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Finding Formulas Involving Hypergeometric Functions by Evaluating and Comparing the Multipliers of the Laplacian on IR^n

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Abstract In this work we give exact formulas for some spectral multipliers of Laplacian on the Euclidian space IR^n. By comparing these multipliers we find old and new formulas involving hypergeometric functions.

Keywords: *hypergeometric function, heat kernel, wave kernel, resolvent kernel, laplace kernel, hankel transform, multiplier*

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1. Introduction

Among the classical equations of mathematical physics the heat equation $\Delta_n u - u_t = 0$, the Schrodinger equation $\Delta_n u - \frac{1}{i} u_t = 0$, the Laplace equation $\Delta_n u + u_{yy} = 0$, the wave equation $\Delta_n u - u_{tt} = 0$ and the Helmoltz equation $\Delta_n \phi + \lambda^2 \phi = \phi_0$, where Δ_n is the Laplace operator in IR^n. These equations are studied with and without initial conditions by several authors since long times [1]. For exemple the heat, the Schrodinger and the Laplace equations are often coupled with the initial condition u(0,x) = u_0(x), the wave equation with initial conditions u(0,x) = u_0(x), u_t(0,x) = u_1(x) and the Helmoltz equation with a boundarie condition at infinity called Sommerfeld radiation condtion ([4,6] and [7]). The aim of this paper is first to give explicit formulas of the Schwartz kernels of the following multipliers called here respectively the weighted heat, Schrodinger, Laplace, wave, resolvent and generalized resolvent kernels on

$$H_n^p(t) := e^{t\Delta_n}\left(\sqrt{-\Delta_n}\right)^p \quad t > 0 \qquad (1.1)$$

$$K_n^p(t) := e^{it\Delta_n}\left(\sqrt{-\Delta_n}\right)^p \quad t > 0 \qquad (1.2)$$

$$L_n^p(y) := e^{-y\sqrt{-\Delta_n}}\left(\sqrt{-\Delta_n}\right)^p \quad y > 0 \qquad (1.3)$$

$$W_n^p(t) := \frac{\sin t\sqrt{-\Delta_n}}{\sqrt{-\Delta_n}}\left(\sqrt{-\Delta_n}\right)^p \quad t > 0 \qquad (1.4)$$

$$\omega_n^p(t) := \cos t\sqrt{-\Delta_n}\left(\sqrt{-\Delta_n}\right)^p \quad t > 0 \qquad (1.5)$$

$$R_n^p(\lambda) := \left(\Delta_n + \lambda^2\right)^{-1}\left(\sqrt{-\Delta_n}\right)^p \quad \mathrm{Im}\,\lambda > 0 \qquad (1.6)$$

$$R_n^{\mu,p}(\lambda) := \left(\Delta_n + \lambda^2\right)^{-1-\mu}\left(\sqrt{-\Delta_n}\right)^p \quad \mathrm{Im}\,\lambda > 0 \qquad (1.7)$$

with $\Delta_n = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian on IR^n.

Note that we can define $\phi\sqrt{-\Delta_n}$ if the function ϕ is such that $F[\phi](\xi)$ is a tempered distribution ([4], p. 149) where F is the Fourier transform (see below for a precise statement).

Recall that the classical heat, Schrodinger, Laplace, wave and resolvent Schwartz kernels are given respectively by ([4], p. 146 et 170-171) and ([6], p. 59).

$$H_n^0(t,x,x') = (4\pi t)^{-n/2}\exp\left(-\frac{|x-x'|^2}{4t}\right) \quad t > 0 \qquad (1.8)$$

$$K_n^0(t,x,x') = (4\pi it)^{-n/2}\exp\left(-\frac{|x-x'|^2}{4it}\right) \qquad (1.9)$$

$$L_n^0(y,x,x') = \frac{\Gamma\left[(n+1)/2\right]}{\pi^{(n+1)/2}}\, y\left(y^2 + |x-x'|^2\right)^{-\frac{n+1}{2}} \qquad (1.10)$$

$$W_n^0(t,x,x') = \frac{1}{2^{(n+1)/2}\pi^{(n-1)/2}}\left(\frac{\partial}{t\partial t}\right)^{(n-1)/2} 1_{\{|x-x'|<t\}} \qquad (1.11)$$

$$n = 2k+1 \qquad t > 0$$

$$W_n^0(t,x,x')=\frac{1}{2^{n/2}\pi^{n/2}}\left(\frac{\partial}{t\partial t}\right)^{(n-2)/2}\left(t^2-|x-x'|^2\right)^{-1/2} \quad (1.12)$$

$n=2k \qquad t>0$

$$\omega_n^0(t,x,x')=\frac{1}{2^{(n+1)/2}\pi^{(n-1)/2}}\frac{\partial}{\partial t}\left(\frac{\partial}{t\partial t}\right)^{(n-1)/2}1_{\{|x-x'|<t\}} \quad (1.13)$$

$n=2k+1 \qquad t>0$

$$\omega_n^0(t,x,x')=\frac{1}{2^{n/2}\pi^{n/2}}\left(\frac{\partial}{t\partial t}\right)^{(n-2)/2}\left(t^2-|x-x'|^2\right)^{-1/2} \quad (1.14)$$

$n=2k \qquad t>0$

$$R_n^0(\lambda,x,x')=\frac{i}{4}\left(\frac{\lambda}{2\pi|x-x'|}\right)^{(n-2)/2}H^{(1)}_{(n-2)/2}\left(\lambda|x-x'|\right) \quad (1.15)$$

$\operatorname{Im}\lambda>0$

where $H_v^{(1)}$ is the Hankel function of the first kind. The end of this section is devoted to the preliminaries on the Fourier transform on $\mathbb{R}^n$ and formulas evaluating the Hankel transform of some functions.

The Fourier transform of a function $f\in L^1(IR^n)$ is defined by the integral

$$F|f|(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{+\infty}e^{-ix.\xi}f(x)dx. \quad (1.16)$$

with $x.\xi=x_1.\xi_1+x_2.\xi_2+\ldots+x_n.\xi_n$ is the inner product on $\mathbb{R}^n$. The Fourier inverse transform is given by

$$F^{-1}|f|(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{IR^n}e^{i\xi.x}f(\xi)d\xi \quad (1.17)$$

Recall also that the Fourier transform of a radial function f on R^n is radial and it can be written in terms of the Hankel transform ([7], p. 226) as

$$F^{-1}|f|(\xi)=|\xi|^{1-n/2}\int_0^{\infty}f(r)J_{n/2-1}(r|\xi|)r^{n/2}dr \quad (1.18)$$

where $J_v(.)$ is the Bessel function of the first kind and order v. For more informations on the Fourier transform the reader can consults the book [7].

Proposition 1.1. *The Schwartz kernel of the operator* $\phi\sqrt{-\Delta_n}$ *is given at last formally by*

$$K_n(\phi,x,x')=(2\pi)^{-n/2}|x-x'|^{1-n/2}\int_0^{\infty}J_{\frac{n-2}{2}}(r|x-x'|)\phi(r)r^{n/2}dr \quad (1.19)$$

The proof of this proposition uses essentially the formula (1.18) and in consequence is left to the reader. Recall the following formulas evaluating some Hankel transform ([3] p. 24, 29, and 30).

$$\int_0^{\infty}x^{\mu}e^{-ax^2}J_v(xy)dx=\frac{\Gamma\left(\frac{v+\mu+1}{2}\right)}{2^{v+1}\Gamma(v+1)}\frac{y^v}{a^{(\mu+v+1)/2}}{}_1F_1\left(\frac{\mu+v+1}{2},v+1;\frac{-y^2}{4a}\right) \quad (1.20)$$

$\operatorname{Re} a>0;\operatorname{Re}(\mu+v)>-1$ with ${}_1F_1(a,c,z)$ is the first kind conuente hypergeometric function.

$$\int_0^{\infty}x^{\mu-1}e^{-ax}J_v(xy)dx=\frac{\Gamma(v+\mu)}{2^v\Gamma(v+1)}\frac{z^v}{a^{\mu+v}}{}_2F_1\left(\frac{\mu+v}{2},\frac{\mu+v+1}{2},v+1;\frac{-z^2}{a^2}\right) \quad (1.21)$$

where ${}_2F_1$ (a,b,c;z) is the Gauss hypergeometric with $\operatorname{Re} a>0,\operatorname{Re}\mu+v>0$.

$$\int_0^{\infty}\frac{x^{\rho-1}J_v(ax)}{(x^2+k^2)^{\mu+1}}dx$$
$$=\frac{\Gamma((\rho+v)/2)\Gamma(\mu+1-(\rho+v)/2)}{2^{v+1}\Gamma(\mu+1)\Gamma(v+1)}a^vk^{\rho+v-2\mu-2}\,{}_1F_2\left((\rho+v)/2,(\rho+v)/2-\mu,v+1,\frac{a^2k^2}{4}\right) \quad (1.22)$$
$$+\frac{\Gamma(-\mu-1+(\rho+v)/2)}{2^{2\mu+3-\rho}\Gamma(\mu+2+(v-\rho)/2)}a^{2\mu+2-\rho}\,{}_1F_2\left(\mu+1,\mu+2+(v-\rho)/2,\mu+2-(\rho+v)/2,\frac{a^2k^2}{4}\right)$$

for a > 0, $-\operatorname{Re}v<\operatorname{Re}\rho<2\operatorname{Re}\mu+7/2$ where

$${}_1F_1(a;c,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n\,n!}z^n \quad (1.23)$$

$${}_2F_1(a,b;c,z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}z^n \quad (1.24)$$

$${}_1F_2(a;b,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n(c)_n\,n!}z^n \quad (1.25)$$

are the hypergeometric functions.

2. Weighted Heat and Schrodinger Evolution Operators on IR^n

In this section we give an exact formula for the Schwartz integral kernel of the weighted heat and Schrodinger evolution operators $e^{t\Delta n}(-\Delta_n)^{p/2}$ and $e^{it\Delta}(-\Delta_n)^{p/2}$ on IR^n.

Theoreme 2.1. *For Rep > -n, the Schwartz integral kernel of the weighted heat evolution operator* $e^{t\Delta}(-\Delta_n)^{p/2}$ *on* IR^n *is given in terms of the first order Kummer conuent hypergeometric function* ${}_1F_1(a, c; z)$ *by*

$$H_n^p(t,x,x')=\frac{\Gamma\left(\frac{n+p}{2}\right)}{(4\pi)^{n/2}\Gamma(n/2)}t^{-\frac{n+p}{2}}\,{}_1F_1\left(\frac{n+p}{2},\frac{n}{2};\frac{-|x-x'|^2}{4t}\right) \quad (2.1)$$

Proof. By making use of the formula (1.19) with $\phi(r)=e^{-tr^2}r^p$ we can write

$$H_n^p(t,x,x')=(2\pi)^{-n/2}|x-x'|^{1-n/2}\int_0^\infty J_{\frac{n-2}{2}}(|x-x'|r)e^{-tr^2}r^{p+n/2}dr \tag{2.2}$$

using the formula (1.20) with $v=\frac{n-2}{2},\mu=p+n/2,a=t$ and $y=|x-x'|$ we get the formula (2.1).

Corollary 2.2. *For Rep >-n, the Schwartz integral kernel of the weighted Schrodinger evolution operator $e^{it\Delta_n}(-\Delta_n)^{p/2}$ on IR^n is given in terms of the first order Kummer conuent hypergeometric function $_1F_1(a, c; z)$ by*

$$K_n^p(t,x,x')=\frac{\Gamma\left(\frac{n+p}{2}\right)}{(4\pi)^{n/2}\Gamma(n/2)}(it)^{-\frac{n+p}{2}}{}_1F_1\left(\frac{n+p}{2},\frac{n}{2};\frac{-|x-x'|^2}{4it}\right) \tag{2.3}$$

3. Weighted Poisson Operator on IR^n

This section is devoted to the computation of the weighted Poisson operator $e^{-y\sqrt{-\Delta_n}}(-\Delta_n)^{p/2}$.

Theoreme 3.1. *For Rep >-n, The Shwartz integral kernel of the weighted Poisson operator $e^{-y\sqrt{-\Delta_n}}(-\Delta_n)^{p/2}$ on IR^n is given in terms of the Gauss hypergeometric function $_2F_1$ by*

$$L_n^p(y,x,x')=\frac{2\Gamma(p+n)}{(4\pi)^{n/2}\Gamma(n/2)}y^{-n-p}{}_2F_1\left(\frac{n+p}{2},\frac{n+p+1}{2};\frac{n}{2},-\frac{|x-x'|^2}{y^2}\right) \tag{3.1}$$

with $_2F_1$ is the Gauss hypergeometric function.

Proof. From the formula (1.19) with $\phi(r)=e^{-yr}r^p$ we get

$$L_n^p(y,x,x')=(2\pi)^{-n/2}|x-x'|^{1-n/2}\int_0^\infty J_{\frac{n-2}{2}}(r|x-x'|)r^{p+n/2}e^{-yr}dr. \tag{3.2}$$

and by the formula (1.21) with $v=(n-2)/2,\mu=p+n/2+1,z=|x-x'|$ and a = y one can easily deduce the formula (3.1) .

Proposition 3.2. *$H_n^p(z,x,x')$ and $L_n^p(y,x,x')$ are respectively the weighted heat and the weighted. Poisson kernels given above then we have*

$$L_n^p(y,x,x')=\frac{y}{\sqrt{\pi}}\int_0^\infty e^{-y^2/4z}z^{-1/2}H_n^p(z,x,x')dz. \tag{3.3}$$

and

$$\int_0^\infty e^{-\frac{y^2}{4}t}t^{(n+p-3)/2}{}_1F_1\left((n+p)/2,n/2,-\frac{|x-x'|^2}{4}t\right)dt = -\frac{2\sqrt{\pi}\Gamma(n+p)}{\Gamma((n+p)/2)}y^{-n-p-1}{}_2F_1\left(\frac{n+p}{2},\frac{n+p+1}{2};\frac{n}{2},-\frac{|x-x'|^2}{y^2}\right) \tag{3.4}$$

Proof. We recall the formula ([6], p. 50)

$$\frac{e^{-y\lambda}}{y}=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-uy^2}u^{-1/2}e^{-\lambda^2/4u}du \tag{3.5}$$

By setting $\lambda=\sqrt{-\Delta_n}$ in (3.5) we can write for the Poisson semi-group

$$e^{-y\sqrt{-\Delta_n}}=\frac{y}{\sqrt{\pi}}\int_0^\infty e^{-y^2/4u}u^{-1/2}e^{u\Delta_n}du \tag{3.6}$$

and this gives the formula (3.3). To prove the formula (3.4) we use the formula (3.3), (2.1) and (3.1).

4. Weighted Wave Evolution Operator on IR^n

In this section we shall compute explicitly the Schwartez integral kernel of the weighted wave evolution operators $\frac{\sin t\sqrt{-\Delta_n}}{\sqrt{-\Delta_n}}(-\Delta_n)^{p/2}$ and $\cos t\sqrt{-\Delta_n}(-\Delta_n)^{p/2}$

Theorem 4.1. *For t > 0 we have*

$$\frac{\sin t\sqrt{-\Delta_n}}{\sqrt{-\Delta_n}}=\sqrt{\frac{\pi}{2}}\frac{t}{2i\pi}\int_{-\infty}^{0+}e^{(1/2)\left(u+t^2\Delta_n/u\right)}u^{-3/2}du \tag{4.1}$$

Here we should note that the integral in (4.1) can be extended over a contour starting at 1, going clockwise around 0, and returning back to 1 without cutting the real negative semi-axis.

Proof. We start by recalling the formulas ([5], p. 73)

$$\sin z=\sqrt{\pi z/2}J_{1/2}(z) \tag{4.2}$$

where $J_v(.)$ is the Bessel function of first kind and of order v given by ([5], p. 83)

$$J_v(\alpha z)=\frac{z^v}{2i\pi}\int_{-\infty}^{0+}e^{(\alpha/2)\left(t-z^2/t\right)}t^{-v-1}dt \tag{4.3}$$

provided that $\mathrm{Re}\,\alpha>0$ and $|\arg z|\le\pi$. Moreover, we have the following formula:

$$\frac{\sin\alpha z}{z}=\sqrt{\frac{\pi}{2}}\frac{\sqrt{\alpha}}{2i\pi}\int_{-\infty}^{0+}e^{(\alpha/2)\left(t-z^2/t\right)}u^{-3/2}du \tag{4.4}$$

Putting α= 1 and replacing the variable z by the symbol $t\sqrt{-\Delta_n}$ (4.4) we obtain the formula (4.1).

Theorme 4.2. *For p > 1-n the Schwartz integral kernel of the weighted wave operator $\frac{\sin t\sqrt{-\Delta_n}}{\sqrt{-\Delta_n}}\left(\sqrt{-\Delta_n}\right)^p$ is given by the following formulas*

$$W_n^p(t,x,y) = \frac{t}{2i\sqrt{2\pi}} \int_{-\infty}^{0+} e^{u/2} H_n^p\left(t^2/2u, x\sqrt{x'}\right) u^{-3/2} du \quad (4.5)$$

and

$$W_n^p(t,x,x') = \begin{cases} C_n^p t^{-(n+p-1)} F\left(\frac{n+p}{2}, \frac{n+p-1}{2}; \frac{n}{2}, \frac{|x-x'|^2}{t^2}\right) & if\ |x-x'| < t \\ & if\ 0 < t < |x-x'| \end{cases} \quad (4.6)$$

where $H_n^p(t,x,x')$ is the weighted heat kernel given in (2.1)

$$C_n^p = \frac{\Gamma((n+p)/2)\Gamma((n+p-1)/2)\sin(n+p-3)\pi/2}{2^{n+3/2}\Gamma(n/2)\pi^{(n+1)/2}}$$

and $_2F_1$ is the classical Gauss hypergeometric function.

Proof. The formula (4.5) is a consequence of (4.1), to prove the formula (4.6) set

$$W_n^p(t,x,x') = \frac{\Gamma\left(\frac{n+p}{2}\right)}{i\Gamma(n/2)2^{(n-p)/2+3/2}\pi^{(n+1)/2}} t^{1-n-p} J \quad (4.7)$$

where

$$J = \int_{-\infty}^{0+} e^{\frac{u}{2}} u^{(n+p-3)/2} {}_1F_1\left(\frac{n+p}{2}, \frac{n}{2}, -\frac{|x-x'|^2}{2t^2}u\right) du \quad (4.8)$$

then we have

$$J = J_1 + J_2 + J_3 \quad (4.9)$$

$$J_1 = \int_{\gamma 1} e^{\frac{u}{2}} u^{(n+p-3)/2} {}_1F_1\left(\frac{n+p}{2}, \frac{n}{2}, -\frac{|x-x'|^2}{2t^2}u\right) du \quad (4.10)$$

$$J_2 = \int_{\gamma 2} e^{\frac{u}{2}} u^{(n+p-3)/2} {}_1F_1\left(\frac{n+p}{2}, \frac{n}{2}, -\frac{|x-x'|^2}{2t^2}u\right) du \quad (4.11)$$

and

$$J_3 = \int_{\gamma 3} e^{\frac{u}{2}} u^{(n+p-3)/2} {}_1F_1\left(\frac{n+p}{2}, \frac{n}{2}, -\frac{|x-x'|^2}{2t^2}u\right) du \quad (4.12)$$

where the paths γ1, γ2 and γ3 are given by

$\gamma 1: z = re^{i\pi};\ \epsilon \le r < \infty$ (above the cut)

$\gamma 2: z = re^{-i\pi};\ \infty > r \ge \epsilon$ (below the cut)

$\gamma 3: z = \epsilon e^{-i\theta};\ -\pi < \theta < \pi$ (rund the small circle)

as $\epsilon \to 0$, we have $J_1 \to e^{i(n+p-3)\pi/2} I$, $J_2 \to -e^{-i(n+p-3)\pi/2} I$ and $J_3 \to 0$.

Adding the integrals estabilishes the following formula

$$J = 2i\sin(n+p-3)\pi/2I \quad (4.13)$$

with

$$I = \int_0^\infty e^{-\frac{u}{2}} u^{(n+p-3)/2} {}_1F_1\left(\frac{n+p}{2}, \frac{n}{2}, \frac{|x-x'|^2}{2t^2}u\right) du \quad (4.14)$$

Recalling the formula [5], p. 24 Reα> 0, Reα >, Rek, Rez > 0

$$\int_0^\infty e^{-\alpha t} t^{z-1} {}_1F_1(a,c,kt)\,dt = \alpha^{-z}\Gamma(z)_2 F_1\left(a,z,c,\frac{k}{\alpha}\right) \quad (4.15)$$

with α = 1/2, a = (n + p)/2, c = n/2, z = (n + p-1)/2 and $k = \frac{|x-x'|^2}{2t^2}$ to get

$$I = \begin{cases} 2^{(n+p-1)/2}\Gamma((n+p-1)/2) F\left(\frac{n+p}{2}, \frac{n+p-1}{2}; \frac{n}{2}, \frac{|x-x'|^2}{t^2}\right) & if\ |x-x'| < t \\ 0 & if\ 0 < t < |x-x'| \end{cases} \quad (4.16)$$

Combining (4.7), (4.13) and (4.16) we get the formula (4.5) .

Corollary 4.3. *The Schwartz integral kernel for the weighted wave evolution operator* $\cos t\sqrt{-\Delta_n}(-\Delta_n)^{p/2}$ *on the Euclidian space can be written on the following form*

$$\omega_n^p(t,x,y) = \begin{cases} c_n^p t^{-(n+p)} F\left(\frac{n+p}{2}, \frac{n+p-1}{2}; \frac{n}{2}, \frac{|x-x'|^2}{t^2}\right) & if\ |x-x'| < t \\ 0 & if\ 0 < t < |x-x'| \end{cases} \quad (4.17)$$

with

$$c_n^p = -\frac{\Gamma((n+p)/2)\Gamma((n+p+1)/2)\sin(n+p-3)\pi/2}{\Gamma(n/2)2^{n+1/2}\pi^{(n+1)/2}}.$$

Proof. In view of the Formula $\omega_n^p(t,x,x') = \frac{d}{dt}W_n^p(t,x,x')$ we can use the formula ([5], p. 41)

$$\frac{d}{dz} z^a {}_2F_1(a,b,c,z) = az^{a-1} {}_2F_1(a+1,b,c,z) \quad (4.18)$$

to obtain the formula (4.17) from the formula (4.6).

5. Weighted Resolvent Operator on IR^n

In this section we give explicit formula for the Schwartz integral kernel of the weighted resolvent operator $R_n^p(\lambda) = \left(\Delta_n + \lambda^2\right)^{-1}(-\Delta_n)^{p/2}$.

Theorem 5.1. *For Imλ > 0 The Schwartz integral kernel for the weighted resolvent operator is given by*

$$R_n^p(\lambda,x,x')=\frac{\Gamma((n+p)/2)\Gamma(1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(n/2)}\left(-\lambda^2\right)^{(n+p)/2-1}{}_1F_2\left((n+p)/2,(n+p)/2,n/2,-\frac{\lambda^2}{4}|x-x'|^2\right)+\frac{\Gamma(-1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(1-p/2)\pi^{n/2}}\left(\frac{|x-x'|^2}{4}\right)^{1-(n+p)/2}{}_1F_2\left(1,2-(n+p)/2,1-p/2,-\frac{\lambda^2}{4}|x-x'|^2\right) \tag{5.1}$$

where ${}_1F_2$ is the hypergeometric series given in (1.25).

Proof. Using the formula (1.19) with $\phi(r)=\left(r^2-\lambda^2\right)^{-1}r^p$ we get

$$R_n^p(\lambda,x,x')=(2\pi)^{-n/2}|x-x'|^{1-n/2}\int_0^\infty\frac{J_{\frac{n-2}{2}}(r|x-x'|)}{r^2-\lambda^2}r^{n/2+p}dr. \tag{5.2}$$

and by the formula (1.22) with $v=(n-2)/2,\mu=0,\rho=n/2+p+1,a=|x-x'|$ and $k^2=-\lambda^2$ we get the result of the theorem.

Proposition 5.2. *Let $R_n^p(\lambda,x,x')$ be the Schwartz kernel of the weighted resolvent operator then we have the following integral representations*

$$R_n^p(\lambda,x,x')=\int_0^\infty e^{\lambda^2t}H_n^p(t,x,x')dt;\quad \operatorname{Re}\lambda^2<0 \tag{5.3}$$

$$R_n^p(\lambda,x,x')=i\int_0^\infty e^{-\lambda^2t}K_n^p(t,x,x')dt;\quad \operatorname{Re}\lambda^2<0 \tag{5.4}$$

$$R_n^p(\lambda,x,x')=\int_0^\infty e^{-\lambda^2t}W_n^p(t,x,x')dt;\quad \lambda\in IR \tag{5.5}$$

$$R_n^p(\lambda,x,x')=\frac{i}{\lambda}\int_0^\infty e^{-it\lambda}\omega_n^p(t,x,x')dt;\quad \lambda\in IR \tag{5.6}$$

where $H_n^p(t,x,x'),K_n^p(t,x,x'),W_n^p(t,x,x'),\omega_n^p(t,x,x')$ are respectively the Schwartz integral kernel of the weighted heat, Schrodinger, and wave evolution operators.

Proof. We use respectively the following formulas

$$\left(a^2+y^2\right)^{-1}=\int_0^\infty e^{-\left(a^2+y^2\right)t}dt\quad \operatorname{Re}a^2>0 \tag{5.7}$$

$$\left(a^2+y^2\right)^{-1}=i\int_0^\infty e^{-\left(a^2+y^2\right)it}dt\quad \operatorname{Re}a^2>0 \tag{5.8}$$

$$\left(a^2+y^2\right)^{-1}=\int_0^\infty e^{-ax\frac{\sin xy}{y}}dx\quad \operatorname{Re}a>0 \tag{5.9}$$

$$\left(a^2+y^2\right)^{-1}=\frac{1}{a}\int_0^\infty e^{-ax}\cos xy dx\quad \operatorname{Re}a>0 \tag{5.10}$$

Corollary 5.3. *For $Re\lambda^2<0$ we have the following formula*

$$\int_0^\infty e^{\lambda^2t}t^{-(n+p)/2}{}_1F_1\left((n+p)/2,n/2,-\frac{|x-x'|}{4t}\right)dt=\Gamma(1-(n+p)/2)\left(-\lambda^2\right)^{(n+p)/2-1}{}_1F_2\left((n+p)/2,(n+p)/2,n/2,-\frac{\lambda^2}{4}|x-x'|^2\right)+\frac{\Gamma(-1-(n+p)/2)\Gamma(n/2)}{\Gamma((n+p)/2)\Gamma(1-p/2)}\left(\frac{|x-x'|^2}{4}\right)^{1-(n+p)/2}{}_1F_2\left(1,2-(n+p)/2,1-p/2,-\frac{\lambda^2}{4}|x-x'|^2\right) \tag{5.11}$$

Proof. This is a consequence of the proposition (5.2) formula (5.3), (5.1) and (2.1).

On can write the weighted resolvent in terms of the weighted wave kernel.

Corollary 5.4. *We have*

$$\int_{|x-x'|}^\infty e^{-i\lambda s}s^{-(n+p-1)}F\left(\frac{n+p}{2},\frac{n+p-1}{2};\frac{n}{2},\frac{|x-y|^2}{s^2}\right)ds=\left(C_n^p\right)^{-1}\left[\frac{\Gamma((n+p)/2)\Gamma(-1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(n/2)}\left(-\lambda^2\right)^{(n+p)/2-1}\times{}_1F_2\left((n+p)/2,(n+p)/2,n/2,-\frac{\lambda^2}{4}|x-x'|^2\right)+\frac{\Gamma(-1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(1-p/2)}\left(\frac{|x-x'|^2}{4}\right)^{-(n+p)/2+1}{}_1F_2\left(1,2-(n+p)/2,1-p/2,-\frac{\lambda^2}{4}|x-x'|^2\right)\right] \tag{5.12}$$

$$\frac{1}{i\lambda}\int_{|x-x'|}^\infty e^{-i\lambda s}s^{-(n+p-1)}F\left(\frac{n+p}{2},\frac{n+p-1}{2};\frac{n}{2},\frac{|x-y|^2}{s^2}\right)ds=\left(C_n^p\right)^{-1}\left[\frac{\Gamma((n+p)/2)\Gamma(-1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(n/2)}\left(-\lambda^2\right)^{(n+p)/2-1}\times{}_1F_2\left((n+p)/2,(n+p)/2,n/2,-\frac{\lambda^2}{4}|x-x'|^2\right)+\frac{\Gamma(-1-(n+p)/2)}{\Gamma(1-p/2)2^{2-n/2-p}}\left(\frac{|x-x'|^2}{4}\right)^{-(n+p)/2+1}{}_1F_2\left(1,2-(n+p)/2,1-p/2,-\frac{\lambda^2}{4}|x-x'|^2\right)\right] \tag{5.13}$$

where C_n^p is as in the theorem 4.2.

Proof. The proof of this corollary can be seen from the proposition 5.2 (5.5), (5.6), (4.6), (4.17) and (5.1).

6. Weighted Generalized Resolvent Operators on IR^n

In this section we generalize some results of the section 5 by give an explicit expression of the weighted generalized resolvent kernels $R_n^{\mu,p}(\lambda) = \left(\Delta_n + \lambda^2\right)^{-1-\mu} \left(-\Delta_n\right)^{p/2}$.

Theorem 6.1. *For Re$\lambda^2 < 0$ The Schwartz kernel of the weighted generalized resolvent operator is given by*

$$R_n^{\mu,p}(\lambda,x,x') = \Gamma((n+p)/2)\Gamma(\mu+1) \frac{\Gamma(\mu+1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(n/2)} \left(-\lambda^2\right)^{-1-\mu+(n+p)/2} {}_1F_2\left((n+p)/2, -\mu+(n+p)/2, n/2, -\frac{\lambda^2}{4}|x-x'|^2\right) + \frac{\Gamma(-\mu-1-(n+p)/2)}{(4\pi)^{n/2}\Gamma(1+\mu-p/2)} \left(\frac{|x-x'|^2}{4}\right)^{\mu-(n+p)/2+1} {}_1F_2\left(1+\mu, \mu+2-(n+p)/2, \mu+1-p/2, -\frac{\lambda^2}{4}|x-x'|^2\right) \quad (6.1)$$

where ${}_1F_2$ is the hypergeometric function given in (1.25).

Proof. Using the formula (1.19) with $\phi(r) = \left(r^2-\lambda^2\right)^{-1-\mu}r^p$ we get

$$R_n^{p,\mu}(\lambda,x,x') = (2\pi)^{-n/2}|x-x'|^{1-n/2}\int_0^\infty \frac{J_{\frac{n-2}{2}}(r|x-x'|)}{\left(r^2-\lambda^2\right)^{\mu+1}} r^{n/2+p}dr. \quad (6.2)$$

and to see the formula (6.1) we use (1.22) with $v = (n-2)/2, \rho = n/2+p+1, a = |x-x'|$ and $k^2 = -\lambda^2$.

Proposition 6.2. *We have the following formula connecting the weighted generalized resolvent kernel to the weighted heat kernel*

$$R_n^{\mu,p}(\lambda,x,x') = \frac{1}{\Gamma(\mu+1)} \int_0^\infty e^{\lambda^2 t} t^\mu H_n^p(t,x,x')dt;\ \mathrm{Re}\,\lambda^2 < 0. \quad (6.3)$$

Proof. We use the formula

$$\left(a^2+y^2\right)^{-1-\mu} = \frac{1}{\Gamma(\mu+1)}\int_0^\infty e^{-\left(a^2+y^2\right)t} t^\mu dt; \quad (6.4)$$

$\mathrm{Re}\, a < 0$.

Corollary 6.3. *We have*

$$\int_0^\infty e^{\lambda^2 t} t^{\mu-(n+p)/2} {}_1F_1\left((n+p)/2, n/2, -\frac{|x-x'|}{4t}\right)dt = \frac{\Gamma(\mu+1-(n+p)/2)\Gamma(\mu+1)}{\Gamma(\mu+1)}\left(-\lambda^2\right)^{-1-\mu+(n+p)/2-1} \times {}_1F_2\left((n+p)/2, -\mu+(n+p)/2, n/2, -\frac{\lambda^2}{4}|x-x'|^2\right) + \frac{\Gamma(-\mu-1-(n+p)/2)\Gamma(n/2)\Gamma(\mu+1)}{\Gamma((n+p)/2)\Gamma(1+\mu-p/2)}\left(\frac{|x-x'|^2}{4}\right)^{\mu-(n+p)/2+1} {}_1F_2\left(1+\mu, \mu+2-(n+p)/2, \mu+1-p/2, -\frac{\lambda^2}{4}|x-x'|^2\right) \quad (6.5)$$

Proof. We use the formulas (6.3), (2.1) and (6.1).

7. Commentaries and Applications

The subject of study of this paper is situated at the meeting point of the partial differential equations and the special functions of the mathematical physics.

Firstly explicit solutions of the following partial differential equations are given in terms of the hypergeometric functions.

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \Delta_n u(t,x) = 0; (t,x) \in IR_+^* \times IR^n \\ u(0,x) = \left(\sqrt{-\Delta_n}\right)^p u_0(x); u_0 \in C_0^\infty\left(IR^n\right) \end{cases}, \quad \left(H_p^n\right)$$

and

$$\begin{cases} \frac{\partial}{\partial y^2}V(t,x) + \Delta_n V(y,x) = 0; (y,x) \in IR_+^* \times IR^n \\ \lim_{y\to 0} V(y,x) = \left(\sqrt{-\Delta_n}\right)^n V_0(x), V_0 \in C_0^\infty\left(IR^n\right) \end{cases}, \quad \left(P_p^n\right)$$

$$\begin{cases} \frac{\partial^2}{\partial t^2}v(t,x) - \Delta_n v(t,x) = 0; \\ (t,x) \in IR_+^* \times IR^n \\ v(0,x) = \left(\sqrt{-\Delta_n}\right)^p v_0(x), \\ v_t(0,x) = \left(\sqrt{-\Delta_n}\right)^p v_1(x); \\ v_0, v_1 \in C_0^\infty\left(IR^n\right) \end{cases} \quad \left(W_p^n\right)$$

$$\left[\Delta_n + \lambda^2\right]R(\lambda,x,x') = \left(\sqrt{-\Delta_n}\right)^p \delta; x \in IR^n \quad \left(R_p^n\right)$$

Secondly some old and new formulas involving the hypergeometric functions are given by comparing these solutions. The formulas (5.1) and (6.1) gives the Laplace transform of Kummer hypergeometric function with argument 1/x and extend the well known formula giving the Laplace transform of the exponential with the argument 1/x.

$$\int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x-x'|^2}{4t}} e^{\lambda^2 t}dt = \frac{i}{4}\left(\frac{\lambda}{2\pi|x-x'|}\right)^{(n-2)/2} H_{(n-2)/2}^{(1)}(\lambda|x-x'|) \quad (7.1)$$

$\mathrm{Im}\,\lambda > 0$

The formulas (5.12) gives the Fourier transform of the Gauss hypergeometric function with argument $1/x^2$ and extend the known formula

$$H_0^{(1)}(\lambda|x-x'|) = \frac{-2i}{\pi}\int_{|x-x'|}^\infty e^{i\lambda s}\left(s^2-|x-x'|^2\right)^{-1/2} ds \quad (7.2)$$

and to the best of our knowledge there is no such relation in the mathematical litterature. Note that the formulas (7.1) and (7.2) are a consequences respectively of the formula

$$H_\nu^{(1)}(z) = \frac{2}{i\sqrt{\pi}\Gamma(1/2-\nu)}(z/2)^{-\nu} \int_1^\infty e^{izt}\left(t^2-1\right)^{-\nu-1/2} dt \quad (7.3)$$

(see Erdely et al [2], p. 83).

$$H_\nu^{(1)}\left(z\sqrt{\alpha^2}\right) = \frac{-i}{\pi} e^{-i\nu\pi/2}\left(\alpha^2\right)^{\nu/2} \int_0^\infty e^{i\frac{z}{2}\left(t+\frac{\alpha^2}{t}\right)} t^{-\nu-1} dt \quad (7.4)$$

Imz > 0 and Imα^2z > 0 (see Magnus et al [5], p. 84).

We can also derive the formulas (7.1) and (7.2) from (5.3) (1.9), (1.15) and (5.5), (1.12), (1.15).

We finish this section by the following corollary.

Corollary 7.1. *We have the following formula connecting the Hankel and the hypergeomtric functions* ${}_1F_2$.

$$H_\nu^{(1)}(x) = (-1)^\nu \frac{\Gamma(-\nu)}{i\pi}(x/2)^\nu \, {}_1F_2\left(\nu+1, \nu+1, \nu+1, -\frac{x^2}{4}\right) + \frac{\Gamma(-2-\nu)}{i\pi}(x/2)^{-\nu} \, {}_1F_2\left(1, 1-\nu, 1, -\frac{x^2}{4}\right) \quad (7.5)$$

Proof. By comparing the formulas (1.15) and (5.1) with p = 0 we obtain (7.5).

Remark 7.2. *By using the formula*

$${}_1F_2\left(a; a, \nu+1, -\frac{z^2}{4}\right) = 2^\nu \Gamma(\nu+1) z^{-\nu} J_\nu(z) \quad (7.6)$$

we can deduce from the formula (7.5) the following classical formula

$$H_\nu^{(1)}(x) = (i\sin\pi\nu)^{-1}\left[J_\nu(x) - J_{-\nu}(x) e^{-i\pi\nu}\right] \quad (7.7)$$

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