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Existence of Weak Solutions for Elliptic Nonlinear System in R^{N}

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Abstract We study the nonuniformly elliptic, nonlinear system $\begin{cases} -\mathrm{div}\left(h_1(x)\nabla u\right) = f(x,u,v) \text{ in } \mathrm{R}^N \\ -\mathrm{div}\left(h_2(x)\nabla v\right) = g(x,u,v) \text{ in } \mathrm{R}^N \end{cases}$ Under growth and regularity conditions on the nonlinearities f and g, we obtain weak solutions in a subspace of the Sobolev space $H^1\left(\mathrm{R}^N,\mathrm{R}^2\right)$ by applying a variant of the Mountain Pass Theorem.

Keywords: *nonuniformly elliptic, nonlinear systems, mountain pass theorem, weakly continuously differentiable functional*

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1. Introduction

We study the nonuniformly elliptic, nonlinear system

$$\begin{cases} -\mathrm{div}\left(h_1(x)\nabla u\right) = f(x,u,v) \text{ in } \mathrm{R}^N \\ -\mathrm{div}\left(h_2(x)\nabla v\right) = g(x,u,v) \text{ in } \mathrm{R}^N \end{cases}$$

where $N \geq 3, h_i \in L^1_{loc}\left(\mathrm{R}^N\right), h_i(x) \geq 1, i = 1,2.$

System (1.1) , There, under appropriate growth and regularity conditions on the functions $f(x,u,v)$ and $g(x,u,v)$, the weak solutions are exactly the critical points of a functional defined on a Hilbert space of functions u,v in $H^1(\mathrm{R}^N)$. In the scalar case, the problem

$$-div(|x|^{\alpha}\nabla u) + b(x)u = f(x,u) \ in \ \mathrm{R}^N$$

with $N \geq 3$ and $\alpha\varepsilon(0,2)$, has been studied by Mihailescu and Radulescu [12]. In this situation, the authors overcome the lack of compactness of the problem by using the the Caffarelli-Kohn-Nirenberg inequality

In this paper, we consider (1.1) which may be a nonuniformly elliptic system. We shall reduce (1.1) to a uniformly elliptic system by using appropriate weighted Sobolev spaces. Then applying a variant of the Mountain pass theorem in [9], we prove the existence of weak solutions of system (1.1) in a subspace of H^1 $(\mathrm{R}^N, \mathrm{R}^2)$.

To prove our main results, we introduce the following some hypotheses:

$(H.1)$ There exists a function $F(x,w) \in C^1\left(\mathrm{R}^N \times \mathrm{R}^2, \mathrm{R}\right)$ such that $\frac{\partial F}{\partial u}(x,w) = f(x,w), \frac{\partial G}{\partial v}(x,w) = g(x,w)$, for all $x \in \mathrm{R}^N$, $w = (u,v) \in \mathrm{R}^2$ $(H.2)$

$$f(x,w), g(x,w) \in C^1\left(\mathrm{R}^N \times \mathrm{R}^2, \mathrm{R}\right),$$
$$f(x,0,0) = g(x,0,0) = 0,$$

for all $x \in \mathrm{R}^N$, there exists a positive constant α such that

$$\left|\nabla f(x,w)\right| + \left|\nabla g(x,w)\right| \leq \alpha |w|^{p-1}$$

for all $x \in \mathrm{R}^N$, $w = (u,v) \in \mathrm{R}^2$.

$(H.3)$ There exists a constant β > 2 such that

$$0 < \beta F(x,w) < wF(x,w)$$

for all $x \in \mathrm{R}^N$, $w \in \mathrm{R}^2 ? \{(0,0)\}$.

Let $H^1\left(\mathrm{R}^N, \mathrm{R}^2\right)$ be the usual Sobolev space under the norm

$$\|w\| = \int_{\mathrm{R}^N}\left(|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2\right)dx,$$
$$w = (u,v) \in H^1\left(\mathrm{R}^N, \mathrm{R}^2\right)$$

Consider the subspace

$$E = \left\{(u,v) \in H^1\left(\mathrm{R}^N, \mathrm{R}^2\right) \, : \, \int_{\mathrm{R}^N}\left(|\nabla u|^2 + |\nabla v|^2\right)dx < \infty\right\}$$

Then E is a Hilbert space with the norm

$$\|w\|_E^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx$$

It is clear that

$$\|w\|_E \geq \gamma \|w\|_{H^1(\mathbb{R}^N,\mathbb{R}^2)}, \forall w \in E, \gamma > 0.$$

and the embeddings $E ? H^1\left(\mathbb{R}^N, \mathbb{R}^2\right) ? L^q\left(\mathbb{R}^N, \mathbb{R}^2\right), 2 \leq q \leq 2^*$ are continuous. moreover, the embedding $E ? L^2\left(\mathbb{R}^N, \mathbb{R}^2\right)$ is compact (see [8]). we now introduce the space

$$H = \left\{(u,v) \in E \ : \ \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right) dx < \infty\right\}$$

endowed with the norm

$$\|w\|_H^2 = \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right) dx$$

1.1. Remark

Since $h_1(x) \geq 1, h_2(x) \geq 1$ for all $x \in \mathbb{R}^N$ we have $\|w\|_E \leq \|w\|_H$ with $\forall w \in H$ and $C_0^\infty\left(\mathbb{R}^N, \mathbb{R}^2\right) \subset H$.

1.2. Proposition

The set H is a Hilbert space with the inner product

$$\langle w_1, w_2 \rangle = \int_{\mathbb{R}^N} \left(h_1(x) \nabla u_1 \nabla u_2 + h_2(x) \nabla v_1 \nabla v_2\right) dx$$

for all $w_1 = (u_1, u_2), w_2 = (v_1, v_2) \in H$.

Proof. It suffices to check that any Cauchy sequences $\{\omega_m\}$ in H converges to $\omega \in H$. Indeed, let $\{\omega_m\} = \{(u_m, v_m)\}$ be a Cauchy sequence in H. Then

$$\lim_{m;k\to\infty} \int_{\mathbb{R}^N} \begin{pmatrix} h_1(x)|\nabla u_m - \nabla u_k|^2 \\ +h_2(x)|\nabla v_m - \nabla v_k|^2 \end{pmatrix} dx = 0$$

and $\{\|\omega_m\|_H\}$ is bounded. Moreover, by Remark 1.1, $\{\omega_m\}$ is also a Cauchy sequence in E. Hence the sequence $\{\omega_m\}$ converges to $\omega = (u,v) \in E$; i.e,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u_m - \nabla u|^2 + h_2(x)|\nabla v_m - \nabla v|^2\right) dx = 0$$

It follows that $\{\nabla \omega_m = (\nabla u_m, \nabla v_m)\}$ converges to $\nabla \omega = (\nabla u, \nabla v)$ and $\{\omega_m\}$ converges to ω in $L^2\left(\mathbb{R}^N, \mathbb{R}^2\right)$. Therefore $\{\nabla \omega_m(x)\}$ converges to $\{\nabla \omega(x)\}$ and $\{\omega_m(x)\}$ converges to $\omega(x)$ for almost everywhere $x \in \mathbb{R}^N$. Applying Fatou's lemma we get

$$\int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right) dx \leq$$
$$\liminf_{m\to\infty} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u_m|^2 + h_2(x)|\nabla v_m|^2\right) dx < \infty$$

Hence $\omega = (u,v) \in H$. Applying again Fatou's lemma

$$0 \leq \lim_{m\to\infty} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u_m - \nabla u|^2 + h_2(x)|\nabla v_m - \nabla v|^2\right) dx$$
$$\leq \lim_{m\to\infty} \left[\liminf_{k\to\infty} \int_{\mathbb{R}^N} \begin{pmatrix} h_1(x)|\nabla u_m - \nabla u_k|^2 \\ +h_2(x)|\nabla v_m - \nabla v_k|^2 \end{pmatrix} dx\right] = 0$$

We conclude that $\{\omega_m(x)\}$ converges to $\omega = (u,v)$ in H.

1.3. Definition

We say that $\omega = (u,v) \in H$ is a weak solution of system (1.1) if

$$\int_{\mathbb{R}^N} \left(h_1(x) \nabla u \nabla \phi + h_2(x) \nabla v \nabla \psi\right) dx$$
$$-\int_{\mathbb{R}^N} \left(f(x,u,v)\phi + g(x,u,v)\psi\right) dx = 0$$

for all $\varphi = (\phi, \psi) \in H$.

Our main result is stated as follows.

1.4. Theorem

Let $(H.1)$ and $(H.3)$ are satisfied, the system (1.1) has at least one non-trivial weak solution in H.

This theorem will be proved by using variational techniques based on a variant of the Mountain pass theorem in [9]. Let us define the functional $J : H \to \mathbb{R}$ given by

$$J(w) = \frac{1}{2} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right) dx$$
$$-\int_{\mathbb{R}^N} F(x,u,v) dx = T(w) - P(w)$$

for $w = (u,v) \in H$,

where

$$T(w) = \frac{1}{2} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right) dx$$
$$P(w) = \int_{\mathbb{R}^N} F(x,u,v) dx$$

2. Existence of weak solutions

In general, due to $h(x) \in L_{lok}^1\left(\mathbb{R}^N\right)$, the functional J may be not belong to $C^1(H)$, (in this work, we do not completely care whether the functional J belongs to $C^1(H)$ or not).This means that we cannot apply directly the Mountain pass theorem by Ambrosetti-Rabinowitz (see [4]), we recall the following concept of weakly continuous differentiability. Our approach is based on a weak version of the Mountain pass theorem by Duc (see [9]).

2.1. Definition

Let J be a functional from a Banach space Y in to $\mathbb{R}$. We say that J is weakly continuously differentiable on

Y if and only if the following conditions are satisfied (i) J is continuous on Y. (ii) For any $u \in Y$, there exists a linear map $DJ(u)$ from Y into R such that $\lim_{t\to 0} \frac{J(u+tv)-J(u)}{t} = \langle DJ(u), v\rangle, v \in Y.$ (iii) For any $v \in Y$, the map $u \mapsto \langle DJ(u), v\rangle$, vi is continuous on Y.

We denote by $C_w^1(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C_w^1(Y)$, where $C^1(Y)$ is the set of all continuously Frechet differentiable functionals on Y. The following proposition concerns the smoothness of the functional J

2.2. Proposition

Under the assumptions of Theorem 1.4, the functional $J(w), w \in H$ given by (1.3) is weakly continuously differentiable on H and

$$\langle DJ(w), \varphi\rangle = \int_{R^N} \left(h_1(x)\nabla u \nabla \phi + h_2(x)\nabla v \nabla \psi\right) dx$$
$$-\int_{R^N} \left(f(x,u,v)\phi + g(x,u,v)\psi\right) dx$$

for all $w = (u,v), \varphi = (\phi,\psi) \in H.$

Proof. By conditions $(H1)$ -- $(H3)$ and the embedding $H ? E$ is continuous, it can be shown (cf. [[5], Theorem A.VI]) that the functional P is well-defined and of class $C^1(H)$. Moreover, we have

$$\langle DP(w), \varphi\rangle = \int_{R^N} \left(f(x,u,v)\phi + g(x,u,v)\psi\right) dx$$

for all $w = (u,v), \varphi = (\phi,\psi) \in H.$

Next, we prove that T is continuous on H. Let $\{\omega_m\}$ be a sequence converging to w in H, where $\omega_m = (u_m, v_m)$, $m = 1,2,\ldots,$ $\omega = (u,v)$ Then

$$\lim_{m\to\infty} \int_{R^N} \left(h_1(x)|\nabla u_m - \nabla u|^2 + h_2(x)|\nabla v_m - \nabla v|^2\right) dx = 0$$

and $\{\|\omega_m\|_H\}$ is bounded. Observe further that

$$\left|\int_{R^N} \left(h_1(x)|\nabla u_m|^2 dx - \int_{R^N} h_1(x)|\nabla u|^2 dx\right)\right|$$
$$= \left|\int_{R^N} h_1(x)\left(|\nabla u_m|^2 - |\nabla u|^2\right) dx\right|$$
$$\le \int_{R^N} h_1(x)\big||\nabla u_m| - |\nabla u|\big|\left(|\nabla u_m| + |\nabla u|\right) dx$$
$$\le \int_{R^N} h_1(x)|\nabla u_m - \nabla u||\nabla u_m| dx$$
$$+\int_{R^N} h_1(x)|\nabla u_m - \nabla u||\nabla u| dx$$
$$\le \left(\int_{R^N} h_1(x)|\nabla u_m - \nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_{R^N} h_1(x)|\nabla u_m|^2 dx\right)^{\frac{1}{2}}$$
$$+\left(\int_{R^N} h_1(x)|\nabla u_m - \nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_{R^N} h_1(x)|\nabla u|^2 dx\right)^{\frac{1}{2}}$$
$$\le \left(\|\omega_m\|_H + \|\omega\|_H\right)\|\omega_m - \omega\|_H.$$

Similarly, we obtain

$$\left|\int_{R^N} \left(h_2(x)|\nabla v_m|^2 dx - \int_{R^N} h_2(x)|\nabla v|^2 dx\right)\right|$$
$$\le \left(\|\omega_m\|_H + \|\omega\|_H\right)\|\omega_m - \omega\|_H$$

From the above inequalities, we obtain

$$|T(\omega_m) - T(\omega)| \le 2\left(\|\omega_m\|_H + \|\omega\|_H\right)\|\omega_m - \omega\|_H \to 0$$

as $m \to \infty$.

Thus T is continuous on H. Next we prove that for all $w = (u,v), \varphi = (\phi,\psi) \in H.$

$$\langle DJ(w), \varphi\rangle = \int_{R^N} \left(h_1(x)\nabla u \nabla \phi + h_2(x)\nabla v \nabla \psi\right) dx$$

Indeed, for any, $w = (u,v), \varphi = (\phi,\psi) \in H,$ any $t \in (-1,1)?\{0\}$ and $x \in R^N$ we have

$$\left|\frac{h_1(x)|\nabla u + t\nabla \phi|^2 - h_1(x)|\nabla u|^2}{t}\right|$$
$$= \left|2\int_0^1 h_1(x)(\nabla u + st\nabla \phi)\nabla \phi ds\right|$$
$$\le 2h_1(x)\left(|\nabla u| + |\nabla \phi|\right)|\nabla \phi|$$
$$\le h_1(x)|\nabla u|^2 + 3h_1(x)|\nabla \phi|^2.$$

Since

$$h_1(x)|\nabla u|^2 \in L^1(R^N), h_1(x)|\nabla \phi|^2 \in L^1(R^N),$$
$$g(x) = h_1(x)|\nabla u|^2 + 3h_1(x)|\nabla \phi|^2 \in L^1(R^N).$$

Applying Lebesgue's Dominated convergence theorem we get

$$\lim_{t\to 0} \int_{R^N} \frac{h_1(x)|\nabla u + t\nabla \phi|^2 - h_1(x)|\nabla u|^2}{t} dx$$
$$= 2\int_{R^N} h_1(x)\nabla u \nabla \phi dx$$

Similarly, we have

$$\lim_{t\to 0} \int_{R^N} \frac{h_2(x)|\nabla v + t\nabla \psi|^2 - h_2(x)|\nabla v|^2}{t} dx$$
$$= 2\int_{R^N} h_2(x)\nabla v \nabla \psi dx$$

Combining (2.2) - (2.3), we deduce that

$$\langle DT(w), \varphi\rangle = \lim_{t\to 0} \frac{T(w + t\varphi) - T(w)}{t}$$
$$= \int_{R^N} \left(h_1(x)\nabla u \nabla \phi + h_2(x)\nabla v \nabla \psi\right) dx$$

Thus T is weakly differentiable on H.

Let $\varphi = (\phi,\psi) \in H$ be fixed. We now prove that the map $w \mapsto \langle DT(w), \varphi\rangle$ is continuous on H. Let $\{\omega_m\}$ be a sequence converging to ω in H. We have

$$|\langle DT(\omega_m), \varphi\rangle - \langle DT(\omega), \varphi\rangle|$$
$$\le \int_{R^N} h_1(x)|\nabla u_m - \nabla u||\nabla \phi| dx$$
$$+\int_{R^N} h_2(x)|\nabla v_m - \nabla v||\nabla \psi| dx$$

It follows by applying Cauchy's inequality that

$$\left|\langle DT(\omega_m),\varphi\rangle - \langle DT(\omega),\varphi\rangle\right| \le 2\|\varphi\|_H \|\omega_m-\omega\|_H \to 0 \text{ as } m\to\infty.$$

Thus the map $w \mapsto \langle DT(w),\varphi\rangle$, i is continuous on H and we conclude that functional T is weakly continuously differentiable on H. Finally, J is weakly continuously differentiable on H.

2.3. Remark

From Proposition 2.2 we observe that the weak solutions of system (1.1) correspond to the critical points of the functional $J(w), w\in H$ given by (1.3) Thus our idea is to apply a variant of the Mountain pass theorem in [9] for obtaining non-trivial critical points of J and thus they are also the non-trivial weak solutions of system (1.1).

2.4. Proposition

The functional $J(w), w\in H$ given by (1.3) satisfies the Palais-Smale condition.

Proof Let $\{\omega_m\}=\{(u_m,v_m)\}$ be a sequence in H such that

$$\lim_{m\to\infty} J(\omega_m)=c,\ \lim_{m\to\infty}\|DJ(\omega_m)\|_{H^*}=0$$

First, we prove that $\{\omega_m\}$ is bounded in H. We assume by contradiction that $\{\omega_m\}$ is not bounded in H. Then there exists a subsequence $\{\omega_{m_j}\}$ of $\{\omega_m\}$ such that $\|\omega_{m_j}\|_H \to\infty$ as $j\to\infty$. By assumption $(H.3)$ it follows that

$$J(\omega_{m_j})-\frac{1}{\mu}DJ(\omega_{m_j}),\omega_{m_j}\rangle=\left(\frac{1}{2}-\frac{1}{\mu}\right)\|\omega_{m_j}\|_H^2 +\left(\frac{1}{\mu}\langle DP(\omega_{m_j}),\omega_{m_j}\rangle - P(\omega_{m_j})\right)\ge\gamma\|\omega_{m_j}\|_H^2$$

where $\gamma=\frac{1}{2}-\frac{1}{\mu}$, This yields

$$\begin{aligned}J(\omega_{m_j}) &\ge \gamma\|\omega_{m_j}\|_H^2+\frac{1}{\mu}\langle DJ(\omega_{m_j}),\omega_{m_j}\rangle\\ &\ge \gamma\|\omega_{m_j}\|_H^2-\frac{1}{\mu}\|DJ(\omega_{m_j})\|_{H^*}\|\omega_{m_j}\|_H\\ &=\|\omega_{m_j}\|_H\left(\gamma\|\omega_{m_j}\|_H-\frac{1}{\mu}\|DJ(\omega_{m_j})\|_{H^*}\right)\end{aligned}$$

Letting $j\to\infty$, since $\|\omega_{m_j}\|_H\to\infty, \|DJ(\omega_{m_j})\|_{H^*}\to 0$, we deduce that $J(\omega_{m_j})\to\infty$, which is a contradiction. Hence $\{\omega_m\}$ is bounded in H.

Since H is a Hilbert space and $\{\omega_m\}$ is bounded in H, there exists a subsequence $\{\omega_{m_k}\}$ of $\{\omega_m\}$ weakly converging to ω in H. Moreover, since the embedding $H ? E$ is continuous, $\{\omega_{m_k}\}$ is weakly convergent to ω in E. We shall prove that

$$T(\omega)\le \liminf_{j\to\infty} T(\omega_{m_k})$$

since $\{\omega_{m_k}\}$ converges weakly to ω in E; i.e,

$$\lim_{k\to\infty}\int_{\mathrm{R}^N}\left[(\nabla u_{m_k}-\nabla u)\nabla\phi+(\nabla v_{m_k}-\nabla v)\nabla\psi\right]dx=0$$

for all $\varphi=(\phi,\psi)\in E$, this implies that $\{\nabla\omega_{m_k}\}$ converges weakly to $\nabla\omega$ in $L^1(\Omega,\mathrm{R}^2)$.

Applying [[16], Theorem 1.6], we obtain

$$T(\omega)\le \liminf_{j\to\infty} T(\omega_{m_k})$$

Thus (2.8) is proved. We now prove that

$$\lim_{k\to\infty}\langle DP(\omega_{m_k}),\omega_{m_k}-\omega\rangle = \lim_{k\to\infty}\int_{\mathrm{R}^N}\nabla F(x,\omega_{m_k}).(\omega_{m_k}-\omega)dx=0$$

Indeed, by $(H.2)$, we have

$$\begin{aligned}&\left|\nabla F(x,\omega_{m_k}).(\omega_{m_k}-\omega)\right|\\ &=\left|f(x,\omega_{m_k}).(u_{m_k}-u)+g(x,\omega_{m_k}).(v_{m_k}-v)\right|\\ &\le\left|\nabla f(x,\theta_1\omega_{m_k})\right|\left|\omega_{m_k}\right|\left|u_{m_k}-u\right|\\ &\quad+\left|\nabla g(x,\theta_2\omega_{m_k})\right|\left|\omega_{m_k}\right|\left|v_{m_k}-v\right|\\ &\le A_1\left|\omega_{m_k}\right|^p\left|u_{m_k}-u\right|+A_2\left|\omega_{m_k}\right|^p\left|v_{m_k}-v\right|\\ &\le A_3\left|\omega_{m_k}\right|^p\left|\omega_{m_k}-\omega\right|, 0<\theta_1,\theta_2<1\end{aligned}$$

where A_i $(i=1,2,3)$ are positive constants.

Set $2^*=\frac{2N}{N-2}, p_1=\frac{2^*}{p-1}, p_2=p_3=\frac{2p_1}{p_1-1}$. We have $p_1>1, 2<p_2,p_3<2^*$ and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$

Therefore,

$$\begin{aligned}&\lim_{k\to\infty}\int_{\mathrm{R}^N}\nabla F(x,\omega_{m_k}).(\omega_{m_k}-\omega)dx\\ &\le A_3\int_{\mathrm{R}^N}\left|\omega_{m_k}\right|^{p-1}\left|\omega_{m_k}-\omega\right|\left|\omega_{m_k}\right|dx\\ &\le A_3\|\omega_{m_k}\|_{L^{2^*}}^{p-1}\|\omega_{m_k}-\omega\|_{L^{p_2}}\|\omega_{m_k}\|_{L^{p_3}}\end{aligned}$$

On the other hand, using the continuous embeddings $H ? E ? L^q(\mathrm{R}^N), 2\le q\le 2^*$ together with the interpolation inequality $\left(\text{where } \frac{1}{p_2}=\frac{\delta}{2}+\frac{1-\delta}{2^*}\right)$, it follows that

$$\left\|\omega_{m_k}-\omega\right\|_{L^{p_2}\left(\mathrm{R}^N\right)} \le \left\|\omega_{m_k}-\omega\right\|^{\delta}_{L^2\left(\mathrm{R}^N\right)} \cdot \left\|\omega_{m_k}-\omega\right\|^{1-\delta}_{L^{2^*}}$$

Since the embedding $E ? L^2\left(\mathrm{R}^N\right)$ is compact we have $\left\|\omega_{m_k}-\omega\right\|_{L^2\left(\mathrm{R}^N\right)} \to 0$ as $k\to\infty$. Hence $\left\|\omega_{m_k}-\omega\right\|_{L^{p_2}\left(\mathrm{R}^N\right)} \to 0$ as $k\to\infty$ and (2.10) is proved.

On the other hand, by (2.10) and (2.1) it follows

$$\lim_{k\to\infty}\langle DT\left(\omega_{m_k}\right), \omega_{m_k}-\omega\rangle = 0$$

Hence, by the convex property of the functional T we deduce that

$$T(\omega) - \lim_{k\to\infty}\sup T\left(\omega_{m_k}\right) = \lim_{k\to\infty}\inf\left[T(\omega) - T\left(\omega_{m_k}\right)\right]$$
$$\ge \lim_{k\to 0} < DT(\omega_{m_k}), \omega_{m_k}-\omega >= 0$$

Relations (2.8) and (2.11) imply

$$T(\omega) = \lim_{k\to\infty} T\left(\omega_{m_k}\right)$$

Finally, we prove that $\left\{\omega_{m_k}\right\}$ converges strongly to ω in H. Indeed, we assume by contradiction $\left\{\omega_{m_k}\right\}$ that is not strongly convergent to ω in H. Then there exist a constant $\varepsilon > 0$ and a subsequence $\left\{\omega_{m_{k_j}}\right\}$ of $\left\{\omega_{m_k}\right\}$ such that $\left\|\omega_{m_{k_j}}-\omega\right\|_H \ge \varepsilon > 0$ for any $j = 1,2,3...$ Hence

$$\frac{1}{2}T\left(\omega_{m_{k_j}}\right) + \frac{1}{2}T(\omega) - T\left(\frac{\omega_{m_{k_j}}+\omega}{2}\right)$$
$$= \frac{1}{4}\left\|\omega_{m_{k_j}}-\omega\right\|_H^2 \ge \frac{1}{4}\varepsilon^2$$

With the same arguments as in the proof of $(2,8)$, and remark that the sequence $\left(\frac{\omega_{m_{k_j}}+\omega}{2}\right)$ converges weakly to ω in E, we have

$$T(\omega) \le \lim_{j\to\infty}\inf T\left(\frac{\omega_{m_{k_j}}+\omega}{2}\right)$$

Hence letting $j\to 1$, from (2.13)) and (2.14) we infer that

$$T(\omega) - \lim_{j\to\infty}\inf T\left(\frac{\omega_{m_{k_j}}+\omega}{2}\right) \ge \frac{1}{4}\varepsilon^2 \quad (2.16)$$

Relations (2.15) and (2.16) imply $0 \ge \frac{1}{4}\varepsilon^2 > 0$, which is a contradiction. Therefore, we conclude that $\left\{\omega_{m_k}\right\}$ converges strongly to ω in H and J satisfies the Palais -Smale condition on H.

To apply the Mountain pass theorem we shall prove the following proposition which shows that the functional J has the Mountain pass geometry.

2.5. Proposition

(i) There exist $\alpha > 0$ and $r > 0$ such that $J(w) \ge \alpha$, for all $w \in H$ with $\|w\|_H = r$. (ii) There exists $w_0 \in H$ such that $\|w_0\|_H > r$ and $J(w_0) < r$.

Proof. (i) From $(H.3)$, it is easy to see that

$$F(x,z) \ge \min_{|s|=1} F(x,s).|z|^{\mu} > 0 \forall x \in \mathrm{R}^N$$

and $|z| \ge 1, z \in \mathrm{R}^2$.

$$0 < F(x,z) \le \max_{|s|=1} F(x,s).|z|^{\mu} \ \forall x \in \mathrm{R}^N and 0 < |z| \le 1.$$

where $\max_{|s|=1} F(x,s) \le C$ in view of $(H.2)$. It follows from (2.18) that

$$\lim_{|z|\to\infty}\frac{F(x,z)}{|z|^2} = 0 \ .$$

uniformly for $x \in \mathrm{R}^N$.

By using the embeddings $H ? E ? L^2\left(\mathrm{R}^N, \mathrm{R}^2\right)$, with simple calculations we infer from (2.19) that $\inf_{\|w\|_H = r} J(w) = \alpha > 0$ for $r > 0$ small enough This implies (i).

(ii) By (2.17), for each compact set $\Omega \subset \mathrm{R}^N$ there exists $\eta = \eta(\Omega)$ such that

$$F(x,z) \ge \eta.|z|^{\mu} \ \text{forall}\ x \in \Omega, |z| > 1.$$

Let $0 \ne \varphi = (\phi,\psi) \in C^1\left(\mathrm{R}^N, \mathrm{R}^2\right)$ having compact support, for $t > 0$, large enough from (2.20) we have

$$J(t\varphi) = \frac{1}{2}t^2\|\varphi\|_H^2 - \int_{\mathrm{R}^N} F(x, t\varphi)\,dx$$
$$\le \frac{1}{2}t^2\|\varphi\|_H^2 - t^{\mu}\eta\int_{\Omega}|\varphi|^{\mu}\,dx.$$

where $\eta = \eta(\Omega)$, $\Omega = (\sup\phi \cup \sup\psi)$. Then (2.21) and $\mu > 2$ imply (ii).

Proof of Theorem 1.4. It is clear that $J(0) = 0$. Furthermore, the acceptable set

$$G = \left\{\gamma \in C([0,1], H) \ : \ \gamma(0) = 0, \gamma(1) = w_0\right\}$$

where w_0 is given in Proposition (2.5) is not empty (it is easy to see that the function $\gamma(t) = tw_0 \in G$. by Proposition. and Propositions (2.2) - (2.5) , all assumptions of the Mountain pass theorem introduced in [8] are satisfied. Therefore there exists $w^* \in H$ such that

$$0 < \alpha \le J\left(w^*\right) = \inf\left\{\max J\left(\gamma\left(\left[0,1\right]\right)\right) \;:\; \gamma \in G\right\}$$

and $\langle DJ\left(w^*\right), \varphi \rangle = 0$ for all $\varphi \in H, i.e, w^*$ is a weak solution of system $\left(1.1\right)$. The solution w^* is a non-trivial solution by $J\left(w^*\right) \ge \alpha > 0.$ The proof is complete.

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