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The Wave Equation with Dynamic Wentzell Boundary Condition in Polygonal and Polyhedral Domains: Observation and Exact Controllability

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Abstract We study in this article the boundary observability and the exact controllability for a problem of transmission. The system is governed by the wave equation with Wentzell dynamic artificial condition on the boundary. The geometrical domains considered are polyhedrons or polygons.

Keywords: *wave equation, Wentzell boundary condition, Wentsell, Ventcel, polygonal domains, polyhedral domains, observation, exact controllability*

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1. Introduction

We consider the problem of observation and control of a system of transmission waves. The system is governed by a three-dimensional (respectively two-dimensional) D' Alembert wave equation in a bounded domain which comprises corners but which is not fissured. On the boundary we have inside an artificial dynamic Wentzell boundary condition coupled with the internal equation by the normal derivative. Let us start by giving a few motivations of interest in the study of observation, control, stabilization and inverse problems with artificial conditions: The condition at the boundary can be seen like a contribution of energy (kinetic energy and potential energy): $\frac{1}{2}(\int_{\partial\Omega} |\partial_t u|^2 + \beta \int_{\partial\Omega} |\nabla_{\partial\Omega} u|^2)$ through entering flow $\left(-\frac{\partial u}{\partial n}\right)$ and which is due to the two-dimensional wave propagation (respectively one-dimensional) on the boundary. The dynamic Wentzell condition can, then, being written in a simple way in the case as of acoustic waves or of vibration longitudinal as follows: $\partial_{t^2} u - \beta \Delta_{\partial\Omega} u = \alpha u + \frac{\partial u}{\partial n}$ in $\partial\Omega$.

The coefficients α and β can, in certain cases, be some appropriate operators, they can also be simply constants containing information on the physical characteristics of the problem as well as information about geometrical characteristics of the domain. Indeed the boundary conditions of the type Wentzell involved in many physical problems: electromagnetic, acoustics, structural mechanics or in heat problems. One of the most important reasons to use such conditions is the presence of thin layers. Thin layer, have in general, very small dimensions compared with characteristic values of the main problem. The numerical calculation of systems with thin layers presents many problems at several levels, and the discretization with a size scale of the thin layer becomes difficult and leads to expensive calculations and sometimes very sketchy. A very interesting alternative, then, is to model the problem with a system that replaces the presence of thin layers by a boundary on which we impose an "artificial" conditions. There is a whole family of artificial conditions sometimes coupling the Dirichlet, Neumann and operators boundaries may be the same as the operator governing equations of the system whose dimensions are dominant. We can cite for example the case of anechoic chambers are used to analyze the waves emitted by telecommunication devices: these are rooms whose walls, floors and ceilings are covered with an absorbent so that reflections on the material walls do not interfere with the experiment. The thickness of the absorbent zone is small compared to the characteristic dimensions of the work-piece and can therefore be modeled as a thin layer. We can also cite the example of radar stealth: a special paint covers the plane and it absorbs the waves emitted by the radar, which allows the aircraft to remain invisible. The relationship between the thickness of the paint layer and the characteristic dimensions of the device is also very low.

The Wentzell condition was first introduced by [1]. For the physical meaning of the Wentzell condition and artificial conditions, as well as for the study of some mathematical and numerical aspects of such problems, the interested reader may consult for example: [2-8] and references in these Articles, and the list is not exhaustive.

Before stating our main result, it may be interesting to give brief state of the art methods in observation and control to locate this work in context and to make a link with related work. The study of many problems of controllability or stabilization or even inverse problem reduces to finding a priori estimates on the solutions of adjoint problems [9,10]. These estimates are clearly relevant to the notion of stable observation, and it constitute an important ingredient in the Hilbert Uniqueness Method (H.U.M) of J.L. Lions [11]. In the case of regular open we often have the choice between the use of so-called global or local multiplier techniques (Rellich) or technical finer microlocal analysis: by classical approach consisting in propagation analysis of singularities [12,13,14,15,16], or by using the microlocal defect measures (or H measures) consisting in analysis propagation analysis of regularity (or compactness) [17,18]. Both techniques (global or microlocal) give relatively satisfactory results, but with one notable difference. Indeed, global techniques (multipliers) are relatively simple but the results are not always optimal especially for the region of observation or control and the time of observation or action. Classical microlocal techniques are by far more difficult against but give finer and often optimal results. The introduction of microlocal defect measures in some cases considerably simplifies demonstrations while keeping the optimality and the accuracy of the results [19]. In the presence of singularities due to the geometry of the domains the questions of observation, control, Stabilization and inverse problems are much more difficult and there are strict restrictions to extend both technical and results[20,21]. For example it is not easy to work with the tools of microlocal analysis in the case where open are not polyhedral convex (this is the case discussed in this article), or in the case of open only Lipchitz [22].

For some results on control or stabilization with artificial conditions we can see for example [10,23-28], and others.

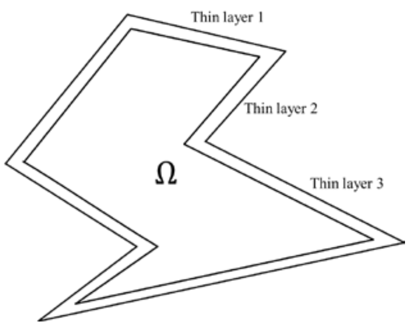

Figure 1. Original Geometry with different thin layer

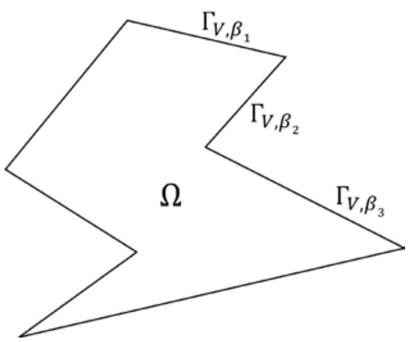

Figure 2. Geometry after transformation with different artificial boundary condition

2. Position of the Problem and Main Result

We note Ω a bounded open domain of $\mathbb{R}^n (n=2,3)$ and Q the cylinder $\Omega\times(0,T)$. We assume that Ω is a polygon (or polyhedron) and we note $\partial\Omega$ its boundary.

We assume that there are two open subsets of the boundary Γ_D and Γ_V such that: $\partial\Omega=\Gamma_D\bigcup\Gamma_V$. Where $\Gamma_D=\bigcup_{i\in D}\Gamma_i$ with $D\subset\mathbb{N}^*$, designating the union of all portions of the boundary where we impose a Dirichlet condition, and where $\Gamma_V=\bigcup_{i\in v}\Gamma_i$ with $V\subset\mathbb{N}^*$, designating the union of all portions of the boundary where we impose a Wentcell condition.

For each portion Γ_i, $i\in V\bigcup D$, we note $(S_{i,j})_{j=1,2}$ its two oriented vertices and ω_j the angle between Γ_i et Γ_{i+1}. We also note for each $i\in V\bigcup D$ by Σ_i the set $\Gamma_i\times[0,T]$. Let us then define $y=y(x,t)$ as the state of the system Q and $\gamma_j y$ the trace on Γ_j ; $j\in V\bigcup D$, satisfying:

$$\partial_{t^2} y-\Delta y=0. \ in\ Q \tag{1}$$

We then act on the system through the controls (v_D, v_V, w) as follows:

$$\gamma_j y=v_{D,j}\ in \sum{}_j;\ j\in D \tag{2}$$

$$\partial_{t^2}(\gamma_j y)-\Delta_{\Gamma_i}(\gamma_j y)=-\frac{\partial y}{\partial n}+v_{V,i}\ in\sum{}_i; i\in V \tag{3}$$

$$\gamma_j y(S_{i,j})=\omega_{i,j}\ in\ (0,T);\ i\in V,\ j\in\{1,N_i\} \tag{4}$$

where N_i denotes the number of vertices of $\Gamma_i, i\in V$.

The system data are:

$$y(x,0)=y^0(x);\ \partial_t y(x,0)=y^1(x)\ \ in\ \Omega \tag{5}$$

$$\gamma_j y(x,0)=p_j^0(x);\ \partial_t\left(\gamma_j y\right)(x,0)=p_j^1(x)\ \ j\in V \tag{6}$$

The control problem is then formulated as follows:

Given $T>0$, can we find for all initial data $\left\{y^0, y^1, p_j^0, p_j^1;\ j\in V\right\}$ in a suitable space, a control functions $\{v_D, v_V, \omega\}$, a geometric region in the boundary, such that the system satisfies:

$$y(x,T)=\partial_t y(x,T)=0\ in\ \Omega \tag{7}$$

$$\gamma_j(x,T)=\partial_t\left(\gamma_j y\right)(x,T)=0\ in\ \Gamma_j;\ j\in V \tag{8}$$

Theorem 1.

There is a time $T_0>0$, a region $\Gamma\subset\partial\Omega$ and a set $\{v_D, v_V, \omega\}$ acting on the system in $\Gamma\times(0,T_0)$ such that the system is brought back to the idle state (7)-(8).

To prove Theorem 1 we will use *Hilbert Uniqueness Method (HUM)* [11] and we proceed as follows:

In Section 3 we study the adjoint problem to the system under question, and we establish results regularity. Then, in Section 4 will be given the estimated observability using the technique of Rellich multipliers and regularity results established in section 3. In Section 5 we will implement the method of Hilbert uniqueness to complete the proof of the main result. In Conclusion of this article we give some remarks concerning the controllability of the transmission system in the case of regular domains and some possible prospects.

3. Regularity of the Adjoint Problem

The adjoint system to (1)-(6) is considered and we denote the corresponding solution φ:

$$\partial_{t^2}\varphi - \Delta\varphi = 0 \text{ in } Q \quad (9)$$

$$\gamma_j\varphi = 0 \text{ in } \sum_j ;\ j \in D \quad (10)$$

$$\partial_{t^2}(\gamma_i\varphi) - \Delta_{\Gamma_i}(\gamma_i\varphi) = -\frac{\partial\varphi}{\partial n} \text{ in } \Sigma_i ; i \in V \quad (11)$$

$$\gamma_j\varphi(S_{i,j}) = 0 \text{ in } (0,T); i \in V,\ j \in \{1, N_i\} \quad (12)$$

Data of homogeneous system are:

$$\varphi(x,0) = \varphi^0(x); \partial_t\varphi(x,0) = \varphi^1(x) \text{ in } \Omega \quad (13)$$

$$\gamma_j y(x,0) = \rho_j^0(x); \partial_t(\gamma_j y)(x,0) = \rho_j^1(x) \quad j \in V \quad (14)$$

The following spaces are then introduced:

$$H(\Omega) = \left\{ \begin{array}{l} (u,v_i) \in H^1(\Omega) \times \prod_{i\in V} H_0^1(\Gamma_i); \\ u|_{\Gamma_i} = v_i, i \in V, u|_{\Gamma_j} = 0,\ j \in D \end{array} \right\} \quad (15)$$

and

$$D(A) = \left\{ \begin{array}{l} (u,v_i) \in H; \Delta u \in L^2(\Omega); \\ \Delta_{\Gamma_i} u \in L^2(\Gamma_i), i \in V \end{array} \right\} \quad (16)$$

Where $D(A)$ is the domain of the operator concerned, we have then the following result:

Theorem 2.

Let $\{\varphi^0, \varphi^1, \rho_j^0, \rho_j^1; j \in V\}$ in $D(A) \times H$, then the solution $(\varphi, \gamma_i\varphi), i \in V$ of the system (9)-(14) is in space $C(0,T;D(A)) \cap C^1(0,T;H)$.

The proof of the theorem 2 is divided into several steps.

3.1. Proof of Theorem 2-Step 1: "Static Problem"

We begin by examining the regularity of the following static problem:

$$-\Delta u = f \text{ in } \Omega \quad (17)$$

$$\gamma_i u = 0 \text{ in } \Gamma_i ; i \in D \quad (18)$$

$$-\Delta_{\Gamma_i}(\gamma_i u) = -\frac{\partial u}{\partial n} \text{ in } \Gamma_i ; i \in V \quad (19)$$

$$\gamma_j\varphi(S_{j,k}) = 0;\ j \in V, k \in \{1, N_j\} \quad (20)$$

3.1.1. Weak Formulation

We endow the space $H(\Omega)$ with the norm:

$$\|u\|_H^2 = \|u\|_{H^1_{\Gamma_D}(\Omega)}^2 + \sum_{j\in V} \|\gamma_j u\|_{H_0^1(\Gamma_j)}^2 \quad (21)$$

Where $\|.\|_{H^1_{\Gamma_D}}$ denotes the norm induced by $H^1(\Omega)$ on the space of functions vanishing on Γ_D. Then we have the following result:

Lemma 1.

Given $f \in L^2(\Omega)$, $g_j \in W_2^{1/2}(\Gamma_j)\ j \in V$, *there exists a unique solution* $u \in H$ *of the following variational formulation*:

$$\int_\Omega \nabla u \nabla v + \sum_{j\in V} \int_{\Gamma_j} \nabla_{\Gamma_j} u \nabla_{\Gamma_j} v = \int_\Omega fv + \sum_{j\in V} \int_{\Gamma_j} g_j v \quad \forall v \in H \quad (22)$$

Proof of lemma 1.

The proof of Lemma 1 is conventional: the bilinear form associated with the formula (22) is coercive and continuous; it then suffices to apply the Lax-Milgram theorem.

3.1.2. Interpretation

The solution u of (22) clearly satisfies the equation (17), and as $f \in L^2(\Omega)$ then $-\Delta u \in L^2(\Omega)$.

Now as u is in $H^1(\Omega)$, this then allows us to give sense of the normal derivative:

$$\left(\frac{\partial u}{\partial n}\right)_{\Gamma_j} \in \left(\mathcal{H}^{1/2}(\Gamma_j)\right)' \quad \forall j \in V \quad (23)$$

Where $\left(\mathcal{H}^{1/2}(\Gamma_j)\right)'$ is the dual space to $\mathcal{H}^{1/2}(\Gamma_j)$ defined by:

$$\mathcal{H}^{1/2}(\Gamma_j) = \left\{ \begin{array}{c} v, \exists\tilde{v} \in H^{1/2}(\mathbb{R}^n), \\ \tilde{v}|_{\Gamma_j} = v \in H^{1/2}(\Gamma_j), \tilde{v} = 0 \text{ otherwise} \end{array} \right\} \quad (24)$$

However, for any $v \in D(\Gamma_j)$, we know (cf. [29]) that there is an extension $w \in H(\Omega)$ satisfying:

$$\gamma_j w = v \text{ in } \Gamma_j ; \gamma_k w = 0 \text{ in } \Gamma_k \ \forall k \neq j \quad (25)$$

We deduce, then, that the solution u of the variational formulation (22) satisfies the equation of surface waves (19) in the sense of $H^{-1}(\Gamma_j)$ for all $j \in V$.

3.1.3. Regularity H^s

We notes $\varpi_j, j \in V \cup D$ the angles between the edges of the boundary. We have then the following regularity result of the static solution:

Lemma 2.

Let $s_0 = 1 + \frac{\pi}{\varpi_{max}}$ *, where* $\varpi_{max} = \max_{j \in V \cup D} (\varpi_j)$*, then*

$$u \in H^s(\Omega) \quad \forall s < s_0 \tag{26}$$

Proof of Lemma 2.

Since $\left(\frac{\partial u}{\partial n}\right)_{\Gamma_j} \in \left(\mathcal{H}^{1/2}(\Gamma_j)\right)' \forall j \in V$, the equation (19) then asserts that $\Delta_{\Gamma_i}(\gamma_i u) \in \left(\mathcal{H}^{1/2}(\Gamma_j)\right)' \forall i \in V$.

On the other hand we know that $\gamma_i u \in H_0^1(\Gamma_i)$, we deduce by interpolation that:

$$\gamma_i u \in H_0^1(\Gamma_i) \cap H^{3/2}(\Gamma_i) \ \forall i \in V.$$

Furthermore, we know that $\gamma_i u = 0$ in Γ_i for all $\in D$. We therefore deduce that:

$$\gamma_i u \in H_0^1(\Gamma_i) \cap H^{3/2}(\Gamma_i) \ \forall i \in V \cup D.$$

We are then led to solve an equation of Laplace with Dirichlet data:

$$-\Delta u \in L^2(\Omega) \tag{27}$$

$$u \in H_0^1(\Gamma_i) \cap H^{3/2}(\Gamma_i) \ \forall i \in V \cup D \tag{28}$$

The compatibility conditions being satisfied:

$$u_{\Gamma_j}(S_{j,k}) = u_{\Gamma_{j+1}}(S_{j+1,k-1});\ j \in V \cup D,\ k \in \{1, N_j\}$$

it is, then, sufficient to apply the Grisvard's result [29] to complete the proof of Lemma 2. For reading paper be self-sufficient we will give an outline of the demonstration:

It is well known that $u \in H^2 \cap U^c$ for every closed neighborhood U of the set of vertices $S_{j,i}$ such as $j \in V$ and $S_{j,i} \in \Gamma_j \cap \Gamma_{j+1}, (j, j+1) \in V \times D$, where $(j, j+1) \in D \times V$.

If it locates, then around one of these peaks is considered and the polar coordinates $(r, \theta_{j,i})$ centered at $S_{j,i}$ and such that the boundary Γ_{j+1} match $\theta_{j,i} = 0$ and the boundary Γ_j with $\theta_{j,i} = \varpi_j$.

Singular solutions of the problem then are writing

$$S_{k,j}(r,\theta) = r^{k\pi/\varpi_j} \sin\left(k\pi/\varpi_j\, \theta\right) \tag{29}$$

The conclusion of the lemma is immediate.

Proposition 1.

For any non-cracked domain (i.e. $\varpi_j < 2\pi$ *), the solution u is in fact in space* $H^{3/2+\varepsilon}(\Omega)$ *and checks the system (17)-(20) in a classical sense since* $\frac{\partial u}{\partial n} \in H^{\varepsilon}(\Gamma_j) \subset L^2(\Gamma_j)$*, and then*:

$$D(A) = \left\{ \begin{array}{l} u \in H^{s_0}(\Omega);\ \Delta u \in L^2(\Omega);\ u_{|\Gamma_D} = 0; \\ u_{|\Gamma_i} \in H_0^1(\Gamma_i);\ -\Delta_{\Gamma_i} + \frac{\partial u}{\partial n} \in L^2(\Gamma_i),\ i \in V \end{array} \right\} \tag{30}$$

Where $s_0 = 1 + \frac{\pi}{\varpi_{max}}$.

3.2. Step 2. "End of the Proof of Theorem 2"

The operator A is self-adjoint and positive, we can then introduce a complete system in $L^2(\Omega)$ formed by eigenfunctions of A i.e.:

$$-\Delta u_k = \lambda_k u_k \ in\ \Omega \tag{31}$$

$$\gamma_i u_k = 0 \ in\ \Gamma_i;\ i \in D \tag{32}$$

$$-\Delta_{\Gamma_i}(\gamma_i u_k) = -\frac{\partial u_k}{\partial n} \ in\ \Gamma_i;\ i \in V \tag{33}$$

$$\gamma_j \varphi(S_{j,k}) = 0;\ j \in V,\ k \in \{1, N_j\} \tag{34}$$

The homogeneous solution of the wave transmission system (9)-(14) can be written on this basis as follows:

$$\varphi(x,t) = \sum_{k=1}^{\infty} (\cos(t\sqrt{\lambda_k})\langle\varphi^0, u_k\rangle + 1/\sqrt{\lambda_k} \sin(t\sqrt{\lambda_k})\langle\varphi^1, u_k\rangle) u_k \tag{35}$$

Where the functions $(u_k)_k$ satisfy:

$$\int_\Omega \nabla u_k \nabla u_l + \int_{\Gamma_V} \nabla_\Gamma u_k \nabla_\Gamma u_l = \delta_k^l \lambda_k \left(\int_\Omega u_k u_l + \int_{\Gamma_V} u_k u_l \right) \tag{36}$$

The symbol δ_k^l denotes the classic Kronecker symbol and hook $\langle .,. \rangle$ denotes the scalar product in L^2.

The operator A is injective and extends in a linear operator of H to H'. This provides continuous and linear dependence on the Cauchy data:

$$\left(\varphi, \varphi_{|\Gamma_V}\right)_{C\left(0,T;D\left(A^m\right)\right)} + \left(\varphi, \varphi_{|\Gamma_V}\right)_{C\left(0,T;D\left(A^{m-1/2}\right)\right)} \le C\left(\left(\varphi^0, \rho^0_{\Gamma_V}\right)_{D\left(A^m\right)} + \left(\varphi^1, \rho^1_{\Gamma_V}\right)_{D\left(A^{m-1/2}\right)} \right) \tag{37}$$

For any $m \ge 1/2$.

Moreover we know that $D\left(A^{1/2}\right) = H(\Omega)$, the result of Theorem 2 is immediate, indeed, it suffices to consider the particular case $m = 1$.

4. Observability Problem of Homogeneous Transmission

4.1. Identity

Let $\left(\varphi,\varphi_{|\Gamma_V}\right)\in D(A)$, and let $q=(q_k)_{1\le k\le n}$ be a sufficiently smooth vector field, we then have the following equality:

$$
\begin{aligned}
-\int_\Omega \varphi(q.\nabla\varphi) &= \sum_{k,j}\int_Q \frac{\partial\varphi}{\partial x_j}\frac{\partial q_k}{\partial x_j}\frac{\partial\varphi}{\partial x_k} \\
&-\tfrac{1}{2}\int_Q div(q)|\nabla\varphi|^2-\int_\Sigma \frac{\partial\varphi}{\partial n}q_k\frac{\partial\varphi}{\partial x_k} \\
&+\tfrac{1}{2}\int_\Sigma\left|\frac{\partial\varphi}{\partial n}\right|^2 q_k n_k+\tfrac{1}{2}\int_\Sigma q_k n_k|\nabla_\Gamma\varphi|^2
\end{aligned}
\quad (38)
$$

But it is possible to define on Γ and almost everywhere $n-1$ vector fields $\left(\tau^k\right)_{1\le k\le n-1}$ constants on the portions of the boundary, and such that $\left\{n(x),\tau^1(x),\dots,\tau^{n-1}(x)\right\}$ realizes an orthonormal basis, outside vertices. We, then, have for any sufficiently smooth function u:

$$\left(\frac{\partial u}{\partial x_j}\right)_{|\partial\Omega}=\frac{\partial u}{\partial n}+\sum_{1\le l\le n-1}\tau_j^l\frac{\partial u}{\partial\tau^l};\ j=1,\dots,n \quad (39)$$

Where τ_j^l is the j^{th} component of τ^l.

If we note:

$$\sigma_j u=\sum_{1\le l\le n-1}\tau_j^l\frac{\partial u}{\partial\tau^l};\ j=1,\dots,n \quad (40)$$

Then $\sigma_j u$ is the $\sigma_j u$ component of $\nabla_\Gamma u$.

Let $\sigma_j^*:L^2(\Gamma)\to\left(H^1(\Gamma_i)\right)'$ be the adjoint Operator of $\sigma_j:H^1(\Gamma)\to L^2(\Gamma)$ and such that:

$$-\Delta_{\Gamma_i}u=\sum_{1\le j\le n}\sigma_j^*\sigma_j u \quad (41)$$

We, then, consider the surface integral $\int_\Sigma\frac{\partial\varphi}{\partial n}q_k\frac{\partial\varphi}{\partial x_k}$:

$$-\int_\Sigma\frac{\partial\varphi}{\partial n}q_k\frac{\partial\varphi}{\partial x_k}=-\int_\Sigma\left|\frac{\partial\varphi}{\partial n}\right|^2 q_k n_k-\int_\Sigma\frac{\partial\varphi}{\partial n}q_k\sigma_k(\varphi) \quad (42)$$

However, in $\Sigma_i, i\in D$, the tangential gradient is identically zero, the second term in the right member of the equation (42) becomes:

$$-\int_\Sigma\frac{\partial\varphi}{\partial n}q_k\sigma_k(\varphi)=\int_{\Sigma_V}\partial_{t^2}\varphi q_k\sigma_k(\varphi)-\int_{\Sigma_V}\Delta_\Gamma\varphi q_k\sigma_k(\varphi) \quad (43)$$

At this level we will do a little more attention and distinguish the case of dimension two and the case of the higher dimension $(n\ge 3)$.

4.1.1. Case of Dimension Two

$$
\begin{aligned}
&\int_{\Sigma_V}\partial_{t^2}\varphi q_k\sigma_k(\varphi)= \\
&\quad-\int_{\Sigma_V}\partial_t\varphi q_k\sigma_k(\partial_t\varphi)+\left[\sum_i\partial_t\varphi|q_k\sigma_k(\varphi)_{\Gamma_i}\right]_0^T
\end{aligned}
\quad (44)
$$

However we have: $\sigma_1(u)=\frac{\partial u}{\partial\tau^1}\tau_1^1$ and $\sigma_2(u)=\frac{\partial u}{\partial\tau^1}\tau_2^1$.

Apply Green's formula on each $\Gamma_i; i\in V$ we obtain the following identity:

$$
\begin{aligned}
-\int_{\Sigma_V}\partial_t\varphi q_k\sigma_k(\partial_t\varphi)&=\tfrac{1}{2}\int_{\Sigma_V}|\partial_t\varphi|^2\,div_\Gamma(q) \\
&-\tfrac{1}{2}\int_0^T\sum_{i\in V}\left[\left(|\partial_t\varphi|^2\,q.\tau\right)\right]_{S_{i,0}}^{S_{i,1}}
\end{aligned}
\quad (45)
$$

where we noted

$$div_\Gamma(u)=\frac{\partial u_1}{\partial\tau^1}\tau_1^1+\frac{\partial u_2}{\partial\tau^1}\tau_1^2=\sum_{1\le i\le 2}\sigma_i(u_i) \quad (46)$$

Regarding the second integral in (43) we have:

$$-\int_{\Sigma_V}\Delta_\Gamma\varphi q_k\sigma_k(\varphi)=-\int_{\Sigma_V}\frac{\partial^2\varphi}{\partial\left(\tau^1\right)^2}\left(q_1\tau_1^1+q_2\tau_1^2\right)\frac{\partial\varphi}{\partial\tau^1} \quad (47)$$

We apply again Green's formula and we obtain:

$$
\begin{aligned}
-\int_{\Sigma_V}\Delta_\Gamma\varphi q_k\sigma_k(\varphi)&=\tfrac{1}{2}\int_{\Sigma_V}|\nabla_\Gamma\varphi|^2\,div_\Gamma(q) \\
&-\tfrac{1}{2}\int_0^T\sum_{i\in V}\left[\left(|\nabla_\Gamma\varphi|^2\,q.\tau\right)\right]_{S_{i,0}}^{S_{i,1}}
\end{aligned}
\quad (48)
$$

This completes the proof in the two-dimensional case.

4.1.2. Case of Dimension 3

In this case the identity (45) remains valid, provided consider:

$$div_\Gamma(u)=\sum_{1\le i\le 3}\sigma_i(u_i) \quad (49)$$

Concerning the second integral, we have:

$$
\begin{aligned}
\int_{\Sigma_V}\sigma_l^*\sigma_l(\varphi)\ q_k(\varphi)&=\int_{\Sigma_V}\sigma_l\sigma_l\left(q_k\sigma_k(\varphi)\right) \\
&-\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\sigma_l(\varphi)n_l^i\sigma_k(\varphi)
\end{aligned}
\quad (50)
$$

Where we noted n_l^i the outer normal to the face Γ_i considered as a variety in $\mathbb{R}^2$.

Now we know that φ is zero on $\partial\Gamma_i$, so the last term in the identity can be rewritten as follows:

$$-\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\sigma_1(\varphi)\nu_l^i\sigma_k(\varphi)=$$
$$-\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\left(\frac{\partial\varphi}{\partial\nu^i}\right)^2\nu_k^i q_k\sigma_k(\varphi) \quad (51)$$

Where we extended the normal of Γ_i into a vector in $\mathbb{R}^3$.

On the other hand we have:

$$\int_{\Sigma_V}\sigma_1\sigma_1\left(q_k\sigma_k(\varphi)\right)=$$
$$\int_{\Sigma_V}\sigma_1\sigma_1(q_k)\sigma_k(\varphi)-\tfrac{1}{2}\int_{\Sigma_V}\left|\nabla_\Gamma\varphi\right|^2\operatorname{div}_\Gamma(q) \quad (52)$$
$$+\tfrac{1}{2}\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\left(\frac{\partial\varphi}{\partial\nu^i}\right)^2\nu_k^i q_k$$

Let us, now, consider and develop the following volume integral:

$$-\int_Q\partial_t\varphi q.\nabla(\partial_t\varphi)=\tfrac{1}{2}\int_Q\left|\partial_t\varphi\right|^2\operatorname{div}(q)-\tfrac{1}{2}\int_\Sigma\left|\partial_t\varphi\right|^2 q.n$$
(53)

finally we obtain the identity:

$$\sum_{k,j}\int_Q\frac{\partial\varphi}{\partial x_k}-\tfrac{1}{2}\int_Q div(q)\left(\left|\nabla\varphi\right|^2+\left|\partial_t\varphi\right|^2\right).$$
$$+\left\langle\partial_t\varphi|.\nabla(\varphi)\right\rangle_\Omega\Big]_0^T+\tfrac{1}{2}\int_{\Sigma_V}div_\Gamma(q)\left(\left|\partial_t\varphi\right|^2-\left|\nabla_\Gamma\varphi\right|^2\right)$$
$$\tfrac{1}{2}\int_\Sigma\left|\frac{\partial\varphi}{\partial n}\right|^2 q_k n_k+\tfrac{1}{2}\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\left(\frac{\partial\varphi}{\partial\nu^i}\right)^2 q.\nu^i$$
$$\tfrac{1}{2}\int_{\Sigma_V}\left(\left|\partial_t\varphi\right|^2-\left|\nabla_\Gamma\varphi\right|^2\right)q.n$$
(54)

Remark 1.

The identity (53) is, a priori, established only for sufficiently smooth solutions, it then extends weak solutions: $(\varphi,\gamma_i\varphi)$ *in* $C(0,T;H)\cap C^1\left(0,T;L^2(\Omega)\times\prod_{i\in V}L^2(\Gamma_i)\right)$ *taking into account the following remark: It is known (cf. (36)) that* φ *is continuously and linearly dependent on the Cauchy data* (φ^0,ρ_i^0) *and* (φ^1,ρ_i^1)*, then it follows that there is a sequence of strong solutions* $((\varphi_n,\gamma_i\varphi_n))_n$ *in*

$$C(0,T;D(A))\cap C^1(0,T;H)\cap C^1\left(0,T;L^2(\Omega)\times\prod_{i\in V}L^2(\Gamma_i)\right)$$

which converges to $(\varphi,\gamma_i\varphi)$ *in*

$$C(0,T;H)\cap C^1\left(0,T;L^2(\Omega)\times\prod_{i\in V}L^2(\Gamma_i)\right).$$

4.2. Boundary Observability

We will show in this section the inequality so called “Observability Inequality” or “Reverse Inequality” which is the central estimate for the implementation of the *Hilbert Uniqueness Method (HUM)*. To do this we need some definitions.

Definitions.

Selecting a current point x_0 of the space and $q(x)=m(x)=x-x_0$. We note Σ^+ the part of the boundary called the shadow area respect to the source x_0 i.e.

$$\Sigma^+=\left\{x\in\partial\Omega;\,m(x).\,n(x)>0\right\} \quad (55)$$

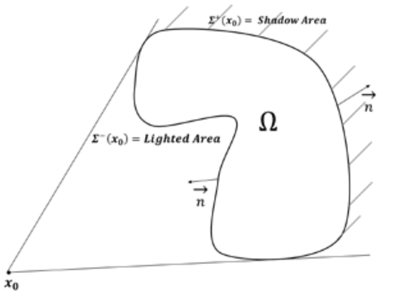

Figure 3. Illustration of Lighted Area and Shadow Area of a smooth Domain

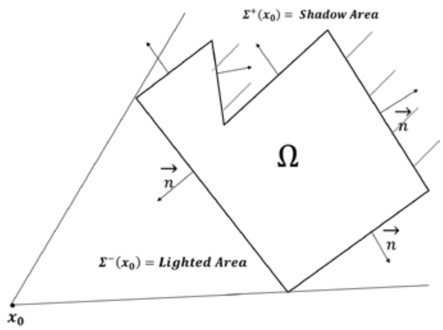

Figure 4. Illustration of Lighted Area and Shadow Area of a Polygonal Domain

We note F the space of the initial data such that the solution φ of the system (9)-(14) satisfies:

$$\left\|\left\{(\varphi^0,\rho_i^0),(\varphi^1,\rho_i^1)\right\}\right\|_F^2=$$
$$\int_{\Sigma_{D,V}^+}\left|\frac{\partial\varphi}{\partial n}\right|^2+\int_{\partial\Sigma_V^+}\left|\frac{\partial\varphi}{\partial\nu}\right|^2+\int_{\Sigma_V}\left(\left|\partial_t\varphi\right|^2+\left|\nabla_\Gamma\varphi\right|^2\right) \quad (56)$$
$$<+\infty$$

We will show that $\|\ \|_F$ defines a norm.

Lemma 3. "Observability"

For any solution φ *of the system (9)-(14) associated with the initial data in F, the quantity* $\|\ \|_F$ *defines a norm. Moreover, we have the following Observability Inequality*:

$$\left(T-T(x_0)\right)E(0\le C\left\|\left\{(\varphi^0,\rho_i^0),(\varphi^1,\rho_i^1)\right\}\right\|_F^2 \quad (57)$$

Where C is a positive constant depending only on the geometry.

Proof of Lemma 3.

We will use the identity (54) with a particular choice of the vector field, ie we will replace $q(x)$ by $m(x)$, where $m(x)=\overrightarrow{x-x_0}$, identity then becomes:

$$\int_Q\left(|\nabla\varphi|^2\left(\frac{2-n}{2}\right)+{}^{n}\!/_{2}|\partial_t\varphi|^2\right)+$$
$$\int_{\Sigma_V}\left(|\nabla_\Gamma\varphi|^2\left(\frac{2-n}{2}\right)+\left(\frac{n-1}{2}\right)|\partial_t\varphi|^2\right)+$$
$$\left.\left\langle\partial_t\varphi|m.\nabla(\varphi)\right\rangle_\Omega\right]_0^T+\left.\sum_{i\in V}\left\langle\partial_t\varphi|m.\sigma(\varphi)\right\rangle_{\Gamma_i}\right]_0^T= \quad (58)$$
$$^1\!/_2\int_\Sigma\left|\frac{\partial\varphi}{\partial n}\right|^2\mathrm{m.n}+{}^1\!/_2\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\left(\frac{\partial\varphi}{\partial v^i}\right)^2 m.v^i+$$
$$^1\!/_2\int_{\Sigma_V}\left(|\partial_t\varphi|^2-|\nabla_\Gamma\varphi|^2\right)\mathrm{m.n}$$

On the other hand we know that:

$$\int_Q\left(|\partial_t\varphi|^2-|\nabla\varphi|^2\right)+\int_{\Sigma_V}\left(|\partial_t\varphi|^2-|\nabla_\Gamma\varphi|^2\right)= \quad (59)$$
$$\left.\langle\partial_t\varphi|\varphi\rangle_\Omega\right]_0^T+\left.\langle\partial_t\varphi|\varphi\rangle_{\Gamma_V}\right]_0^T$$

Then we put (59) in (58) and obtain:

$$TE(0)+{}^1\!/_2\int_{\Sigma_V}|\nabla_\Gamma\varphi|^2+\left.\left\langle\partial_t\varphi|m.\nabla\varphi+\frac{n-1}{2}\varphi\right\rangle_\Omega\right]_0^T+$$
$$\left.\left\langle\partial_t\varphi|m.\sigma(\varphi)+\frac{n-1}{2}\varphi\right\rangle_{\Gamma_V}\right]_0^T=$$
$$^1\!/_2\int_{\Sigma_V}|\partial_t\varphi|^2+\left(|\partial_t\varphi|^2-|\nabla_\Gamma\varphi|^2\right)\mathrm{m.n}$$
$$+{}^1\!/_2\int_\Sigma\left|\frac{\partial\varphi}{\partial n}\right|^2\mathrm{m.n}+{}^1\!/_2\sum_{i\in V}\int_0^T\int_{\partial\Gamma_i}\left(\frac{\partial\varphi}{\partial v^i}\right)^2 m.v^i$$
(60)

If we note

$$\left.\left\langle\partial_t\varphi|m.\nabla\varphi+\frac{n-1}{2}\varphi\right\rangle_\Omega\right]_0^T+\left.\left\langle\partial_t\varphi|m.\sigma(\varphi)+\frac{n-1}{2}\varphi\right\rangle_{\Gamma_V}\right]_0^T$$

by Y, we then have the following increases:

$$|\mathrm{Y}|\le$$
$$\alpha\left\{\int_\Omega|\partial_t\varphi|^2+\int_{\Gamma_V}|\partial_t\varphi|^2\right\}+$$
$$\frac{1}{4\alpha}\left\{\int_\Omega\left(m.\nabla\varphi+\frac{n-1}{2}\varphi\right)^2+\int_{\Gamma_V}\left(m.\sigma(\varphi)+\frac{n-1}{2}\varphi\right)^2\right\}$$
(61)

with $R(x_0)=\frac{T(x_0)}{2}=\max_\Omega\max_\Omega|m(x)|$, and we have:

$$\int_\Omega\left|m.\nabla\varphi+\frac{n-1}{2}\varphi\right|^2+\int_{\Gamma_V}\left|m.\sigma(\varphi)+\frac{n-1}{2}\varphi\right|^2\le$$
$$R^2(x_0)\left(\int_\Omega|\nabla\varphi|^2+\int_{\Gamma_V}|\nabla_\Gamma\varphi|^2\right)+$$
$$+R^2(x_0)\left(\frac{n-1}{2}\right)^2\left(\int_\Omega|\varphi|^2+\int_{\Gamma_V}|\varphi|^2\right) \quad (62)$$
$$+R^2(x_0)(n-1)\left(\int_\Omega\varphi\, m\nabla\varphi+\int_{\Gamma_V}\varphi\, m\quad(\varphi)\right)$$

Now observe that:

$$(n-1)\left(\int_\Omega\varphi\, m.\nabla\varphi+\int_{\Gamma_V}\varphi\, m.\sigma(\varphi)\right)=$$
$$-\frac{(n-1)}{2}\left(\int_\Omega n|\varphi|^2+\int_{\Gamma_V}(n-1)|\varphi|^2\right)+\frac{n-1}{2}\int_{\Gamma_V}m.n|\varphi|^2 \quad (63)$$

And by choosing $\alpha=\frac{R(x_0)}{2}$ we obtain:

$$|\mathrm{Y}|\le R(x_0)E(0)+\frac{n-1}{4R(x_0)}\int_{\Gamma_V}m.n|\varphi|^2 \quad (64)$$

Where:

$$\left|\left.\left\langle\partial_t\varphi|m.\nabla\varphi+\frac{n-1}{2}\right\rangle\varphi_\Omega\right]_0^T+\right.$$
$$\left.\left.\partial_t\varphi|m.\sigma(\varphi)+\frac{n-1}{2}\varphi_{\Gamma_V}\right]_0^T\right|\le T(x_0)E(0)+ \quad (65)$$
$$\frac{n-1}{4R(x_0)}\left(\int_{\Gamma_V}m.n|\varphi(0)|^2+\int_{\Gamma_V}m.n|\varphi(T)|^2\right)$$

The second term on the right in the above inequality is bounded by $C\int_{\Sigma_{V(x_0)}}|\partial_t\varphi|^2$, which gives the existence of a constant $C>0$ and a time $T(x_0)$ such that:

$$(T-T(x_0))E(0)\le$$
$$C\left\{\int_{\Sigma^+}\left|\frac{\partial\varphi}{\partial n}\right|^2+\int_{\Sigma_V(x_0)}|\partial_t\varphi|^2+|\nabla_\Gamma\varphi|^2\right. \quad (66)$$
$$\left.+\int_0^T\int_{\partial\Sigma_V^+(x_0)}\left|\frac{\partial\varphi}{\partial v}\right|^2\right\}$$

It is now easy to see that $\|.\|_F$ clearly defines a norm finer than the energy norm.

In addition we have the following frame:

$$D(A)\times H \subset F \subset H\times L^2(\Omega)\times \prod_{i\in V} L^2(\Gamma_i) \qquad (67)$$

Otherwise if we consider the solutions θ of the following system:

$$\partial_{t^2}\theta - \Delta\theta = \text{f } in\ Q \qquad (68)$$

$$\gamma_j\theta = 0\ in\ \Sigma_j;\ j\in D \qquad (69)$$

$$\partial_{t^2}(\gamma_i\theta) - \Delta_{\Gamma_i}(\gamma_i\theta) = -\frac{\partial\theta}{\partial \text{n}} + \text{g } in\ \Sigma_i;\ i\in V \qquad (70)$$

$$\gamma_j\theta(S_{i,j}) = 0\ in\ (0,T);\ i\in V,\ j\in\{1,N_i\} \qquad (71)$$

$$\theta(x,0) = \theta^0(x);\ \partial_t\theta(x,0) = \theta^1(x)\ in\ \Omega \qquad (72)$$

$$\begin{aligned}&\gamma_j\theta(x,0) = \gamma_j\theta^0(x);\\ &\partial_t(\gamma_j\theta)(x,0) = \gamma_j\theta^1(x)\ j\in V\end{aligned} \qquad (73)$$

where (θ^0,θ^1) are supposed to be in the space *F*, and $\{f,g\}$ are in the space $L^1(0,T;H)$. Then θ satisfies the following proprieties:

$$\begin{aligned}&\frac{\partial\theta}{\partial n}\in L^2\left(\Sigma^+(\text{x}_0)\right);\ \theta\in H_0^1(\Sigma_\text{V})\\ &and\ \frac{\partial\theta}{\partial\nu^i}\in L^2\left((0,\text{T})\times\partial\Gamma_\text{i}^+\right)\end{aligned} \qquad (74)$$

Indeed, it suffices to write the solution $\theta=\theta_1+\theta_2$, where θ_1 corresponds to the homogeneous solution of the Cauchy data, and θ_2 in the second member.

Inclusions above are true for θ_1 by definition of space *F*. On the other hand we have:

$$\theta_2\in C(0,T;D(A))\cap C^1(0,T;H) \qquad (75)$$

This then implies that:

$$\frac{\partial\theta_2}{\partial n}\in C\left(0,T;H^{s-3/2}(\Gamma)\right)\subset L^2(\Sigma) \qquad (76)$$

$$\partial_t\theta_2\in C\left(0,T;H^{s-3/2}(\Gamma)\right)\subset L^2(\Sigma) \qquad (77)$$

$$\nabla_\Gamma\theta_2\in\left(C\left(0,T;H^{s-3/2}(\Gamma)\right)\right)^{n-1}\subset\left(L^2(\Sigma)\right)^{n-1} \qquad (78)$$

$$\frac{\partial\theta_2}{\partial\nu^i}\in L^2\left((0,\text{T})\times\partial\Gamma_\text{i}^+\right) \qquad (79)$$

Which completes the proof of the announced result.

5. Transposition Method

We begin this section by the second main ingredient for the implementation of *Hilbert Uniqueness Method* and which is based on the results previously shown:

Lemma 4.

For all (u_0,u_1) in $L^2(\Omega)\times H'$, v_D in $L^2(\Sigma_D^+)$, v_V in $L^2(\Sigma_V)$ and v_s in $L^2(\partial\Sigma_V^+)$, there is a unique $u\in L^\infty(0,T;H')$ and $(\psi_1,-\psi_0)\in F'$ solutions of:

$$\begin{aligned}&\int_Q \text{uf} + \int_{\Sigma_V}\text{ug} +\\ &\langle\psi_1,\theta_0\rangle_\Omega + \sum_i\langle\overline{\psi_1},\rho_i^0\rangle_{\Gamma_i} - \langle\psi_0,\theta\rangle_{1\Omega} + \sum_i\langle\overline{\psi_0},\rho_i^1\rangle_{\Gamma_i} =\\ &\langle u_1,\theta_0\rangle_\Omega + \sum_i\langle q_1,\rho_i^0\rangle_{\Gamma_i} - \langle u_0,\theta_1\rangle_\Omega + \sum_i\langle q_0,\rho_i^1\rangle_{\Gamma_i} +\\ &-\int_{\Sigma_D^+} v_D\frac{\partial\theta}{\partial n} + \int_{\Sigma_V} v_D\theta - \int_{\partial\Sigma_V^+} v_s\frac{\partial\theta}{\partial\nu}\end{aligned} \qquad (80)$$

For all $(f,g)\in L^1(0,T;H)$ and $\{(\theta^0,\theta^1),(\rho_i^0,\rho_i^1)\}\in F$.

Proof of Lemma 4.

Based on the foregoing, the right side in the identity (80) defines a continuous linear form on $\text{L}^1(0,\text{T};\text{H})$, which establishes the existence and uniqueness of the solution u in $\text{L}^\infty(0,\text{T};\text{H}')$.

We are now able to prove our main result:

The End of the Proof of Theorem 1.

First we solve the homogeneous problem with Cauchy data in *F*, there $(\varphi,\gamma_j\varphi)\in C(0,T;H)\cap C^1\left(0,T;L^2(\Omega)\times\prod_{i\in V}L^2(\Gamma_i)\right)$ solution of the system (9)-(14). Then we know that there $(\varphi,\gamma_j\varphi)\in C(0,T;H')$ solution of:

$$\begin{aligned}&\int_Q \text{uf} + \int_{\Sigma_V}\text{ug} = -\int_{\Sigma_D^+}\frac{\partial\varphi}{\partial n}\frac{\partial\theta}{\partial n}\\ &-\int_{\Sigma_V}\partial_t\varphi\partial_t\theta - \int_{\Sigma_V}\nabla_\Gamma\varphi\nabla_\Gamma\theta - \int_{\partial\Sigma_V^+}\frac{\partial\varphi}{\partial\nu}\frac{\partial\theta}{\partial\nu}\end{aligned} \qquad (81)$$

For all $(f,g)\in L^1(0,T;H)$ and θ solution of the system (68)-(73).

We define the operator Λ as follows:

$$\begin{aligned}&\Lambda\left\{(\varphi_0,\rho_i^0),(\varphi_1,\rho_i^1)\right\} =\\ &\left\{(\psi(0),\gamma_i\psi(0)),(-\partial_t\psi(0),-\gamma_i\partial_t\psi(0))\right\}\end{aligned} \qquad (82)$$

It is a continuous linear operator from *F* in *F’* which verifies more:

$$\left\langle \Lambda\left\{\left(\varphi_0,\rho_i^0\right),\left(\varphi_1,\rho_i^1\right)\right\},\left\{\left(\varphi_0,\rho_i^0\right),\left(\varphi_1,\rho_i^1\right)\right\}\right\rangle = \left\|\left\{\left(\varphi_0,\rho_i^0\right),\left(\varphi_1,\rho_i^1\right)\right\}\right\|_F^2 \quad (83)$$

Λ is an isomorphism of F in F'.

Given $\left\{\left(u_1,q_i^1\right),\left(-u_0,-q_i^0\right)\right\}\in F'$, then there is $\left\{\left(\varphi_0,\rho_i^0\right),\left(\varphi_1,\rho_i^1\right)\right\}\in F$ such as:

$$\Lambda\left\{\left(\varphi_0,\rho_i^0\right),\left(\varphi_1,\rho_i^1\right)\right\}=\left\{\left(u_1,q_i^1\right),\left(-u_0,-q_i^0\right)\right\} \quad (84)$$

We conclude, then, posing:

$$\gamma_i u=\frac{\partial\varphi}{\partial n}\ in\ \Sigma_D^+;\ i\in D$$

$$\partial_{t^2}u-\Delta_{\Gamma_i}u=-\frac{\partial u}{\partial n}-\partial_{t^2}\varphi-\Delta_{\Gamma_i}\varphi\ \text{sur}\ \Sigma_V^+;\ i\in V$$

$$u\left(S_{i,j}\right)=\frac{\partial\varphi}{\partial v^i}\ i\in V,\ j\in\{1,N_i\}.$$

6. Conclusion

In this article, we established an observation and exact controllability result of a wave boundary problem with Wentzell condition in the presence of geometric singularity of polygonal or polyhedral types. We can notice that the extension of the results in case of smooth domains does not cause any difficulty [9]. However, it would be interesting to examine the problem using microlocal analysis, and eventually find a sharp and sufficient conditions on the control region and time control. Otherwise as we noted in [9] the case where Wentzell waves propagate with a velocity smaller than wave velocity in the interior (i.e. $\beta < 1$) is more interesting. Indeed the bicharacteristics solutions of the Hamiltonian flow on the boundary are different from those on the inside, and the habitual geometric control condition is not sufficient in this case to observe and control all the energy of waves of systems. Another interesting problem would be to examine the convergence of Wentzell control system to the Dirichlet control system (respectively Neumann) when the parameter α tends to infinity (respectively to zero) and to extend the results obtained [30] in the case of Robin-Fourier control.

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