{-i\lambda A_2^{-1}x}\frac{E-\chi}{2}\end{pmatrix} e^{-\lambda^2 t}F(\lambda)d\lambda,$$

where

$$F(\lambda) = \int_{-\infty}^{0}\begin{pmatrix} e^{-i\lambda E\xi}d\xi \\ -(E-\chi)(E+\chi)^{-1}e^{i\lambda E\xi}\end{pmatrix} f_1(A_1\xi)d\xi$$
$$+\int_0^{\infty} 2(E+\chi)^{-1}e^{-i\lambda E\xi} f_2(A_2\xi)d\xi.$$

If we change the order of integration we obtain vector version of Poisson's formula [14].

$$u_1 =$$
$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{0}\begin{pmatrix} e^{-\frac{(A_1^{-1}x-E\xi)^2}{4t}} - \\ e^{-\frac{(A_1^{-1}x+E\xi)^2}{4t}}(E-\chi)(E+\chi)^{-1}\end{pmatrix} f_1(A_1\xi)d\xi$$
$$+\frac{1}{2\sqrt{\pi t}}\int_0^{\infty} e^{-\frac{(A_1^{-1}x-E\xi)^2}{4t}} 2(E+\chi)^{-1} f_2(A_2\xi)d\xi,$$

$$u_2 = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{0} e^{-\frac{(A_2^{-1}x-E\xi)^2}{4t}} 2\chi(E+\chi)^{-1} f_1(A_1\xi)d\xi +$$
$$\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}\begin{pmatrix} e^{-\frac{(A_2^{-1}x-E\xi)^2}{4t}} + \\ e^{-\frac{(A_2^{-1}x+E\xi)^2}{4t}}(E-\chi)(E+\chi)^{-1}\end{pmatrix} f_2(A_2\xi)d\xi.$$

6. Vector Dirichlet Problem

In this section we can solve Dirichlet problem for vector Laplace equation on real axis and real composite axis. Dirichlet problem on real axis is given by

$$u_{yy} + A^2 u_{xx} = 0, y > 0, x \in R,$$
$$u(0,x) = f(x),$$
$$u = \begin{pmatrix} u_1 \\ \dots \\ u_N \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \dots \\ f_N \end{pmatrix},$$

It solution is

$$u = F^{-1}\left[e^{-|\lambda|y}\tilde{f}(\lambda)\right]$$
$$= \frac{A^{-1}}{2\pi}\int_{-\infty}^{\infty} e^{iA^{-1}\lambda x}\left[e^{-|\lambda|y}\int_{-\infty}^{\infty} e^{-iA^{-1}\lambda\xi} f(\xi)d\xi\right]d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Ay\left(A^2y^2 + E(x-\xi)^2\right)^{-1} f(\xi)d\xi.$$

Dirichlet problem on real composite axis for vector Laplace equation is given by

$$u_{i,yy} + A^2 u_{i,xx} = 0, y > 0, x \in R_1,$$

with boundary conditions

$$u_i(0,x) = f_i(x),$$
$$u_i = \begin{pmatrix} u_{i,1} \\ \dots \\ u_{i,N} \end{pmatrix}, f_i = \begin{pmatrix} f_{i,1} \\ \dots \\ f_{i,N} \end{pmatrix},$$

with internal boundary conditions at the point $x = 0$

$$u_1 = u_2, \lambda_1 u_1' = \lambda_2 u_2'.$$

The solution of this problem be as follows

$$u = F_1^{-1}\left[e^{-|\lambda|y}F(\lambda)\right].$$

If we change order of integration we obtain vector version of Poisson's formula [14]

$$u_1 = \int_{-\infty}^{0}\begin{pmatrix} G(y, A_1^{-1}x - E\xi) \\ -G(y, A_1^{-1}x + E\xi)(E-\chi)(E+\chi)^{-1}\end{pmatrix} f_1(A_1\xi)d\xi$$
$$+\int_0^{\infty} G(y, A_1^{-1}x - E\xi)2(E+\chi)^{-1} f_2(A_2\xi)d\xi, x < 0$$
$$u_2 = \int_{-\infty}^{0} G(y, A_1^{-1}x - E\xi)2\chi(E+\chi)^{-1} f_1(A_1\xi)d\xi$$
$$+\int_0^{\infty}\begin{pmatrix} G(y, A_2^{-1}x - E\xi) \\ +G(y, A_2^{-1}x + E\xi)(E-\chi)(E+\chi)^{-1}\end{pmatrix} f_2(A_2\xi)d\xi,$$
$$x > 0,$$

where

$$G(y,B) = y(Ey^2 + B^2)^{-1}$$

is Poisson kernel [14].

References

[1] Uflyand I. S. (1967) *Integral transforms in the problem of the theory of elasticity*. Leningrad. Science, p. 402

[2] Uflyand I. S. (1967) *On some new integral transformations and their applications to problems of mathematical physics*. Problems of mathematical physics. Leningrad, p. 93-106

[3] M.P.Lenjuk, *Integral'noe preobrazovanie Fur'e na kusochno-odnorodnoj poluprjamoj*, Izv. vuzov. Matematika, Volume 4,1989, p. 14-18.

[4] L.S.Najda, *Gibridnye integral'nye preobrazovanija tipa Hankelja-Lezhandra,* Mat. metody analiza dinam. sistem, Har'kov, Volume 8, 1984, p. 132-135.

[5] V.S.Procenko, A.I.Solov'jov, *Gibridnye integral'nye preobrazovanija i ih prilozhenija v teorii uprugosti neodnorodnyh sred*, Prikladnaja mehanika, Issue 1, Volume 13, 1982, p. 62-67.

[6] E. Mogileva, O. Yaremko, *Hermite functions with discontinuous coefficients for the solution of fractal diffusion retrospective problems*, International journal of applied mathematics and informatics,Issue 3, Volume 7, 2013, p. 78-86.

[7] F.M. Mors, G. Fishbah, *Methods of theoretical physics*, New York, McGraw-Hill, 1953. Part I, 998 pp.

[8] O.E. Yaremko, *Matrix integral Fourier transforms for problems with discontinuous coefficients and transformation operators*, Reports of Academy of Sciences, Volume. 417, Issue 3, 2007, p. 323-325.

[9] Bavrin I.I., Matrosov V.L., Jaremko O. E. (2006) *Operators of transformation in the analysis, mathematical physics and Pattern recognition*. Moscow, Prometheus, p 292.

[10] Brejsuell R., (1990) Hartley *transform, Moscow,* World, p 584.

[11] Vladimirov V. S., Zharinov V.V., (2004) *The equations of mathematical physics*, Moscow, Phys mat lit, p 400.

[12] Gantmaxer F.R., (2010) *Theory matrix*. Moscow, Phys. Mat. Lit., p 560.

[13] Sneddon I., (1955) *Fourier Transform*, Moscow.

[14] *Physical encyclopedia*. The editor-in-chief A. M. Prokhorov, D.M. Alekseev, Moscow, (1988-1998).

[15] Jaremko O. E., (2007) *Matrix integral Fourier transform for problems with discontinuous coefficients and conversion operators*. Proceedings of the USSR Academy of Sciences. p. 323-325.