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Fourier Transform Methods for Partial Differential Equations

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Abstract The purpose of this seminar paper is to introduce the Fourier transform methods for partial differential equations. The introduction contains all the possible efforts to facilitate the understanding of Fourier transform methods for which a qualitative theory is available and also some illustrative examples was given. The resulting Fourier transform maps a function defined on physical space to a function defined on the space of frequencies, whose values quantify the "amount" of each periodic frequency contained in the original function then inverse Fourier transform reconstructs the original function from its transform.

Keywords: *fourier transform, partial differential equations*

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1. Introduction

The Fourier transform is the natural extension of Fourier series to a function f(x) of infinite period [4]. This paper develops one of the fundamental topics in analysis and in PDEs, namely orthogonal expansions.

Definition: The set of functions $\{Y_n(x):n=0,1,\ldots\}$ each of which is piecewise continuous in an infinite or a finite interval $[\alpha,\beta]$, is said to be orthogonal in $[\alpha,\beta]$ with respect to the weight function $r(x)>0$, if $Y_m, Y_n = \int_\alpha^\beta r(x) Y_m(x) Y_n(x)\,dx = 0$ for all $m \neq n$ and $\int_\alpha^\beta r(x) Y_n^2(x)\,dx \neq 0$ for all n.

We shall always assume that r(x) has only a finite number of zeros in $[\alpha,\beta]$ and the integrals $\int_\alpha^\beta r(x) Y_n(x)\,dx, n = 0,\ldots$ exist.

Definition: The norm of a function $Y_n(x)$ is denoted by $\|Y_n\|$ defined as the inner product of a function with itself and written

$$Y_n = \left(\int_\alpha^\beta r(x) Y_n^2(x)\,dx\right)^{\frac{1}{2}} \geq 0.$$

A real - valued function $Y_n(x)$ is called square - integrable on the interval in $[\alpha,\beta]$ with respect to the weight function r(x) when

$$\int_\alpha^\beta r(x) Y_n^2(x)\,dx < +\infty$$

The orthogonal set $\{Y_n(x):n=0,1,\ldots\}$ in $[\alpha,\beta]$ with respect to the weight function $r(x)$ is said to be orthonormal set if $\int_\alpha^\beta r(x) Y_n^2(x)\,dx = 1$ for all n [8].

If $\{Y_n(x)\}$ is an orthonormal set of functions then

$$Y_m, Y_n = \int_\alpha^\beta r(x) Y_m(x) Y_n(x)\,dx = \delta_{mn} \qquad (1.1)$$

Where $\delta_{mn} = \begin{cases} 1 \text{if } m = n \\ 0 \text{if } m \neq n \end{cases}$ is the Kronecker delta [6].

Thus, orthonormal functions have the same properties as orthogonal functions, but, in addition, they have been normalized [5], i.e., each function $Y_n(x)$ of the orthogonal set has been divided by the norm of that function, which is defined as $Y_n = \left(\int_\alpha^\beta r(x) Y_n^2(x)\,dx\right)^{\frac{1}{2}}$. Hence if an orthogonal set of functions $\{Y_n(x)\} \neq 0$ is defined on the interval $[\alpha,\beta]$, with $\|Y_n\|$we can always construct an orthonormal set of functions $X_n(x)$ by defining $X_n(x) = \frac{Y_n(x)}{Y_n}, \alpha \leq x \leq \beta$ [1].

In fact in view of (1.1),

$$X_m, X_n = \int_\alpha^\beta \frac{r(x) Y_m(x) Y_n(x)\,dx}{Y_m . Y_n} = \frac{1}{Y_m . Y_n} \varphi_m, \varphi_n = \delta_{mn}$$

and hence $\|X_n\|=1$ for all n.

Example: The set of functions $Y_n(x) = \left\{\sin\frac{n\pi x}{L}\right\}, \text{for} n = 1, 2, \ldots$, is a set of orthogonal

functions over the interval 0<X<L With respect to the weight function $r(x)=1$ This is shown by calculating the inner products

$$Y_m, Y_n = \left(\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right) = \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx$$

$$= \frac{L}{\pi}\left[\frac{\sin(m-n)\frac{\pi x}{L}}{2(m-n)} - \frac{\sin(m+n)\frac{\pi x}{L}}{2(m+n)}\right]_0^L = 0, m \neq n,$$

and The norm squared for each function is given by

$$Y_m, Y_m = Y_m{}^2 = \int_0^L \sin^2\frac{m\pi x}{L}dx = \frac{1}{2}\int_0^L\left(1-\cos\frac{2m\pi x}{L}\right)dx$$

$$= \left[\frac{x}{2} - \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\right]_0^L = \frac{L}{2}.$$

For m=1,2,… These results are written in inner product notation as

$$\left(\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right) = \frac{L}{2}\delta_{mn} = \begin{cases} 0 \text{ if } m \neq n \\ \frac{L}{2} \text{ if } m = n \end{cases}$$

The seminar paper discusses a periodic function which can be expanded in terms of an infinite sum of sines and cosines in which most functions encountered in engineering are periodic functions.

Definition: Fourier trigonometric series of a function of f(x) defined on is defined on -L≤X≤L, is the infinite trigonometric series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right),$$

Whose coefficients are given by the inner product formulae

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt, n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt, n \geq 1$$

However, if the function f(x) is odd, then since

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt = 0, n \geq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt = \frac{2}{\pi}\int_{o}^{\pi} f(t)\sin ntdt, n \geq 1$$

the Fourier trigonometric series reduces to the Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin ntdt, n \geq 1$$

Thus, we conclude that if f(x) is odd, or defined only on(0, π) and we make its odd extension then the Fourier sine series can be obtained Exactly, in the same way if f(x) is even, or defined only on (0, π) and we make its even extension then since

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos ntdt, n \geq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt = 0, n \geq 1$$

the Fourier trigonometric series reduces to the Fourier cosine series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad (1.2)$$

Where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos ntdt, n \geq 0.$$

Example: We shall find the Fourier cosine series of the function $f(x) = x, 0 < x < \pi$.

Clearly,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\cos 0tdt = \frac{2}{\pi}\int_0^{\pi} tdt = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos ntdt = \frac{2}{\pi}\left[t\frac{\sin nt}{n}\Big|_0^{\pi,} - \int_0^{\pi}\frac{\sin nt}{n}dt\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{\cos n\pi - 1}{n^2}\right] = \frac{2}{\pi}\frac{(-1)^n}{n^2}, n \geq 1.$$

Thus, from (1.2), we have

$$x \sim \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{(-1)^n}{n^2}\cos nx, 0 < x < \pi.$$

Theorem: (Fourier's theorem) [see [7]]Let f(x) and f'(x) be piecewise continuous in the interval [-L,L]. Then, the Fourier trigonometric series of f(x) converges to $\frac{f(-x^+) + f(x^-)}{2}$ at each point in the open interval (-L,L) and at x=±L the series converges to $\frac{f(-L^+) + f(L^-)}{2}$.

Example: Consider the function

$$f(x) = \begin{cases} 0, x\epsilon[-\pi, 0) \\ 1, x\epsilon[0, \pi) \end{cases}$$

Clearly, $f(x)\epsilon C_P{}^1(-\pi, \pi)$ and has a single jump discontinuity at 0. For this function, the Fourier trigonometric coefficients are a_0=1, a_n=0, $b_n = \frac{(1-(-1)^n)}{n\pi}$.

Thus, we have

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1) = F(x), \text{say.} \quad (1.3)$$

From Fourier's theorem in (1,3) the equality F(x)= f(x) holds at each point in the open intervals ($-\pi$, 0) and (0, π) where as at x=0 the right–hand side is 1/2 which is the same as $\frac{f\left(-0^{+}\right)+f(0^{-})}{2}$ Also, at x=$\pm\pi$ the right–hand side is again 1/2 which is the same as $\frac{f\left(-\pi^{+}\right)+f(\pi^{-})}{2}$.

The Fourier integral is a natural extension of Fourier trigonometric series in the sense that it represents a piecewise smooth function whose domain is semi-infinite or infinite [1]

A periodic function f(x) defined in a finite interval (-L,L) can be expressed in Fourier series by extending this concept, non periodic functions defined in $-\infty<x<\infty$ (for all x) can be expressed as a Fourier integral.

Let $f_p(x)$ be a periodic function of period 2p that can be represented by a Fourier series

$$f_L(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n cos\omega_n x+b_n \sin\omega_n x\right),\omega_n=\frac{n\pi}{L}$$

Where

$$a_0=\frac{1}{2L}\int_{-L}^{L}f_L(x)dt$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f_L(x)\cos\omega_n tdt, n\geq 0$$

$$b_n=\frac{1}{p}\int_{-L}^{L}f_L(x)\sin\omega_n dt, n\geq 1.$$

The problem we shall consider is what happens to the above series when L $\rightarrow\infty$ for this we insert a_n and b_n, to obtain

$$f_L(x)=\frac{1}{2L}\int_{-L}^{L}f_L(t)dt+\frac{1}{L}\sum_{n=1}^{\infty}\left[\begin{array}{l}\cos\omega_n x\int_{-L}^{L}f_L(t)\cos\omega_n tdt\\ +\sin\omega_n x\int_{-L}^{L}f_L(t)\sin\omega_n tdt\end{array}\right]$$

We now set $\triangle\omega=\omega_{n+1}-\omega_n=\frac{(n+1)\pi}{L}-\frac{n\pi}{L}=\frac{\pi}{L}$

Then $\frac{1}{L}=\frac{\triangle\omega}{\pi}$, and we may write the Fourier series in the form

$$f_L(x)=\frac{1}{2L}\int_{-L}^{L}f_L(t)dt$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\begin{array}{l}(\cos\omega_n x)\Delta\omega\int_{-L}^{L}f_L(t)\cos\omega_n tdt\\ +(\sin\omega_n x)\Delta\omega\int_{-L}^{L}f_L(t)\sin\omega_n dt\end{array}\right] \quad (1.4)$$

This representation is valid for any fixed p, arbitrarily large, but fixed.

We now let L$\rightarrow\infty$ and assume that the resulting nonperiodic function is $f(x)=\lim_{L\rightarrow\infty}f_L(x)$ is absolutely integrable on the x-axis, i.e., $\int_{-\infty}^{\infty}\left|f(x)\right|dx<\infty$. Then, $\frac{1}{L}\rightarrow 0$, and the value of the first term on the right side of (1.4) approaches zero. Also, $\Delta\omega=\frac{\pi}{L}\rightarrow 0$ and the infinite series in (1.4) becomes an integral from 0 to ∞, which represent f(x), i.e,

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\begin{array}{l}cos\omega x\int_{-\infty}^{\infty}f(t)cos\omega tdt\\ +\sin\omega x\int_{-\infty}^{\infty}f(t)sin\omega tdt\end{array}\right] \quad (1.5)$$

Now if we introduce the notations

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)cos\omega tdt, B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)sin\omega tdt \quad (1.6)$$

Then (1.5) can be written as

$$f(x)=\int_{0}^{\infty}\left[A\omega cos\omega x+B\omega sin\omega x\right]d\omega \quad (1.7)$$

This representation of f(x) is called Fourier integral.

Theorem: (Fourier Integral Theorem): Let f(x), $-\infty<x<\infty$ be piecewise continuous on each finite interval, and $\int_{-\infty}^{\infty}\left|f(x)\right|dx<\infty$, ($-\infty$,$\infty$) i.e., f is absolutely integrable on($-\infty$,∞). Then, f(x) can be represented by a Fourier integral (1.7).

Further, at each x. $\int_{0}^{\infty}\left[A(\omega)cos\omega x+B(\omega)sin\omega x\right]d\omega=\frac{f(x+0)+f(x-0)}{2}$

Example: Find the Fourier integral representation of the single pulse function

$$f(x)=\begin{cases}1 \text{ if } |x|<1\\ 0 \text{ if } |x|>1.\end{cases}$$

From (1.6) we have

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)cos\omega tdt=\frac{1}{\pi}\int_{-1}^{1}cos\omega tdt=\frac{2sin\omega}{\pi\omega}$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)sin\omega tdt=0.$$

Thus, (1.7) gives the representation

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{cos\ \omega x\ sin\ \omega}{\omega}d\omega. \quad (1.8)$$

Now from this Theorem it is clear that

$$\int_{0}^{\infty}\frac{cos\omega xsin\omega}{\omega}d\omega.=\begin{cases}\frac{\pi}{2}\ if\ |x|<1\\ \frac{\pi}{2}\frac{(1+0)}{2}=\frac{\pi}{4}\ if\ x=\pm 1\\ 0\ if\ |x|>1.\end{cases} \quad (1.9)$$

This integral is called Dirichlet's discontinuity factor.
Setting x=0 in (1.9) yields the important integral

$$\int_0^\infty \frac{sin\omega}{\omega} d\omega = \frac{\pi}{2} \qquad (1.10)$$

known as the Dirichlet integral.

Theorem 1: (**Fourier Cosine Integral Theorem**): If f(x) satisfies the Dirichlet's conditions on the non negative real line and is absolutely integralble on (0, ∞), then $f(x) = \int_0^\infty A(\omega) cos\omega t dt$, where $A(\omega) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(t) cos\omega t dt$

Theorem 2: (**Fourier Sine Integral Theorem):** If f(x) satisfies the Dirichlet's conditions on the non negative real line and is absolutely integralble on (0, ∞), then $f(x) = \int_0^\infty B(\omega) sin\omega x d\omega$; where $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t) sin\omega t dt$.

Indeed, if f(x) is an even function, then B(ω)=0 in (3) and $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t) cos\omega t dt$ and the Fourier integral (1.7) reduces to the Fourier cosine integral,

$$f(x) = \int_0^\infty A(\omega) cos\ \omega t d\omega \qquad (1.11)$$

Similarly, if f(x) is odd, then in (1.6).we have $A(\omega) = 0$ and $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t) sin\omega t dt$. and the Fourier integral (1.7) reduces to the Fourier sine integral

$$f(x) = \int_0^\infty B(\omega) sin\ \omega x dt d\omega \qquad (1.12)$$

Example: Express $f(x) = \begin{cases} 1\, if\, 0 \le x \le \pi \\ 0\, if\ \ x > \pi \end{cases}$ as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1-\cos\cos(\pi\omega)}{\omega} \sin\sin(x\omega d\omega).$$

The Fourier sine integral for

$$f(x) = \frac{2}{\pi}\int_0^\infty sin\omega x d\omega \int_0^\infty f(t) sin\omega t dt$$

$$= \frac{2}{\pi}\int_0^\infty sin\omega x d\omega \int_0^\pi f(t) sin\omega t dt$$

$$= \frac{2}{\pi}\int_0^\infty sin\omega x d\omega \left| \frac{-cos\omega x}{\omega} \right|_0^\pi$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-cos\omega\pi}{\omega} sin\omega x d\omega.$$

$$\frac{2}{\pi}\int_0^\infty \frac{1-cos\omega\pi}{\omega} sin\omega x d\omega = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2}\, if\, 0 \le x < \pi \\ 0 \quad if\ \ x > \pi \end{cases}$$

At $x = \pi$, which point of discontinuity of f(x),the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi-0)+f(\pi+0)}{2}\right] = \frac{\pi}{2}.\frac{1+0}{2} = \frac{\pi}{4}$$

We note that (1.5) is the same as

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\left[cos\omega x cos\omega x + \sin\omega t sin\omega x\right] dt d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(t)\cos(\omega x - \omega t)\right] d\omega$$

The integral in bracket is an even function of ω, we denote it by $\widetilde{F}(\omega)$.

Since $\cos(\omega x - \omega t)$ is an even function of ω, the function f does not depend on ω, and we integrate with respect to t (not ω), the integral of $\widetilde{F}(\omega)$ from $\omega=0$ to ∞ is 1/2 times the integral of $\widetilde{F}(\omega)$ from -∞ to ∞. Thus,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos(\omega x - \omega t) dt\right] d\omega \qquad (1.13)$$

From the above argument it is clear that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos(\omega x - \omega t) dt\right] d\omega = 0. \qquad (1.14)$$

A combination of (1.13) and (1.14) gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) e^{i\omega(x-t)} dt d\omega \qquad (1.15)$$

This is called the complex Fourier integral.
From the above representation of f(x), we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt\right] e^{i\omega x} d\omega. \qquad (1.16)$$

Definition: In the Fourier integral of f(x) in the complex form given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt\right] e^{i\omega x} d\omega$$

the expression in bracket is a function of ω, is denoted by F(ω) or F(f) and is called Fourier transform of f. Now writing x for t we get

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (1.17)$$

And with this (1.16) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \qquad (1.18)$$

The representation (1.18) is called the inverse Fourier transform of F(ω). Finally, as in Theorem-1, if f(x), -∞<x<∞ is piecewise continuous on each finite interval,

and $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. Then, the Fourier transform (1.17) of f(x) exists. Further, at each x,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega = \frac{1}{2}\left[f(x+0) + f(x-0) \right]$$

Example: Find the Fourier transform of the square wave function

$$f(x) = \begin{cases} 0, x < a \\ 1, a \le x \le b \\ 0, x > b \end{cases}$$

From (1.17), we have

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{dx} = \begin{cases} \dfrac{e^{-i\omega b} - e^{-i\omega a}}{-\sqrt{2\pi \mathrm{i}\omega}}, \omega \ne 0 \\ \dfrac{b-a}{\sqrt{2\pi}}, \omega = 0. \end{cases}$$

Further it follows that

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{b} \frac{e^{-i\omega b} - e^{-i\omega a}}{-\sqrt{2\pi \mathrm{i}\omega}} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}\omega = \begin{cases} 0, x < a \\ \dfrac{1}{2}, x = a \\ 1, a < x < b \\ \dfrac{1}{2}, x = b \\ 0, x < b \end{cases}$$

The basic technique for solving partial differential equations (PDE) on a bounded spatial domain is the Fourier method [1].

The seminar paper deals with the problem of the Fourier transform methods for partial differential equations considering first problems in infinite domains which can be effectively solved by finding the Fourier transform or the Fourier sine or cosine transform of the unknown function. However, for such problems usually the method of separation of variables does not work because the Fourier series are not adequate to yield complete solutions. This is due to the fact that often these problems require a continuous superposition of separated solutions

In this seminar paper we begin by motivating the construction by investigating how Fourier series behave as the length of the interval goes to infinity. Therefore, this paper develops the theory of the Fourier transform methods for partial differential equations in which a qualitative theory exists.

The main objective of this paper is to discuss Fourier transform methods for partial differential equations (PDEs) which often help full in approximation to the true situation and that a more realistic model would include some of the periodic functions can be written in terms of an infinite sum of sines and cosine series by using Fourier transforms which were complicated when we are using Fourier series.

2. Transforms of Partial Derivatives

Definition: In the Fourier integral of f(x) in the complex form given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} dt \right] \mathrm{e}^{\mathrm{i}\omega x} d\omega$$

the expression in bracket is a function of ω, is denoted by F(ω) or F(f) and is called Fourier transform of f. Now writing x for t we get

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{dx} \qquad (2.1)$$

And with this (1.18) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \qquad (2.2)$$

The representation (2.2) is called the inverse Fourier transform of F(ω). Finally, as in Theorem-1, if f(x), -∞<x<∞ is piecewise continuous on each finite interval, and $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. Then, the Fourier transform (2.1) of f(x) exists. Further, at each x,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega = \frac{1}{2}\left[f(x+0) + f(x-0) \right]$$

Example: Find the Fourier transform of the square wave function

$$f(x) = \begin{cases} 0, x < a \\ 1, a \le x \le b \\ 0, x > b \end{cases}$$

From (2.1), we have

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{dx} = \begin{cases} \dfrac{e^{-i\omega b} - e^{-i\omega a}}{-\sqrt{2\pi \mathrm{i}\omega}}, \omega \ne 0 \\ \dfrac{b-a}{\sqrt{2\pi}}, \omega = 0. \end{cases}$$

Further it follows that

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{b} \frac{e^{-i\omega b} - e^{-i\omega a}}{-\sqrt{2\pi \mathrm{i}\omega}} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}\omega = \begin{cases} 0, x < a \\ \dfrac{1}{2}, x = a \\ 1, a < x < b \\ \dfrac{1}{2}, x = b \\ 0, x < b \end{cases}$$

Theorem (Convolution Theorem): Suppose that f(x) and g(x) are piecewise continuous, bounded, and absolutely integrable functions on the x-axis. Then

$$F(f * g) = \sqrt{2\pi} F(f) F(g) \qquad (2.3)$$

Where f*g is the convolution of functions f and g defined as

$$(f*g)(x)=\int_{-\infty}^{\infty} f(t)g(x-t)dt=\int_{-\infty}^{\infty} g(t)f(x-t)dt$$

Proof: By the definition and an interchange of the order of integration, we have

$$F(f*g)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(T)g(x-T)e^{-i\omega x}dTdx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(T)g(x-T)e^{-i\omega x}dxdT.$$

Now we make the substitution x-T=v, so that x=T+v and

$$F(f*g)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(T)g(v)e^{-i\omega(T+v)}dvdT$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(T)e^{-i\omega T}dT\int_{-\infty}^{\infty} g(v)e^{-i\omega v}dv$$

$$=\sqrt{2\pi}F(f)G(g)$$

By taking the inverse Fourier transform on both sides of (2.3) and writing $F(f)=\hat{f}(\omega)$ and $F(g)=\hat{g}(\omega)$, and noting that $\sqrt{2\pi}$ and $\frac{1}{\sqrt{2\pi}}$ cancel each other, we obtain

$$(f*g)(x)=\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}d\omega \qquad (2.4)$$

The solution of a IBVP consisting of a partial differential equation together with boundary and initial conditions can be solved by the Fourier transform method. In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

Suppose that u(x,t) is a function of two variables x and t, where -∞<x<∞ and t>0 Because of the presence of two variables, care is needed in identifying the variable with respect to which the Fourier transform is computed. For example, for fixed t, the function u(x,t) becomes a function of the spatial variable x, and as such, we can take its Fourier transform with respect to the x variable. We denote this transform by $\hat{u}(\omega,t)$ Thus,

$$F(u(x,t))(\omega)=\hat{u}(\omega,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx$$

This transform is called Fourier transform in the x variable [2]. To illustrate the use of this notation we compute some very useful transforms.

2.1. Fourier Transform and Partial Derivatives

Given u(x,t) with -∞<x<∞ and t>0, we have

$$F(\frac{\partial}{\partial t}(u(x,t))(\omega)=\frac{d}{dt}\hat{u}(\omega,t);$$

$$F(\frac{\partial^n}{\partial t^n}(u(x,t))(\omega)=\frac{d^n}{dt^n}\hat{u}(\omega,t),n=1,2,...;$$

$$F(\frac{\partial}{\partial x}(u(x,t))(\omega)=i\omega\hat{u}(\omega,t);$$

$$F(\frac{\partial^n}{\partial x^n}(u(x,t))(\omega)=(i\omega)^n\hat{u}(\omega,t),n=1,2,...$$

To prove (i) we start with the right side and differentiate under the integral sign with respect to t:

$$\frac{d}{dt}\hat{u}(\omega,t)=\frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}u(x,t)e^{-i\omega x}dx$$

The last expression is the Fourier transform of $\frac{\partial}{\partial t}u(x,t)$ as a function of x, and (i) follows. Repeated differentiation under the integral sign with respect to t yields (ii).

2.2. Fourier Sine and Cosine Transform

For an even function $f(x)$ Fourier integral is the Fourier cosine integral (1.15) where $A(\omega)$ is given by (1.16). We set $A(\omega)=\sqrt{\frac{2}{\pi}}F_c(\omega)$, where c indicates cosine. Then replacing t by x

We get

$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos(\omega x)dx \qquad (2.5)$$

and

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_c(\omega)\cos\omega x d\omega \qquad (2.6)$$

Formula (2.5) gives from f(x) a new function $F_c(\omega)$ called the Fourier cosine transform of f(x) whereas (2.6) gives back f(x) from $F_c(\omega)$, and we call it the inverse Fourier cosine transform of $F_c(\omega)$. Relations (2.5) and (2.6) together form a Fourier cosine transform pair.

Similarly, for an odd function f(x) the Fourier sine transform is and the inverse Fourier sine transform is

$$F_s(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin(\omega x)dx \qquad (2.7)$$

and the inverse Fourier sine transform is

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_s(\omega)\sin\omega x d\omega \qquad (2.8)$$

Example 1: Find the Fourier cosine transform of $f(x)=e^{-\frac{ax^2}{2}}\ (a>0), x>0$

Solution

$$F_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{ax^2}{2}}\cos\omega x dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}\cos\omega x dx\left(\text{the integral is even}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}e^{-i\omega x}dx\begin{pmatrix}\text{the imaginary part integrates}\\ \text{to zero because it is odd}\end{pmatrix}$$

$$=F(e^{-\frac{ax^2}{2}})(\omega)\left(\text{by definition of Fourier transform}\right)$$

$$=\frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}\quad by$$

Example 2: Consider the function

$$f(x)=\begin{cases} sinx & \text{if } 0\le x\le\pi \\ 0 & \text{if} x>\pi. \end{cases}$$

Express f using an inverse Fourier cosine and then an inverse sine transform.

Solution: We first start by computing the Fourier cosine transform. From (2.8),

$$F_c(\omega)=\sqrt{\frac{2}{\pi}}sinxcos(\omega x)dx=\sqrt{\frac{2}{\pi}}\frac{\cos\cos(\pi\omega)+1}{1-\omega^2}.$$

Using (2.6), we obtain the inverse cosine transform representation $\frac{2}{\pi}$

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\cos\cos(\pi\omega)+1}{1-\omega^2}\cos\omega x d\omega(x\ge 0)$$

We compute the sine transform similarly by using (2.6).

$$F_s(\omega)=\sqrt{\frac{2}{\pi}}\frac{\cos\cos(\pi\omega)}{1-\omega^2}$$

and thus the inverse sine transform representation

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\sin\sin(\pi\omega)}{1-\omega^2}\sin\omega x d\omega(x\ge 0)$$

2.2.1. Cosine and Sine Transforms of Derivatives of Functions.

If f(x) is absolutely integrable on the positive x-axis and piecewise continuous on every finite interval, then the Fourier cosine and sine transforms of f exist. Furthermore, it is clear that F_c and F_s are linear operators, i.e.,

$$F_c(af+bg)=aF_c(f)+bF_c(f)$$
$$and\ F_s(af+bg)=aF_s+bF_s$$

Theorem: Let f(x) be continuous and absolutely integrable on the x-axis, let f'(x) be piecewise continuous on each finite interval, and let f(x)→0 as x→∞. Then,

$$F_c\left(f'(x)\right)=\omega F_s[f(x)]-\sqrt{\frac{2}{\pi}}f(0) \qquad (2.9)$$

$$F_s\left(f'(x)\right)=-\omega F_s\left(f(x)\right)\sqrt{\ } \qquad (2.10)$$

Proof: To show (i), we integrate by parts, to obtain

$$F_C\left(f'(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega\cos\omega x dx$$

$$=\sqrt{\frac{2}{\pi}}\left[f(x)\cos\cos\omega x\Big|_0^\infty+\omega\int_0^\infty f(x)\sin\sin\omega x dx\right]$$

$$==-\sqrt{\frac{2}{\pi}}f(0)+\omega F_s\left(f(x)\right).$$

Also we integrate by parts, to obtain

$$F_C\left(f'(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin\omega x dx$$

$$\sqrt{\frac{2}{\pi}}[f(x)\sin\sin\omega x|_0^\infty-\omega\int_0^\infty f(x)\cos\omega\cos\omega x dx]$$

$$=-\omega F_c\left(f(x)\right)$$

Similarly, (a)

$$F_C\left(f''(x)\right)=-\omega^2F_s\left(f'(x)\right)-\sqrt{\frac{2}{\pi}}f'(0)$$

By formula (ii) with f' instead of f gives

$$F_C\left(f''(x)\right)=\omega-\left(f'(x)\right)-\sqrt{\frac{2}{\pi}}f'(0)$$

$$=\omega(-\omega F_c\left(f(x)\right)-\sqrt{\frac{2}{\pi}}f'(0)$$

$$=-\omega^2F_c\left(f(x)\right)-\sqrt{\frac{2}{\pi}}f'(0)$$

(b) $F_s\left(f''(x)\right)=-\omega^2F_s\left(f(x)\right)-\sqrt{\frac{2}{\pi}}\omega f'(0)$

By formula (2.10) with f' instead of f gives

$F_s\left(f''(x)\right)=-\omega F_c\left(f'(x)\right)$ hence by (2.9)

$$F_s\left(f''(x)\right)=-\omega\left[\omega F_s\left(f(x)\right)-\sqrt{\frac{2}{\pi}}f(0)\right]$$

$$=-\omega^2F_s\left(f(x)\right)\sqrt{\frac{2}{\pi}}\omega f(0)$$

We have by the Fourier transform,

$$F\left(f^n(x)\right)=\int_{-\infty}^{\infty}f^n(x)e^{i\omega x}dx=f^n(\omega) \qquad (2.11)$$

By similar procedure we can find a relation between the sine and cosine Fourier transforms of the derivatives of a function, such as

$$F_C{}^n(\omega)=\int_0^\infty\frac{d^nf}{dx^n}\cos\omega\cos\omega x dx=$$

$$[\frac{d^{n-1}f}{dx^{n-1}}\cos\omega\cos\omega x]_0^\infty+n\int_0^\infty\frac{d^{n-1}f}{dx^{n-1}}\sin\omega\sin\omega x d\omega \quad (2.12)$$

$\left(\text{integrating by part}\right)$

$$=-\alpha_{n-1}+f_s{}^{n-1}(\omega)$$

Under the assumptions, $\frac{d^{n-1}f}{dx^{n-1}} \to 0$ as $x \to \infty$ and $\frac{d^{n-1}f}{dx^{n-1}} \to \alpha_{n-1}$ as x→0

Similarly, integrating,

$$f_s^{\,m}(\omega) = \int_0^\infty \frac{d^n f}{dx^n} \sin\omega \sin\omega x dx = -\omega f_c^{\,n-1}(\omega) \quad (2.13)$$

Equations, (2.12) and (2.13) yield, $f_c^{\,m}(\omega) = -\alpha_{n-1} - n^2 f_c^{\,n-2}(\omega)$ repeating the procedure $f_c(\omega)$ may be expressed as the sum of a's and either $f_c(\omega)$ or $f_s'(\omega)$ or $f_c'(\omega)$. $f_c(\omega)$ will occur when x is odd and in that case we can write $\alpha_o = \omega f_s(\omega)$ in place of $f_c'(\omega)$.

We thus have

$$f_c^{\,2n}(\omega) = \sum_{r=0}^{n-1} (-1)^r \alpha_{2n-2r-1} \omega^{2r} + (-1)^n \omega^{2n} f_c(\omega)$$

And

$$f_c^{\,2n+1}(\omega) = \sum_{r=0}^{n} (-1)^r \alpha_{2n-2r-1} \omega^{2r} + (-1)^n n^{2n+1} f_s(\omega)$$

Similar procedure with help of (2.12) and (2.13) will yield $f_s^{\,n}(\omega) = \omega\alpha_{n-2} - \omega^2 f_s^{\,n-2}(\omega)$

$$f_s^{\,2m}(\omega) = -\sum_{r=1}^{n} (-1)^r \alpha_{2n-2r} \omega^{2r-1} + (-1)^n \omega^{2n} f_s(\omega)$$

and

$$f_s^{\,2n+1}(\omega) = -\sum_{r=1}^{n} (-1)^r \alpha_{2n-2r+1} \omega^{2r-1} + (-1)^{n+1} \omega^{2n+1} f_s(\omega)$$

Similarly the following results are easily deducible,

(i) $\int_0^\infty \frac{d^2 f}{dx^2} \cos\cos x\omega dx = -\omega^2 F_c f(\omega)$

When =0, $\frac{dF}{dx} = \frac{d^3F}{dx^3} = 0$

(ii) $\int_0^\infty \frac{d^4 f}{dx^4} \cos\omega \cos\omega x dx = \omega^4 F_c(\omega)$

(iii) $\int_0^\infty \frac{d^2 f}{dx^2} \sin\omega \sin\omega x dx = \omega^2 F_s(\omega)$

when x=0, $F = \frac{d^2F}{dx^2} = 0$

(iv) $\int_0^\infty \frac{d^4 f}{dx^4} \sin\omega \sin\omega x dx = \omega^4 F_c(\omega)\omega^2$

Example: We found that $F_c\left(e^{-x^2}\right)(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$

Applying (ii) with $f(x) = e^{-x^2}$, we obtain

$$F_s\left(-2xe^{-x^2}\right) = -\omega F_c\left(e^{-x^2}\right) = -\omega \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$$

Hence $F_s\left(xe^{-x^2}\right) = \frac{\omega}{2\sqrt{2}} e^{-\frac{\omega^2}{4}}$

2.2.2. Convolution theorems for Fourier sine and cosine transform

Theorem: Let $F_c(\omega)$ and $G_c(\omega)$ be the Fourier cosine transform of f(x) and g(x), respectively, and let $F_s(\omega)$ and $G_s(\omega)$ be the Fourier sine transform of f(x) and g(x) respectively

Then

$$F^{-1}\left[F_c(\omega)G_c(\omega)\right] = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(s)\left[f(|x-s|) + f(x+s)\right] ds$$

We have

$$\int_0^\infty F_c(\omega) G_c(\omega) cos\omega x d\omega$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty F_c(\omega) cos\omega x d\omega \int_0^\infty g(s) cos\omega s ds$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{g(s)} ds \int_0^\infty F_c(\omega) cos\omega x cos\omega s d\omega$$

$$= \frac{1}{2} \frac{2\sqrt{\infty}}{\sqrt{\pi}} \int_0^\infty g(s) ds \int_0^\infty F_c(\omega)[cos\omega(|x-s|) + cos\omega(x+s)] ds$$

$$= \frac{1}{2} \int_0^\infty g(s) \sqrt{\left[f(|x-s|) + f(x+s)\right]} ds$$

Example: (Convolution with Cosine)

Suppose that f is integrable and even f(-x)= f(x) for all x and let $g(x) = cos\ ax$ Show that, for all real numbers a; $f * g(x) = \cos(ax)(a)$.

Solution From the definition and the fact that ; f*g=g*f, we have

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t) \cos\left[a(x-t)\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) \begin{bmatrix} \cos\cos(ax)\cos\cos(at) \\ +\sin\sin(ax)\sin(at) \end{bmatrix} dt$$

Since f is even, the product $f(t) sinat$ is odd, hence $\int_{-\infty}^\infty f(t) sinat = 0$, and so

$$f * g(x) = \cos(ax) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) \cos\cos(at) dt$$

$$= \cos(ax) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) \begin{bmatrix} \cos\cos(ax)\cos\cos(at) \\ -i\sin(at) \end{bmatrix} dt$$

$$= \cos\cos(ax) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) e^{-iat} dt = \cos(ax)\ \tilde{f}(a)$$

2.3. The Fourier Transform method.

We summarize the Fourier transform method as follows:

Step 1: Fourier transform the given boundary value problem in u(x,t) and get ordinary differential equation in $\hat{u}(\omega,t)$ in the variable t.

Step 2: solve the ordinary differential equation and find $\hat{u}(\omega,t)$.

Step 3: inverse Fourier transform $\hat{u}(\omega,t)$ to get u(x,t).

This method is successful in treating a variety of partial differential equations, but it has its limitations, since we have to assume that the functions in the problem and its solution have Fourier transforms. Nevertheless, the method offers us opportunities beyond these limitations, as we now illustrate.

The method of solution is best explained through the following example.

Example 1: We will show how the Fourier transform applies to the heat equation. We consider the heat flow problem of an infinitely long thin bar insulated on its lateral surface, which is modeled by the following initial-value problem

$$\begin{aligned} &u_t = c^2 u_{xx}, -\infty < x < \infty, t > 0, c > 0 \\ &\text{u and } u_x \text{ finite as } |x| \to \infty,\ t > 0 \\ &u(x,0) = f(x), -\infty < x < \infty, \end{aligned} \tag{2.14}$$

where the function f is piecewise smooth and absolutely integrable in (-∞,∞).

Let $\hat{u}(\omega,t)$ be the Fourier transform of u(x,t). Thus, from the Fourier transform pair, we have

$$u(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t) e^{i\omega x} d\omega$$

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx.$$

Assuming that the derivatives can be taken under the integral, we get

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \hat{u}(\omega,t)}{\partial t} e^{i\omega x} d\omega$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t)(i\omega) e^{i\omega x} d\omega$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t)(i\omega)^2 e^{i\omega x} d\omega$$

In order for u(x,t) to satisfy the heat equation, we must have

$$0 = \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\partial \hat{u}(\omega,t)}{\partial t} + c^2 \omega^2 \hat{u}(\omega,t) \right] e^{i\omega x} d\omega.$$

Thus, $\hat{u}$ must be a solution of the ordinary differential equation

$$\frac{\partial \hat{u}}{\partial t} + c^2 \omega^2 \hat{u} = 0.$$

The initial condition is determined by

$$\hat{u}(\omega,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = F(\omega).$$

Therefore, we have

$$\hat{u}(\omega,t) = F(\omega) e^{-\omega^2 c^2 t}$$

and hence

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-\omega^2 c^2 t} e^{i\omega x} d\omega. \tag{2.15}$$

Now since $\sqrt{}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2 c^2 t} e^{i\omega x} d\omega = \sqrt{2\pi} \frac{e^{-x^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}},$$

if in (2.4) we denote $F(\omega) = \hat{f}(\omega)$ and $\hat{g}(\omega) = e^{-\omega^2 c^2 t}$ then from (2.5) it follows that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mu) \sqrt{2\pi} \frac{e^{-(x-\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} d\mu \\ &= \int_{-\infty}^{\infty} f(\mu) \frac{e^{-(x-\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} d\mu. \end{aligned} \tag{2.16}$$

This formula is due to Gauss.

This formula is due to Gauss and Weierstrass.

For each μ the function $u(x,t) \to e^{-(x-\mu)^2/(4c^2 t)} / \sqrt{4\pi c^2 t}$ is a solution of the heat equation and is called the fundamental solution. Thus, (2.16) gives a representation of the solution as a continuous superposition of the fundamental solution.

The standard normal distribution function Φ is defined as

$$e^{-(x-\mu)^2/(4c^2 t)} / \sqrt{4\pi c^2 t}$$

This is a continuous increasing function with $\Phi(-\infty) = 0, \Phi(0) = \frac{1}{2}, \Phi(\infty) = 1$, If a<b, then we can write

$$\int_a^b \frac{e^{\frac{-(x-\mu)^2}{4c^2 t}}}{\sqrt{4\pi c^2 t}} d\mu = \frac{1}{\sqrt{2\pi}} \int_{\frac{x-b}{\sqrt{2c^2 t}}}^{\frac{x-a}{\sqrt{2c^2 t}}} e^{\frac{-z^2}{2}} dz, \frac{(x-\mu)}{\sqrt{2c^2 t}} = z$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(x-a)}{\sqrt{2c^2 t}}} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(x-b)}{\sqrt{2c^2 t}}} e^{\frac{-z^2}{2}} dz \\ &= \Phi\left(\frac{x-a}{\sqrt{2c^2 t}} \right) - \Phi\left(\frac{x-b}{\sqrt{2c^2 t}} \right) \end{aligned} \tag{2.17}$$

From (2.16) and (2.17) it is clear that the solution of the problem

$$u_t = c^2 u_{xx,} -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x) = \begin{cases} 0, x < a \\ L, a \le x \le b \\ 0, x > b \end{cases}$$

Can be written as

$$u(x,t) = L\Phi\left(\frac{x-a}{\sqrt{2c^2 t}}\right) - L\Phi\left(\frac{x-b}{\sqrt{2c^2 t}}\right)$$

Now using the properties $\Phi(-\infty) = 0, \Phi(0) = \frac{1}{2}, \Phi(\infty) = 1,$ we can verify that

$$\lim_{t\to 0} u(x,t) = \begin{cases} 0, x < a \\ \frac{L}{2}, x = a \\ L, a < x < b \\ \frac{L}{2}, x = b \\ 0, x > b \end{cases}$$

Example 2: Consider the problem

$$u_t = c^2 u_{xx,} x > 0, t > 0, u(x,0) = 0, t > 0 \qquad (2.18)$$

u and u_x finite as $x \to \infty, t > 0$

$$u(x,0) = f(x), x > 0$$

which appears in heat flow in a semi–infinite region. In (2.18) the function f is piecewise smooth and absolutely integrable in [0, ∞)

We define the odd function

$$\overline{f}(x) = \begin{cases} f(x), x > 0 \\ -f(-x), x < 0. \end{cases}$$

Then from (2.16) we have

$$= \int_{-\infty}^{\infty} \frac{e^{-(x-\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} \overline{f}(\mu) d\mu$$

$$= \int_{-\infty}^{0} \frac{e^{-(x-\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} \overline{f}(\mu) d\mu + \int_{0}^{\infty} \frac{e^{-(x-\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} \overline{f}(\mu) d\mu$$

In the first integral we change μ to -μ and use the oddness of $\overline{f}$, to obtain

$$\int_{-\infty}^{0} \frac{e^{-(x-\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} \overline{f}(\mu) d\mu = -\int_{0}^{\infty} \frac{e^{-(x+\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} f(\mu) d\mu.$$

Thus, the solution of the problem (2.18) can be written as

$$u(x,t) = \int_{0}^{\infty} \frac{e^{-(x-\mu)^2/(4c^2 t)} - e^{-(x+\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} f(\mu) d\mu.$$

The above procedure to find the solution of (1.18) is called the method of images.

In an analogous way it can be shown that the solution of the problem

$$u_t = c^2 u_{xx,} x > 0, t > 0, u_x(x,0) = 0, t > 0 \qquad (2.19)$$

u and u_x finite as $x \to \infty, t > 0$

$$u(x,0) = f(x), x > 0$$

can be written as

$$u(x,t) = \int_{0}^{\infty} \frac{e^{-(x-\mu)^2/(4c^2 t)} - e^{-(x+\mu)^2/(4c^2 t)}}{\sqrt{4\pi c^2 t}} f(\mu) d\mu.$$

Here, of course, we need to extend f(x) to an even function

$$\overline{f}(x) = \begin{cases} f(x), x > 0 \\ f(-x), x < 0. \end{cases}$$

In (2.19) the physical significance of the condition $u_x(x,0) = 0$ is that there is a perfect insulation, i.e., there is no heat flux across the surface.

Example 3: Consider the initial-value problem for the wave equation

$$u_t = c^2 u_{xx,} -\infty < x < \infty, t > 0, c > 0$$

u and u_x finite as

$$|x| \to \infty, t > 0 \qquad (2.20)$$

$$u(x,0) = f_1(x), -\infty < x < \infty,$$
$$u_t(x,0) = f_2(x), -\infty < x < \infty,$$

where the functions f_1 and f_2 are piecewise smooth and absolutely integrable in (-∞,∞).

To find the solution of this problem, we introduce the Fourier transforms

$$F_j(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f_j(x) dx, j = 1,2$$

and its inversion formulas

$$f_j(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} F_j(\omega) d\omega, j = 1,2.$$

We also need the Fourier representation of the solution u(x,t),

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t) e^{i\omega x} d\omega$$

Where $\hat{u}(\omega,t)$ is an unknown function, which we will now determine. For this, we substitute this into the differential equation (2.20), to obtain

$$0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\partial^2 \hat{u}(\omega,t)}{\partial t^2} + c^2\omega^2 \hat{u}(\omega,t) \right] e^{i\omega x} d\omega.$$

Thus, $\hat{u}$ must be a solution of the ordinary differential equation

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2\omega^2 \hat{u} = 0$$

whose solution can be written as

$$\hat{u}(\omega,t) = c_1(\omega)cos\omega ct + c_2(\omega)sin\omega ct$$

To find $c_1(\omega)$ and $c_2(\omega)$, we note that

$$f_1(x) = u(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c_1(\omega)e^{i\omega x}d\omega$$

$$f_2(x) = \frac{\partial u(x,0)}{\partial t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \omega c c_2(\omega)e^{i\omega x}d\omega$$

and hence $F_1(\omega) = c_1(\omega)$ and $F_2(\omega) = \omega c c_1(\omega)$.

Therefore, it follows that

$$\hat{u}(\omega,t) = F_1(\omega)cos\omega ct + \frac{F_2(\omega)}{\omega c}sin\omega ct$$

and hence the Fourier representation of the solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F_1(\omega)cos\omega ct + \frac{F_2(\omega)}{\omega c}sin\omega ct\right]e^{i\omega x}d\omega.$$

Now since $cos = \frac{e^i + e^{-i}}{2}, sin = \frac{e^i - e^{-i}}{2i}$ we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\omega)(cos\omega ct)e^{i\omega x}dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\omega)(e^{i\omega x} + e^{-i\omega x})e^{i\omega x}d\omega$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\omega)(e^{i\omega(x+ct)} + e^{i\omega(x-ct)})d\omega$$

$$= \frac{1}{2}\left[f_1(x+ct) + f_1(x-ct)\right]$$

Similarly,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_2(\omega)\frac{sin\omega ct}{\omega c}e^{i\omega x}dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_2(\omega)\frac{e^{i\omega ct} - e^{-i\omega ct}}{i\omega c}e^{i\omega x}d\omega$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_2(\omega)\frac{e^{i\omega(x+ct)} - e^{i\omega(x-ct)}}{i\omega c}d\omega$$

$$= \frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_2(\omega)\left(\int_{x-ct}^{x+ct} e^{i\omega\xi}d\xi\right)d\omega$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega\xi}F_2(\omega)d\omega\right]d\xi$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi.$$

Putting these together yields d' Alembert's formula

$$u(x,t) = \frac{1}{2}[f_1(x+ct) + f_1(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi$$

Example 4: Consider the following problem involving the Laplace equation in a half-plane:

$$u_{xx} + u_{yy} = 0, -\infty < x < \infty, y > 0$$
$$u(x,0) = f(x), -\infty < x < \infty$$
$$|u(x,y)| \le M, -\infty < x < \infty, y > 0,$$

where the function f is piecewise smooth and absolutely integrable in (-∞,∞)

If $f(x) \to 0\ as\ |x| \to \infty$, then we also have the implied boundary conditions

$$\lim_{|x|\to\infty,} u(x,y) = 0, \lim_{y\to+\infty,} u(x,y) = 0.$$

For this, we let

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega,$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

and

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\omega,y)e^{i\omega x}d\omega.$$

We find that

$$0 = u_{xx,} + u_{yy} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[-\omega^2\hat{u}(\omega,y) + \frac{\partial^2\hat{u}(\omega,y)}{\partial y^2}\right]e^{i\omega x}d\omega$$

Thus, $\hat{u}$ must satisfy the ordinary differential equation

$$\frac{\partial^2\hat{u}}{\partial y^2} = \omega^2\hat{u}$$

and the initial condition $\hat{u}(\omega,0) = F(\omega)$ for each ω.

The general solution of the ordinary differential equation is $c_1e^{\omega y} + c_2e^{-\omega y}$. If we impose the initial condition and the boundedness condition, the solution becomes

$$\hat{u}(\omega,y) = \begin{Bmatrix} F(\omega)e^{-\omega y}, \omega \ge 0 \\ F(\omega)e^{\omega y}, \omega < 0 \end{Bmatrix} = F(\omega)e^{-|\omega|y}$$

Thus, the desired Fourier representation of the solution is

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-|\omega|y}e^{i\omega x}d\omega.$$

To obtain an explicit representation, we insert the formula for F(ω) and formally interchange the order of integration, to obtain

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(\xi)e^{-i\omega\xi}d\xi\right)e^{-|\omega|y}e^{i\omega x}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{i\omega(x-\xi)}e^{-|\omega|y}d\omega\right)f(\xi)d\xi.$$

Now the inner integral is

$$\int_{-\infty}^{\infty}e^{i\omega(x-\xi)}e^{-|\omega|y}d\omega=2\,\mathrm{Re}\int_{0}^{\infty}e^{i\omega(x-\xi)}e^{-\omega y}d\omega$$

$$=2\,\mathrm{Re}\frac{1}{y-i(x-\xi)}=\frac{2y}{y^2+(x-\xi)^2}$$

Therefore, the solution u(x,y) can be explicitly written as

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{y^2+(x-\xi)^2}f(\xi)d\xi. \qquad (2.21)$$

This representation is known as Poisson's integral formula.

In particular, for

$$u(x,0)=f(x)=\begin{cases}1, a<x<b\\ 0, \text{otherwise}\end{cases}$$

Thus, (2.21) become

$$u(x,y)=\frac{1}{\pi}\int_{a}^{b}\frac{y}{y^2+(x-\xi)^2}d\xi=\frac{1}{\pi}\int_{a}^{b}\frac{d\xi/y}{\frac{(x-\xi)^2}{y^2}+1}$$

Using the substitution $v=\frac{x-\xi}{y}$ we have $d\xi=ydv$, so that

$$u(x,y)=\frac{1}{\pi}\int_{(a-x)/y}^{(b-x)/y}\frac{1}{1+v^2}dv$$

$$\frac{1}{\pi}\left(\tan^{-1}\frac{b-x}{y}-\tan^{-1}\frac{a-x}{y}\right)$$

2.4. The Fourier Sine and Cosine Transform Methods

We will motivate the introduction of Fourier sine and cosine transforms by considering a simple physical problem. In order to use the Fourier sine and cosine transform to solve a partial differential equation:

If the boundary conditions are of the Dirichlet type: where the function value is prescribed on the boundary, then the Fourier sine transform is used.

If the boundary conditions are of the Neumann type: where the derivatives of function is prescribed on boundary, then Fourier cosine transform is applied.

In either case, the PDE reduces to an ODE in Fourier transform which is solved. Then the inverse Fourier sine (or cosine) transforms will give the solution to the problem.

2.4.1. Infinite Fourier cosine and sine Transform Method

To solve a partial differential equation (containing a second derivative) defined on a semi-infinite interval x≥0, using Fourier cosine transform, $\frac{\partial f}{\partial x}(0)$ must be known. In case, $\left[\frac{\partial u(x,t)}{\partial x}\right]_{x=o}$ is given then we employ cosine transform to remove $\frac{\partial^2 u}{\partial x^2}$.

Definition: The infinite Fourier cosine transform of a function f(x) for 0<x<∞,is defined as $F_c(n)=\int_0^{\infty}f(x)\sin\sin nxdx.$ n being a positive integer.

Here f(x) is called as the inverse Fourier cosine transform of $F_c(n)$ and is defined as

$$f(x)=\frac{2}{\pi}\int_0^{\infty}F_c(n)\cos nxdx.$$

Similarly, the Fourier sine transform may be used for semi-infinite problems if f(0) is given.

Furthermore, problems are more readily solved if the boundary conditions are homogeneous. Thus, if f(0)=0, separation of variable motivates the use of sines only. Similarly, $\frac{\partial f}{\partial x}(0)=0$ implies the use of cosine.

Definition: The infinite Fourier sine transform of a function f(x) of x such that 0<x<∞ is denoted by $F_s(n)$, n being a positive integer and is defined as

$$F_s(n)=\int_0^{\infty}f(x)\sin\sin nxdx$$

Here f(x) is called as the inverse Fourier sines transform of $F_s(n)$ and defined as

$$f(x)=\frac{2}{\pi}\int_0^{\infty}F_s(n)\sin nxdx$$

Example: We shall employ the Fourier sine transform to find the solution of the following problem involving the Laplace equation in a semi-infinite strip:

$$u_{xx}+u_{yy}=0, 0<x<\infty, 0<y<b$$
$$u(x,0)=f(x), 0<x<\infty,$$
$$u(0,y)=0, 0<y<b$$
$$u(x,b)=0, 0<x<\infty,$$

where the function f is piecewise smooth and absolutely integrable in [0, ∞). We shall also need the boundary conditions $\lim_{x\to\infty,}u(x,y)=0$ and $\lim_{x\to\infty,}u_x(x,y)=0$

For this, we let

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_s(\omega)\sin\omega xd\omega,$$

$$F_s(\omega)==\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\omega xdx$$

and

$$u(x,y)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{u}_s(\omega,y)\sin\omega xd\omega.$$

We find that

$$0 = u_{xx,} + u_{yy} = \int_0^\infty \left[-\omega^2 \hat{u}_s(\omega, y) + \frac{\partial^2 \hat{u}_s(\omega, y)}{\partial y^2} \right] sin\omega x d\omega.$$

Thus, $\hat{u}$ must satisfy the ordinary differential equation

$$\frac{\partial^2 \hat{u}_s}{\partial y^2} = \omega^2 \hat{u}_s$$

and hence

$$\hat{u}_s(\omega, y) = c_1(\omega) cosh\ \omega y + c_2(\omega) sinh\ \omega y.$$

Now the boundary condition $\hat{u}_s(\omega, b) = 0$ yields

$$c_1(\omega) = -c_2(\omega) \frac{sinh\ \omega b}{cosh\ \omega b}$$

Thus, we have

$$\hat{u}_s(\omega, y) = -c_2(\omega) \frac{sinh\ \omega b}{cosh\ \omega b} cosh\ \omega y + c_2(\omega) sinh\ \omega y$$
$$= c_2(\omega) \frac{sinh\ \omega(y-b)}{cosh\ \omega b}$$

Now since $\hat{u}_s(\omega, 0) = F_s(\omega)$, we find $c_2(\omega) = -F_s(\omega) \frac{sinh\ \omega b}{sinh\ \omega b}$, and therefore

$$\hat{u}_s(\omega, y) = F_s(\omega) \frac{sinh\ \omega(b-y)}{sinh\ \omega b}$$

This gives the solution

$$u(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\omega) \frac{sinh\ \omega(b-y)}{sinh\ \omega b} sin\omega x d\omega$$
$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) sin\omega t \frac{sinh\ \omega(b-y)}{sinh\ \omega b} sin\omega x dt d\omega$$

2.4.2. Finite Fourier cosine and sine Transform Method

When the domain of the physical problem is finite, it is generally not convenient to use the transforms with an infinite range of integration. In many cases, finite Fourier transform can be used with advantage.

Definition: The finite Fourier sine transforms of $f(x), 0 < x < L$ is defined as

$$F_s(n) = \int_0^L f(x) \sin \sin \left(\frac{n\pi x}{L} \right) dx$$

Where n≥0 is an integer. The function f(x) is then called the inverse finite Fourier sine transform of $F_s(n)$ and is given by

$$f(x) = \frac{2}{L} \sum_{n=1}^\infty F_s(n) \sin F_s(n) \sin \left(\frac{n\pi x}{L} \right) \qquad (2.22)$$

Definition: The finite Fourier cosine transforms of $f(x), 0 < x < L$ is defined as

$$F_c(n) = \int_0^L f(x) \cos \left(\frac{n\pi x}{L} \right) dx,$$

Where n≥0 is an integer. The function f(x) is then called the inverse finite Fourier cosine transform of $F_c(n)$ and is given by

$$f(x) = \frac{1}{L} F_c(0) + \frac{2}{L} \sum_{n=1}^\infty F_c(n) \cos(n) \cos \left(\frac{n\pi x}{L} \right)$$

Finite Fourier transforms are useful in solving partial differential equations. For this, we note that

$$\int_0^L \frac{\partial u(x,t)}{\partial x} \sin \frac{n\pi x}{L} dx$$
$$= u(x,t) \sin \frac{n\pi x}{L} \Big]_0^L - \frac{n\pi}{L} \int_0^L u(x,t) \cos \frac{n\pi x}{L} dx$$

And hence

$$F_s\left(\frac{\partial u}{\partial x} \right) = -\frac{n\pi}{L} F_c(n)$$

and similarly,

$$F_c\left(\frac{\partial u}{\partial x} \right) = \frac{n\pi}{L} F_s(n) - [u(0,t) - u(L,t) \cos n\pi],$$
$$F_s\left(\frac{\partial^2 u}{\partial x^2} \right) = -\frac{n\pi}{L} F_c\left(\frac{\partial u}{\partial x} \right) \qquad (2.23)$$
$$= -\frac{n^2\pi^2}{L^2} F_s(n) + \frac{n\pi}{L} \left[u(0,t) - u(L,t) \cos n\pi \right]$$

$$F_c\left(\frac{\partial^2 u}{\partial x^2} \right) = -\frac{n^2\pi^2}{L^2} F_c(n) - \left[u_x(0,t) - u_x(L,t) \cos n\pi \right]$$

Example: Find the solution of the problem

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 4, t > 0$$
$$u(x,0) = 2x, 0 < x < 4$$
$$u(0,t) = u(4,t) = 0, t > 0.$$

Taking the finite Fourier sine transform with L = 4 of both sides of the partial differential equation gives

$$\int_0^4 \frac{\partial u}{\partial x} \sin \frac{n\pi x}{4} dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{4} dx.$$

Writing $\hat{u}$ for $F_s(n)$ and using (2.23) with $u(0,t) = 0, u(4,t) = 0$ leads to

$$\frac{d\hat{u}(n,t)}{dt} = -\frac{n^2\pi^2}{16} \hat{u},$$

which can be solved to obtain $\hat{u}(n,t) = ce^{-n^2\pi^2/16}$.

Now taking the finite Fourier sine transform of the condition u(x=0)=2x we have

$$\hat{u}(n,0)=\int_0^4 2x\sin\frac{n\pi x}{4}dx$$

$$=\left[2x\left(-\frac{\cos \mathrm{n}\pi x/4}{\mathrm{n}\pi/4}\right)-2\left(-\frac{\sin \mathrm{n}\pi x/4}{n^2\pi^2/16}\right)\right]\Bigg|_0^4$$

$$=-\frac{32}{\mathrm{n}\pi}\cos \mathrm{n}\pi$$

Since $c=\hat{u}(n,0)$ it follow that

$$\hat{u}(n,t)=-\frac{32}{\mathrm{n}\pi}\cos \mathrm{n}\pi e^{-n^2\pi^2 t/16}$$

Thus, from (2.22) we get

$$u(x,t)=-\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{\cos \mathrm{n}\pi}{\mathrm{n}}e^{-n^2\pi^2 t/16}\sin\frac{n\pi x}{4}$$

3. Conclusion

However, Physical problems never In this work, when modeling problems over regions that extended very far in at least one direction, we often idealized the situation to that of a problem having infinite extent in one or more directions, where any boundary conditions that would have applied on the far-away boundaries are discarded in favor of simple boundedness conditions on the solution as the appropriate variable is sent to infinity. Such problems were mathematically modeled by differential equations defined on infinite regions. For one-dimensional problems we distinguish two types of infinite regions: infinite intervals extending from -∞ to ∞ and semi-infinite intervals extending from one point (usually the origin) to infinite (usually +∞) are infinite, but by introducing a mathematical model with infinite extent, we are able to determine behavior of problems in the situations in which the influence of actual boundaries are expected to be negligible. Thus the seminar paper developed the Fourier transform method and applied it to solve: heat flow problem of an infinitely long thin bar insulated on its lateral surface, heat flow in a semi–infinite region, wave equation, Laplace equation in a half-plane and in a semi-infinite strip, and some partial differential equation on the entire real line. Even though a survey of this seminar paper shows that what is actually studied Fourier transform method to PDE is that, we taken the Fourier transform of PDE and its initial and boundary conditions to reduce it into an ODE. We then solved this ODE for the transformed function. We inverted this function to determine the solution to our PDE. This is not just a method that is specific to the Fourier transform because this method also works for the Laplace transform and in general for many integral transforms. The integrals defining the Fourier transform and its inverse are remarkably alike, and this symmetry was often exploited, for example when assembling appendix given for Fourier transforms. One condition on this is that the variable you taken to the integral transform its domain must match the range of integration of the integral transform. The type of boundary and initial conditions that are given should also played a role in which transform should be used. In case, the Fourier transform is used to analyze boundary value problems on the entire line.

The extension of Fourier methods to the entire real line leads naturally to the Fourier transform, an extremely powerful mathematical tool for the analysis of non-periodic functions.

It is reasonable to expect a Fourier transform method apply to solve different forms of partial differential equations such as

Telegraph equation: $u_{tt}+(\alpha+\beta)u_t+\alpha\beta u=c^2u_{xx}$ for the case $\alpha,\beta>0$.

The Fourier transform is of fundamental importance in a broad range of applications, including both ordinary and partial differential equations, quantum mechanics, signal and image processing, control theory, and probability, to name but a few.

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