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On a Nonlocal Problem with the Second Kind Integral Condition for a Parabolic Equation

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Abstract In this article we consider a nonlocal problem with the second kind integral condition for a parabolic equation. Under some conditions on initial data we proved existence and uniqueness of a generalized solution applying the method of a priori estimates and a parameter continuation method.

Keywords: *nonlocal condition, nonlocal problem, parabolic equation, parameter continuation method*

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1. Introduction

Over the previous years nonclassical problems for partial differential equations have been widely used for a description a number of phenomena in modern physics and technology. Nonclassical problems with nonlocal conditions include relations between boundary values of an unknown solution and its derivatives and their values at internal points of a domain. Nonlocal problems with integral conditions which are naturally generalization of discrete nonlocal conditions can be considered as mathematical models the processes with inaccessible boundary.

One of the initial works devoted to problems with integral condition for second-order partial differential equations is the paper of Cannon [2] where the nonlocal problem was used for modeling of heat conduction process. Later, nonlocal problems with integral conditions were investigated in [1,3,4,6]. Note here some resent results [7,8,9,10].

It was found that the choice of the method of research depends considerably on the kind of nonlocal conditions.

Usually, we understand nonlocal condition of the II kind as correlations connecting values of the solution, and possibly, its derivatives on some inner manifold and at boundary points. If these correlations do not contain values of the solution on the boundary, then they are said to be nonlocal conditions of the I kind [11]. In the case of one spatial variable difficulties generated by the I kind integral condition can be avoided through reducing the problem to the problem with the II kind integral condition, and hence, in such case nonlocal problems with the II kind conditions are of considerable interest.

Motivated by this, in the present article we study a nonlocal problem with the II kind integral condition for parabolic equation. The main aim of the paper is to demonstrate a nonlinear functional analysis technique which is applicable for investigation of such nonlocal problems.

2. Preliminaries

Let $Q_T = \{(x,t) : x \in (0,l), t \in (0,T)\}$. Consider the equation

$$u_t - u_{xx} + c(x,t)u = f(x,t) \tag{1}$$

and set a problem: determine a solution $u(x,t)$ of the equation (1) in Q_T satisfying the initial condition

$$u(x,0) = \phi(x), 0 \le x \le l, \tag{2}$$

the boundary condition

$$u(0,t) = 0, 0 \le t \le T, \tag{3}$$

and the nonlocal condition

$$u(l,t) = \int_0^l K(x,t)u(x,t)\,dx \tag{4}$$

Definition. Following [5], we define as $W_2^{2,1}(Q_T)$ a Hilbert space which consists of all elements of $L_2(Q_T)$ such that $u_t, u_x, u_{xx} \in L_2(Q_T)$ with the norm:

$$\|u\|^2 = \int_{Q_T} \left(u^2 + u_t^2 + u_x^2 + u_{xx}^2\right) dxdt$$

Definition. A function $u(x,t) \in W_2^{2,1}(Q_T)$ is said to be a solution of the problem (1)-(4) if $u(x,0) = \phi(x)$, $u(0,t) = 0$, $u(l,t) = \int_0^l K(x,t)u(x,t)\,dx$ and for all $\eta(x,t) \in L_2(Q_T)$ the function $u(x,t)$ satisfies the integral identity

$$\int_{Q_T}(u_t-u_{xx}+cu)\eta(x,t)dxdt=\int_{Q_T}f(x,t)\eta(x,t)dxdt \quad (5)$$

3. Results

Theorem. Let $c(x,t)\in C_2^{2,1}(\overline{Q}_T)$, $f(x,t)\in L_2(Q_T)$, $\phi(x)\in W_2^1(0,l)$, $K(x,t)\in C_2^1(\overline{Q}_T)$ and $K(x,0)=0$, $\forall x\in[0,l]$. Then there exists a unique solution of the problem (1)-(4).

The proof of the theorem is organized as follows. First we obtain energy estimates which imply uniqueness of the solution. To prove the existence we use a parameter continuation method.

3.1. Energy Estimate

We take an arbitrary number $\lambda\in[0,1]$ and consider the problem (1)-(3) with the nonlocal condition

$$u(l,t)=\lambda\int_0^l K(x,t)u(x,t)\,dx \quad (6)$$

Assume that a solution of the problem (1)-(3), (6) exists. First we multiply (1) by u_{xx} and integrate over Q_τ, $0\le\tau\le T$. As a result after integration by parts we obtain the equality

$$\begin{aligned}&\frac{1}{2}\int_0^l u_x^2\,dx+\int_{Q_\tau}u_{xx}^2\,dxdt=\int_0^\tau u_x(l,t)u_t(l,t)dt\\&-\int_{Q_\tau}fu_{xx}\,dxdt+\int_0^\tau c(l,t)u(l,t)u_x(l,t)dt-\int_{Q_\tau}cu_x^2\,dxdt \quad (7)\\&+\frac{1}{2}\int_0^l\phi_x^2dx+\frac{1}{2}\int_0^\tau c_x(l,t)u^2(l,t)dt-\int_{Q_\tau}c_{xx}u^2\,dxdt\end{aligned}$$

Our next aim is to derive estimates of the right-hand side part of (7). Applying the ε-inequality to the first, second and third terms of (7) we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_x^2\,dx+\int_{Q_\tau}u_{xx}^2\,dxdt\le\frac{1}{2\rho}\int_0^\tau u_x^2(l,t)dt+\frac{\rho}{2}\int_0^\tau u_t^2(l,t)dt\\&+\frac{1}{2}\int_0^\tau u_x^2(l,t)dt+\frac{1}{2}\int_0^\tau c^2(l,t)u^2(l,t)dt+2\int_{Q_\tau}f^2\,dxdt\\&+\frac{1}{8}\int_{Q_\tau}u_{xx}^2\,dxdt+c_1\int_{Q_\tau}u_x^2\,dxdt+\frac{1}{2}\int_\Omega\phi_x^2dx \quad (8)\\&+\frac{c_2}{2}\int_0^\tau u^2(l,t)dt+\frac{c_3}{2}\int_{Q_\tau}u^2dxdt\end{aligned}$$

Observe further that for a solution of (1)-(3), (6) the following estimates are valid.

$$u^2(l,t)\le\lambda^2\left(\int_0^l K^2(x,t)dx\right)\left(\int_0^l u^2(x,t)dx\right)\le K_0\int_0^l u^2(x,t)dx \quad (9)$$

$$\begin{aligned}u_t^2(l,t)&\le 2\lambda^2K_1\int_0^l u^2(x,t)dx+2\lambda^2K_0\int_0^l u_t^2(x,t)dx\\&\le 2K_1\int_0^l u^2(x,t)dx+2K_0\int_0^l u_t^2(x,t)dx\end{aligned} \quad (10)$$

$$\int_0^\tau u_x^2(l,t)dt\le\int_{Q_\tau}\left(\varepsilon u_{xx}^2+a(\varepsilon)u_x^2\right)dxdt \quad (11)\text{[p. 77]},$$

where the constants $K_0,K_1>0$ are such that

$$\int_0^l K^2(x,t)dx\le K_0,\int_0^l K_t^2(x,t)dx\le K_1,0<t<T,$$

$\varepsilon>0$ is arbitrary and $a(\varepsilon)$ defines by l,ε.

From the inequalities (8)-(11) we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_x^2\,dx+\int_{Q_\tau}u_{xx}^2\,dxdt\\&\le\frac{1}{2\rho}\left(\int_{Q_\tau}\left(\varepsilon u_{xx}^2+a(\varepsilon)u_x^2\right)dxdt\right)\\&+\frac{1}{2}\int_{Q_\tau}\left(\varepsilon_1u_{xx}^2+a(\varepsilon_1)u_x^2\right)dxdt\\&+2\int_{Q_\tau}f^2\,dxdt+\frac{1}{2}\int_\Omega\phi_x^2dx+c_1\int_{Q_\tau}u_x^2\,dxdt \quad (12)\\&+\left(\frac{c_1^2K_0}{2}+\frac{c_2K_0}{2}+\frac{c_3}{2}\right)\int_{Q_\tau}u^2dxdt\\&+\rho K_0\int_{Q_\tau}u_t^2dxdt+\rho K_1\int_{Q_\tau}u^2dxdt\end{aligned}$$

Second, we multiply (1) we multiply (1) by u_t and integrate over Q_τ, $0\le\tau\le T$. After integration by parts we have

$$\begin{aligned}&\int_{Q_\tau}u_t^2\,dxdt+\frac{1}{2}\int_0^l u_x^2\,dx+\frac{1}{2}\int_0^l cu^2\,dx\\&=\int_{Q_\tau}fu_t\,dxdt+\frac{1}{2}\int_{Q_\tau}c_tu^2\,dxdt\\&+\int_0^\tau u_x(l,t)u_t(l,t)dt+\frac{1}{2}\int_0^l c(x,0)\phi^2dx+\frac{1}{2}\int_0^l\phi_x^2dx\end{aligned}$$

Applying the estimates (9)-(11) we obtain

$$\begin{aligned}&\int_{Q_\tau}u_t^2\,dxdt+\frac{1}{2}\int_0^l u_x^2\,dx+\frac{c_0}{2}\int_0^l u^2\,dx\le\int_{Q_\tau}f^2\,dxdt\\&+\frac{1}{4}\int_{Q_\tau}u_t^2\,dxdt+\frac{1}{2\rho}\left(\int_{Q_\tau}\left(\varepsilon u_{xx}^2+a(\varepsilon)u_x^2\right)dxdt\right)\\&+\frac{\rho}{2}\left(2K_0\int_{Q_\tau}u_t^2dxdt+2K_1\int_{Q_\tau}u^2dxdt\right)\\&+\frac{c_4}{2}\int_{Q_\tau}u^2\,dxdt+\frac{1}{2}\int_0^l\left(\phi_x^2+c_1\phi^2\right)dx\end{aligned}$$

Choose $\varepsilon=\frac{\rho}{4}$. Then

$$\frac{3}{4}\int_{Q_\tau} u_t^2\,dxdt+\frac{1}{2}\int_0^l\left(u_x^2+c_0u^2\right)dx\le\int_{Q_\tau} f^2\,dxdt$$
$$+\frac{1}{8}\int_{Q_\tau} u_{xx}^2\,dxdt+\frac{a(\varepsilon)}{2\rho}\int_{Q_\tau} u_x^2\,dxdt+\frac{1}{2}\int_0^l\left(\phi_x^2+c_1\phi^2\right)dx \quad (13)$$
$$+\left(\frac{c_4}{2}+\rho K_1\right)\int_{Q_\tau} u^2dxdt+\rho K_0\int_{Q_\tau} u_t^2dxdt.$$

It follows from (12), (13) that

$$\frac{3}{4}\int_{Q_\tau} u_t^2\,dxdt+\frac{1}{2}\int_0^l\left(2u_x^2+c_0u^2\right)dx+\frac{1}{2}\int_{Q_\tau} u_{xx}^2\,dxdt$$
$$\le 3\int_{Q_\tau} f^2\,dxdt+\frac{1}{2}\int_0^l\left(2\phi_x^2+c_1\phi^2\right)dx$$
$$+\left(c_1+\frac{a(\varepsilon_1)}{2}+\frac{a(\varepsilon)}{\rho}\right)\int_{Q_\tau} u_x^2\,dxdt+2\rho K_0\int_{Q_\tau} u_t^2dxdt$$
$$+\left(\frac{c_1^2K_0}{2}+\frac{c_2K_0}{2}+\frac{c_3}{2}+\frac{c_4}{2}+2\rho K_1\right)\int_{Q_\tau} u^2dxdt$$

Let $\rho=\frac{1}{8K_0}$. Then we obtain

$$\int_{Q_\tau}\left(u_t^2+u_{xx}^2\right)dxdt+\int_0^l\left(u_x^2+u^2\right)dx \le m\left(F(\tau)+\int_{Q_\tau}\left(u^2+u_x^2\right)dxdt\right) \quad (14)$$

where $m=\frac{m_1}{m_2}$, $m_1=\min\{1,c_0\}$, $m_2=\max\{6,k_1,k_2\}$

$$k_1=2c_1+a(\varepsilon_1)+16K_0a(\varepsilon),$$
$$k_2=c_1^2K_0+c_2K_0+c_3+c_4+\frac{K_1}{2K_0},$$
$$F(\tau)=\int_{Q_\tau} f^2\,dxdt+\int_0^l\left(\phi_x^2+\phi^2\right)dx.$$

It follows from (14) that

$$\int_0^l\left(u_x^2(x,\tau)+u^2(x,\tau)\right)dx\le m\left(F(\tau)+\int_{Q_\tau}\left(u^2+u_x^2\right)dxdt\right)$$

Now by Gronwall's lemma we conclude that

$$\forall\tau\in[0,T],\int_0^l\left(u_x^2(x,\tau)+u^2(x,\tau)\right)dx\le me^{m\tau}F(\tau)$$

and hence,

$$\int_{Q_\tau}\left(u^2+u_x^2\right)dxdt\le\left(e^{mT}-1\right)F(T). \quad (15)$$

From (14) it also follows that $\forall\tau\in[0,T]$

$$\int_{Q_\tau}\left(u_t^2+u_{xx}^2\right)dxdt\le m\left(F(\tau)+\int_{Q_\tau}\left(u^2+u_x^2\right)dxdt\right)$$

Therefore, the inequality (15) implies that

$$\int_{Q_\tau}\left(u^2+u_x^2+u_t^2+u_{xx}^2\right)dxdt\le M^2$$

where M defines by $K_0,K_1,c_i,i=\overline{1,4},f(x,t),\phi(x)$ and does not depend on the chosen $\lambda\in[0,1]$, $M=0$ when $f(x,t),\phi(x)=0$, that is

$$\|u\|_{W_2^{2,1}}\le M \quad (16)$$

3.1. Uniqueness

Assume that the problem (1)-(3), (6) has two different solution $u_1(x,t)$ and $u_2(x,t)$. Then the function $u=u_1-u_2$ is a solution of the problem (1)-(3), (6) with $f(x,t),\phi(x)=0$. From the estimate (16) it immediately follows that $\|u\|_{W_2^{2,1}}\le 0$, and hence, $u_1=u_2$ a.e. in $W_2^{2,1}(Q_T)$.

3.3. Existence

To prove the existence part we shall apply the parameter continuation method.

Let $\lambda\in[0,1]$ be an arbitrary number. Denote by Λ the set of such λ for which the problem (1)-(3) with nonlocal condition (6)

$$u(l,t)=\lambda\int_0^l K(x,t)u(x,t)\,dx$$

has a solution in $W_2^{2,1}(Q_T)$.

If such a set Λ is non-empty, open and closed then $\Lambda=[0,1]$, and therefore, there exists a solution $u(x,t)$ of the problem (1)-(3), (6) for any $\lambda\in[0,1]$.

First, we note that Λ is non-empty. Indeed, for the case $\lambda=0$ the problem (1)-(3), (6) becomes

$$u_t-u_{xx}+c(x,t)u=f(x,t),$$
$$u(x,0)=\phi(x),0\le x\le l,$$
$$u(0,t)=0,0\le t\le T,u(l,t)=0.$$

It is possible to see that there exists a solution $u(x,t)\in W_2^{2,1}(Q_T)$ of this problem [[5], pp. 273-276].

Our next aim is to prove that the set Λ is closed. To this end we consider a sequence $\{\lambda_n\}\subset\Lambda$ which converges to some λ_0 as $n\to\infty$. We shall show that $\lambda_0\in\Lambda$.

Indeed, to each λ_n there corresponds a function $u_n\in W_2^{2,1}(Q_T)$ that satisfies the integral identity (5) and conditions (2), (3), (6). Thus, the estimate (16) is also valid for u_n, that is $\|u_n\|_{W_2^{2,1}} \le M$, where the constant M does not depend on n and λ_n.

Therefore, as $W_2^{2,1}(Q_T)$ is a Hilbert space, so there exists a subsequence u_{n_k} and a function $u \in W_2^{2,1}(Q_T)$ such that

$$\begin{aligned} & u_{n_k} \to u, \\ & \left(u_{n_k}\right)_t \to u_t, \quad \text{weakly in } L_2(Q_T) \text{ and} \\ & \left(u_{n_k}\right)_{xx} \to u_{xx} \\ & \left.\begin{aligned} u_{n_k}(0,t) \to u(0,t), \\ u_{n_k}(l,t) \to u(l,t) \end{aligned}\right. \quad \text{a.e. on } [0,T]. \end{aligned}$$

Thereafter, letting $k \to \infty$ in the equalities

$$u_{n_k}(x,0) = \phi(x), u_{n_k}(0,t) = 0,$$

$$u_{n_k}(l,t) = \lambda_{n_k} \int_0^l K(x,t) u_{n_k}(x,t)\,dx,$$

$$\int_{Q_T} \left(\left(u_{n_k}\right)_t - \left(u_{n_k}\right)_{xx} + c u_{n_k} \right) \eta(x,t)\,dxdt = \int_{Q_T} f(x,t)\eta(x,t)\,dxdt$$

it is easy to see that $u \in W_2^{2,1}(Q_T)$ is a solution of the problem (1)-(3), (6) and hence, $\lambda_0 \in \Lambda$.
This implies that Λ is closed.

The next step is to prove that the set Λ is open. To this end we take $\lambda_0 \in \Lambda$ and shall prove that $\lambda = \lambda_0 + \bar{\lambda} \in \Lambda$ for small enough $|\bar{\lambda}|$.

We introduce an operator $A: W_2^{2,1}(Q_T) \to W_2^{2,1}(Q_T)$ in the following way. Let $v(x,t) \in W_2^{2,1}(Q_T)$ and $\eta(x,t)$ be a solution of the integral equation

$$\eta(x,t) - \lambda_0 \int_0^x K(\xi,t)\eta(\xi,t)\,d\xi = \bar{\lambda} \int_0^x K(\xi,t) v(\xi,t)\,d\xi \quad (17)$$

For every fixed $t \in [0,T]$ the equation (17) is Volterra equation with respect to the function $\eta(x,t)$. We note that as the function $g(x,t) = \bar{\lambda} \int_0^x K(\xi,t)v(\xi,t)\,d\xi$ belongs to $L_2(Q_T)$ and the kernel $K(x,t)$ is bounded, so there exists a unique solution $\eta(x,t)$ of (17) such that $\eta(x,t) \in L_1(0,l)$ for every fixed $t \in [0,T]$.

Observe that

$$\left| \lambda_0 \int_0^x K(\xi,t)\eta(\xi,t)\,d\xi \right| \le \left(\int_0^l K^2(x,t)\,dx \right)^{1/2} \left(\int_0^l \eta^2(x,t)\,dx \right)^{1/2}$$

Therefore the solution of (17) is represented as a sum of two functions

$$\eta(x,t) = g(x,t) + \lambda_0 \int_0^x K(\xi,t)\eta(\xi,t)\,d\xi,$$

where the first function is an element of $L_2(Q_T)$ and second one is bounded. Hence, $\eta(x,t) \in L_2(Q_T)$.

Also it is easy to prove that $\eta(x,t) \in W_2^{2,1}(Q_T)$. Indeed, we formally differentiate (17) with respect to t and consider $\eta_t(x,t)$ as a solution of the integral equation

$$\eta_t(x,t) - \lambda_0 \int_0^x K(\xi,t)\eta_t(\xi,t)\,d\xi = \lambda_0 \int_0^x K_t(\xi,t)\eta(\xi,t)\,d\xi + \bar{\lambda} \int_0^x \left(K_t(\xi,t)v(\xi,t) + K(\xi,t)v_t(\xi,t) \right) d\xi$$

with a bounded kernel and right-hand side part from $L_2(Q_T)$. Therefore, repeating the above arguments we see that $\eta_t(x,t) \in L_2(Q_T)$.

Next, we differentiate (17) with respect to x and obtain that $\eta_x(x,t)$ can be defined as

$$\eta_x(x,t) = \lambda_0 K(x,t)\eta(x,t) + \bar{\lambda} K(x,t)v(x,t) \quad (18)$$

By the properties of the functions $K(x,t)$, $v(x,t)$, $\eta(x,t)$ it means that $\eta_x(x,t) \in L_2(Q_T)$. Now we differentiate (18) with respect to x and get

$$\begin{aligned} \eta_{xx}(x,t) &= \lambda_0 \left(K_x(x,t)\eta(x,t) + K(x,t)\eta_x(x,t) \right) \\ &+ \bar{\lambda} \left(K_x(x,t)v(x,t) + K(x,t)v_x(x,t) \right) \end{aligned} \quad (19)$$

Hence, using the properties of $K(x,t)$, $v(x,t)$, $\eta(x,t)$ we conclude that $\eta_{xx}(x,t) \in L_2(Q_T)$.
Therefore, the solution of the Volterra equation $\eta(x,t) \in W_2^{2,1}(Q_T)$.

Next, we define a function $\omega(x,t) \in W_2^{2,1}(Q_T)$ as a solution of the problem

$$\omega_t - \omega_{xx} + c(x,t)\omega = f_1(x,t) \quad (20)$$

$$\omega(x,0) = \phi_1(x) \quad (21)$$

$$\omega(0,t) = 0 \quad (22)$$

$$\omega(l,t) = \lambda_0 \int_0^l K(x,t)\omega(x,t)\,dx \quad (23)$$

where $f_1(x,t) = f(x,t) - c\eta(x,t) + \eta_{xx}(x,t) - \eta_t(x,t)$,

$$\phi_1(x) = \phi(x) - \lambda_0 \int_0^x K(\xi,0)\eta(\xi,0)\,d\xi - \bar{\lambda} \int_0^x K(\xi,0)v(\xi,0)\,d\xi$$

By the assumption, $\lambda_0 \in \Lambda$, and hence, there exists the solution $\omega(x,t) \in W_2^{2,1}(Q_T)$ of the problem (21)-(23).

Now consider the function $u(x,t) = \omega(x,t) + \eta(x,t)$. It is easy to see that $u(x,t) \in W_2^{2,1}(Q_T)$ satisfies (1)-(3) and

$$\begin{aligned} u(l,t) &= \lambda_0 \int_0^l K(x,t)\omega(x,t)\,dx + \lambda_0 \int_0^l K(x,t)\eta(x,t)\,dx \\ &+ \bar{\lambda} \int_0^l K(x,t)v(x,t)\,d\xi \\ &= \lambda_0 \int_0^l K(x,t)u(x,t)\,dx + \bar{\lambda} \int_0^l K(x,t)v(x,t)\,d\xi. \end{aligned} \quad (24)$$

Therefore, $u(x,t)\in W_2^{2,1}(Q_T)$ is a solution of the problem (1), (3), (24). We shall consider this function a mapping Av: $u=Av$.

Now our aim is to prove that the introduced operator A is a contraction mapping for small enough $|\bar{\lambda}|$. To this end, we consider functions $v_1(x,t)$, $v_2(x,t)\in W_2^{2,1}(Q_T)$ and let $\eta_1(x,t)$, $\eta_2(x,t)$ be the corresponding solutions of the integral equation (18) and $\omega_1(x,t)$, $\omega_2(x,t)$ be the solutions of the problem (21)-(24). We define $u_1(x,t)=\omega_1(x,t)+\eta_1(x,t)$, $u_2(x,t)=\omega_2(x,t)+\eta_2(x,t)$.

Let us denote $v=v_1-v_2$, $\eta=\eta_1-\eta_2$, $\omega=\omega_1-\omega_2$, $u=u_1-u_2$. Then $\omega(x,t)$ is a solution of the problem (20)-(23) with $f_1(x,t)=-c\eta(x,t)+\eta_{xx}(x,t)-\eta_t(x,t)$ and $\phi_1(x)=0$.

Following the proof of the estimate (16) we obtain

$$\|\omega\|^2_{W_2^{2,1}}\le M_1\|f_1\|^2\le \le \tilde{M}\left(\|\eta\|^2_{L_2(Q_T)}+\|\eta_t\|^2_{L_2(Q_T)}+\|\eta_{xx}\|^2_{L_2(Q_T)}\right) \tag{25}$$

where the constant M_1 depends on m, T and $\tilde{M}$ depends on m, T, c.

The next step is to obtain an estimate of each norm on the right-hand side of (25). Observe that (17) implies

$$\eta^2(x,t)\le 2\lambda_0^2\left(\int_0^x K(\xi,t)\eta(\xi,t)d\xi\right)^2 +2\bar{\lambda}^2\left(\int_0^x K(\xi,t)v(\xi,t)d\xi\right)^2.$$

Applying the Cauchy inequality we have

$$\eta^2(x,t)\le 2K_0^2\int_0^x\eta^2(\xi,t)d\xi+2\bar{\lambda}^2K_0^2\int_0^l v^2(\xi,t)d\xi \tag{26}$$

Now by Gronwall's lemma we conclude that

$$\int_0^x\eta^2(\xi,t)d\xi\le\bar{\lambda}^2\left(e^{2K_0l}-1\right)\int_0^l v^2(\xi,t)d\xi$$

and (27)

$$\int_0^l\eta^2(\xi,t)d\xi\le\bar{\lambda}^2\left(e^{2K_0l}-1\right)\int_0^l v^2(\xi,t)d\xi,$$

and hence,

$$\int_0^T\int_0^l\eta^2(\xi,t)d\xi dt\le\bar{\lambda}^2\left(e^{2K_0l}-1\right)\int_0^T\int_0^l v^2(\xi,t)d\xi dt$$

Therefore, $\|\eta\|^2_{L_2(Q_T)}\le\bar{\lambda}^2\left(e^{2K_0l}-1\right)\|v\|^2_{L_2(Q_T)}\le$

$$\le\bar{\lambda}^2\left(e^{2K_0l}-1\right)\|v\|^2_{W_2^{2,1}(Q_T)} \tag{28}$$

Observe further that

$$\eta_t^2(x,t) \le 4\lambda_0^2\left(\int_0^x K_t(\xi,t)\eta(\xi,t)d\xi\right)^2+4\lambda_0^2\left(\int_0^x K(\xi,t)\eta_t(\xi,t)d\xi\right)^2 +4\bar{\lambda}^2\left(\int_0^x K_t(\xi,t)v(\xi,t)d\xi\right)^2+4\bar{\lambda}^2\left(\int_0^x K(\xi,t)v_t(\xi,t)d\xi\right)^2$$

Applying the Cauchy inequality and (27) we obtain

$$\eta_t^2(x,t) \le 4K_0\int_0^x\eta_t^2(\xi,t)d\xi+4K_1\bar{\lambda}^2\left(e^{2K_0l}-1\right)\int_0^l v^2(\xi,t)d\xi \tag{29}$$
$$+4K_1\bar{\lambda}^2\int_0^l v^2(x,t)dx+4K_0\bar{\lambda}^2\int_0^l v_t^2(x,t)dx$$

Using the Gronwall's lemma for (29) we obtain

$$\int_0^x\eta_t^2(\xi,t)d\xi \le\bar{\lambda}^2\left(\frac{K_1e^{2K_0l}}{K_0}\int_0^l v^2(x,t)dx+\int_0^l v_t^2(x,t)dx\right)\left(e^{4K_0x}-1\right)$$

This inequality implies that

$$\|\eta_t\|^2_{L_2(Q_T)}\le\bar{\lambda}^2\bar{K}^2\left(e^{4K_0l}-1\right)\|v\|^2_{W_2^{2,1}(Q_T)} \tag{30}$$

where $\bar{K}^2=\max\left\{1,\dfrac{K_1e^{2K_0l}}{K_0}\right\}$.

Now it remains to obtain an estimate for $\|\eta_{xx}\|^2_{L_2(Q_T)}$. To this end first we find an estimate for $\|\eta_x\|^2_{L_2(Q_T)}$. From (19) it follows that $\eta_x^2\le2\lambda_0^2K^2\eta^2+2\bar{\lambda}^2K^2v^2$, which implies

$$\int_0^T\int_0^l\eta_x^2dxdt\le2\int_0^T\int_0^l K^2\eta^2dxdt+2\bar{\lambda}^2\int_0^T\int_0^l K^2v^2dxdt.$$

Thus, $\|\eta_x\|^2_{L_2(Q_T)}\le4k_1^2\bar{\lambda}^2e^{2K_0l}\|v\|^2_{W_2^{2,1}(Q_T)}$, where k_1 is such that $|K(x,t)|\le k_1$.

Now we consider (20) and conclude that

$$\eta_{xx}^2\le4\left(K_x^2\eta^2+K\eta_x^2\right)+4\bar{\lambda}^2\left(K_x^2v^2+Kv_x^2\right).$$

Therefore,

$$\int_0^T\int_0^l\eta_{xx}^2(x,t)dxdt\le4k_2^2\int_0^T\int_0^l\eta^2(x,t)dxdt +4k_1^2\int_0^T\int_0^l\eta_x^2(x,t)dxdt+4\bar{\lambda}^2k_2^2\int_0^T\int_0^l v^2(x,t)dxdt +4\bar{\lambda}^2k_2^2\int_0^T\int_0^l v_x^2(x,t)dxdt$$

and hence,

$$\|\eta_{xx}\|^2_{L_2(Q_T)} \le 4\bar{k}^{-2}\bar{\lambda}^{2}Q^2\|v\|^2_{W_2^{2,1}(Q_T)} \qquad (31)$$

where $\bar{k}^{-2} = \max\left\{k_1^2, k_2^2\right\}$, $Q^2 = \max\left\{1, 2k_1^2 e^{2K_0 l} - 1\right\}$.

It immediately follows from (28), (30), (31) that

$$\|\omega\|^2_{W_2^{2,1}} \le \bar{\lambda}^{2} R\|v\|^2_{W_2^{2,1}(Q_T)}$$

Finally, we conclude that

$$\|Av\|_{W_2^{2,1}} = \|u\|_{W_2^{2,1}} = \|\omega + \eta\|_{W_2^{2,1}} \le \left|\bar{\lambda}\right| \; |R_1|\|v\|_{W_2^{2,1}(Q_T)}$$

where R_1 depends on m, T, $c(x,t)$, $K(x,t)$.

Now if $\left|\bar{\lambda}\right|$ is small enough, then $\left|\bar{\lambda}\right||R_1| < 1$ and hence, the operator $A: W_2^{2,1}(Q_T) \to W_2^{2,1}(Q_T)$ is a contraction mapping. Taking into account that the space $W_2^{2,1}(Q_T)$ is complete we conclude that there exists a unique fixed point $u(x,t) \in W_2^{2,1}(Q_T)$ which satisfies (1)-(3) and the condition

$$u(l,t) = \left(\lambda_0 + \bar{\lambda}\right)\int_0^l K(x,t)u(x,t)dx \qquad (32)$$

Therefore, for small enough $\left|\bar{\lambda}\right|$ the problem (1)-(3), (32) has a solution $u(x,t) \in W_2^{2,1}(Q_T)$.

This implies that $\lambda_0 + \bar{\lambda} \in \Lambda$, and hence the set Λ is open.

Finally, we conclude that $\Lambda = [0,1]$, and the problem (1)-(3), (6) has a solution $u(x,t) \in W_2^{2,1}(Q_T)$ for any $\lambda \in [0,1]$, that is, there exists a solution of the problem (1)-(3), (6) for $\lambda = 1$, and hence, the existence part for the problem (1)-(4) has been proved.

4. Conclusion

In the present research the investigation of the nonclassical problem with the second type integral condition has been demonstrated. While dealing with such problems the main question is about a choice of the most powerful and suitable research method. This choice essentially depends on the kind of a nonlocal condition. In this article we have shown that a nonlinear functional analysis method can be applied to study nonlocal problems and existence of the introduced solution has been proved by a parameter continuation method. To prove the uniqueness part we used an a priori estimate method.

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