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# On a New Formulas for a Direct and Inverse Cauchy Problems of Heat Equation

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**Abstract** In this paper a solution of the direct Cauchy problems for heat equation is founded in the form of Hermite polynomial series. A well-known classical solution of direct Cauchy problem is represented as Poisson's integral. The author reveals, the formulas obtained by him for solution of the inverse Cauchy problems have a symmetry with respect to the formulas for corresponding direct Cauchy problems. Obtained formulas for solution of the inverse problems can serve as a basis for reg-ularizing computational algorithms while well-known classical formula for the solution of the inverse Cauchy problem can't be a basis for regu-larizing computational algorithms.

***Keywords:*** *heat equation, direct/inverse Cauchy problem, well-posed/ill-posed problem, hermite polynomials, poisson integral.*

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## 1. Introduction

In this paper a direct and an inverse Cauchy problems for the heat equation are solved at Cartesian and polar coordinates. The inverse Cauchy problem for the heat equation consists of to reconstruct a priori unknown initial condition of the dynamic system from its known final condition. In 1939 French mathematician Jacques Hadamard defined the problem is called well-posed if a solution exists, the solution is unique, the solution's behavior hardly changes when there's a slight change in the initial condition. The problems are called ill-posed or not well-posed if at least one of these three conditions is not fulfilled. The most often, the third condition so called the stability condition of solution is violated for ill-posed problems. In this case there is a paradoxical situation: the problem is mathematically defined but solution cannot be obtained by conventional methods. In mathematics the vast majority of inverse problems is not well-posed: small perturbations of the initial data (observations) may correspond to an arbitrarily large perturbations of the solution. A classic example of ill-posed problem is the inverse Cauchy problem (retrospective problem) for the heat equation. The direct Cauchy problem for the heat equation is well-posed most frequently. In this work the solution for direct Cauchy problems found in the form of the Hermite polynomial series. A wellknown classical solution for the direct Cauchy problem is represented in the form of Poisson's integral. We noticed the solution of inverse Cauchy problem possess the symmetry with respect to solution of the direct Cauchy problem. Formulas for the inverse problem obtained in this paper can serve as a basis for regularizing computational algorithm. Previously known classical formula for the solution of the inverse problem did not have such properties, that's why classical formula can't serve as a basis for convergent algorithm. The main result for the direct problems is formulas (5), (6), (9), (23), (24), (25). The main result for the inverse problems is formulas (13), (14), (15), (27), (28), (29).

## 2. The Cauchy Problem (Direct Problem) for the Heat Equation

The solution $u(\tau, x)$ of the Cauchy problem with the initial thermal field $f(x)$ for an infinite bar we will get as the Hermite polynomial series. In order to get this result, we use the well-known analytic solution $u(\tau, x)$ in the form of Fourier's integral, [1]:

$$u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2 t}e^{i\lambda x}\left(\int_{-\infty}^{\infty}e^{-i\lambda\xi}f(\xi)\,d\xi\right)d\lambda$$

where $f(x)$ - the initial thermal field, $u(\tau, x)$ - thermal field in time $\tau$ and at the point $x$.

Write the last equality in the form

$$u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2(\tau+\beta)}e^{i\lambda x}\left(\int_{-\infty}^{\infty}e^{\lambda^2\beta}e^{-i\lambda\xi}f(\xi)\,d\xi\right)d\lambda \quad (1)$$

where $\beta > 0$.

The function $e^{\lambda^2\tau+i\lambda x}$ is a generating function for the Hermite polynomials, [1], this means that

$$e^{\lambda^2\beta-i\lambda\xi}=\sum_{j=0}^{\infty}\frac{(-i\lambda)^j\beta^{\frac{j}{2}}}{j!}H_j\left(\frac{\xi}{2\sqrt{\beta}}\right) \quad (2)$$

where

$$H_j(z) = (-1)^j e^{z^2} \frac{d^j}{dz^j}\left(e^{-z^2}\right) \tag{3}$$

the Hermite polynomials.

In accordance with (2) the formula (1) takes the form

$$u(\tau,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2(\tau+\beta)} e^{i\lambda x} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} \beta^{\frac{j}{2}} \int_{-\infty}^{\infty} H_j\left(\frac{\xi}{2\sqrt{\beta}}\right) f(\xi)\, d\xi\, d\lambda .$$

If we may change the order of integration, then compute inner integral with respect to λ. We use the Poisson's integral, [13]. We get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2(\tau+\beta)} e^{i\lambda x} d\lambda = \frac{e^{-\frac{x^2}{4(\tau+\beta)}}}{2\sqrt{\pi(\tau+\beta)}} \tag{4}$$

Differentiating both sides of ( 4 ) with respect to $x$ we obtain the value of the desired integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2(\tau+\beta)} e^{i\lambda x} d\lambda = (-1)^j \frac{d^j}{dx^j}\left[\frac{e^{-\frac{x^2}{4(\tau+\beta)}}}{2\sqrt{\pi(\tau+\beta)}}\right].$$

On the basis of Hermite polynomials (3) the last formula can be written

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2(\tau+\beta)} e^{i\lambda x} d\lambda = \frac{e^{-\frac{x^2}{4(\tau+\beta)}}}{2\sqrt{\pi(\tau+\beta)}} \frac{1}{\left(2\sqrt{\tau+\beta}\right)^j} H_j\left(\frac{x}{2\sqrt{\tau+\beta}}\right).$$

Finally, we get an analytical representation for the thermal field in time τ and at the point $x$

$$u(\tau,x) = \frac{e^{-\frac{x^2}{4(\tau+\beta)}}}{2\sqrt{\pi(\tau+\beta)}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\tau+\beta})^j} H_j\left(\frac{x}{2\sqrt{\tau+\beta}}\right) \frac{\beta^{\frac{j}{2}}}{j!} f_j \tag{5}$$

where

$$f_j = \int_{-\infty}^{\infty} H_j\left(\frac{\xi}{2\sqrt{\beta}}\right) f(\xi)\, d\xi .$$

**Remark.** By interchanging the variables x and $\xi$ in (5) we get another version of the thermal field:

$$u(\tau,x) = \frac{e^{-\frac{x^2}{4\tau}}}{2\sqrt{\pi\tau}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\tau})^j} H_j\left(\frac{x}{2\sqrt{\tau}}\right) \frac{(\tau+\beta)^{\frac{j}{2}}}{j!} f_j \tag{6}$$

where

$$f_j = \int_{-\infty}^{\infty} H_j\left(\frac{\xi}{2\sqrt{\tau+\beta}}\right) f(\xi)\, d\xi .$$

Now we get the third new formula. To do this, the formula (1) can be written in the form

$$u(\tau,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2(\tau+\beta)} \left(\int_{-\infty}^{\infty} e^{\lambda^2\beta} e^{-i\lambda(x-\xi)} f(\xi) d\xi\right) d\lambda ,$$

where $\beta > 0$.

In view of (2) we get

$$u(\tau,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2(\tau+\beta)} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} \beta^{\frac{j}{2}} \int_{-\infty}^{\infty} H_j\left(\frac{x-\xi}{2\sqrt{\beta}}\right) f(\xi)\, d\xi\, d\lambda .$$

To simplify last formula, we change the order of integration and compute inner integral with respect to λ, substitute $x = 0$ in (4). We obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2(\tau+\beta)} d\lambda = \frac{1}{\left(2\sqrt{\tau+\beta}\right)^{j+1}} H_j(0) \tag{7}$$

Taking into account the well-known formula from [1]

$$H_{2j}(0) = \frac{(-1)^j (2j)!}{2^j j!}, H_{2j+1}(0) = 0;\ j = 0,1,2,\ldots \tag{8}$$

finally we get an analytical representation of the thermal field

$$u(\tau,x) = \sum_{j=0}^{\infty} \frac{1}{\left(2\sqrt{\tau+\beta}\right)^{2j+1}} \frac{(-1)^j \beta^j}{2^j j!} f_{2j} \tag{9}$$

where

$$f_{2j} = \int_{-\infty}^{\infty} H_{2j}\left(\frac{x-\xi}{2\sqrt{\beta}}\right) f(\xi)\, d\xi .$$

## 2.1. The Inverse Cauchy Problem for the Heat Equation

The inverse problem for the heat equation of an infinite bar is to find the unknown initial distribution $f(x)$ of thermal field by the known temperature field $u(\tau, x)$ [3,4,5,7,8]. This problem leads to the solving of first - type Fredholm integral equation:

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) f(\xi)\, d\xi = u(\tau,x) \tag{10}$$

The left side of equation (10) is Poisson's integral, [15]. As it is shown in [1,2] the solution of equation (10) is:

$$f(x) = \frac{1}{\sqrt{x}} \sum_{j=0}^{\infty} \frac{u^{(j)}(0)}{(2\sqrt{\tau})^{n+1} j!} H_j\left(\frac{x}{2\sqrt{\tau}}\right) \tag{11}$$

where $H_j(z)$ - Hermite polynomials, (3).

Formula (11) contains a derivatives of an arbitrarily high order so formula (11) can't serve as a basis for the regularizing computational algorithm. Consequently, it is

actual to find the new formulas without derivatives for the solving of the equation (10).

As in section-1, we obtain three new formulas. We get the solution of equation (10) by the Fourier transform integral method from [1,9,11,12]

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\lambda^2\tau}e^{i\lambda x}\left(\int_{-\infty}^{\infty}e^{-i\lambda\xi}u(\tau,\xi)d\xi\right)d\lambda .$$

If $\beta > 0$, then the last formula takes the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}e^{\lambda^2(\tau+\beta)}e^{i\lambda x}\left(\int_{-\infty}^{\infty}e^{-i\lambda\xi}u(\tau,\xi)d\xi\right)d\lambda \tag{12}$$

Because of formula (2) we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}e^{i\lambda x}\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}(\tau+\beta)^{\frac{j}{2}}\int_{-\infty}^{\infty}H_j\left(\frac{\xi}{2\sqrt{\tau+\beta}}\right)u(\tau,\xi)d\xi d\lambda .$$

We will change the order of integration and calculate inner integral with respect to λ. The Poisson integral (4) [15] is used in this calculations. We get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}e^{i\lambda x}dx=\frac{e^{-\frac{x^2}{4\beta}}}{2\sqrt{\pi\beta}} .$$

We calculate the value of an integral by $j$ - time differentiating with respect to $x$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta}e^{i\lambda x}d\lambda=(-1)^j\frac{d^j}{dx^j}\left[\frac{e^{-\frac{x^2}{4\beta}}}{2\sqrt{\pi\beta}}\right] .$$

On the basis of (3) we can write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta}e^{i\lambda x}d\lambda=\frac{e^{-\frac{x^2}{4\beta}}}{2\sqrt{\pi\beta}}\frac{1}{(2\sqrt{\beta})^j}H_j\left(\frac{x}{2\sqrt{\beta}}\right) .$$

Finally, first new formula for the initial distribution of the thermal field takes the form

$$f(x)=\frac{e^{-\frac{x^2}{4\beta}}}{2\sqrt{\pi\beta}}\sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\beta})^j}H_j\left(\frac{x}{2\sqrt{\beta}}\right)\frac{(\tau+\beta)^{\frac{j}{2}}}{j!}u_j \tag{13}$$

where

$$u_j=\int_{-\infty}^{\infty}H_j\left(\frac{\xi}{2\sqrt{\tau+\beta}}\right)u(\tau,\xi)d\xi .$$

**Remark.** By interchanging variables x and ξ in (13) we get a second new formula for the solution of equation (10):

$$f(x)=\sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\beta})^j}H_j\left(\frac{x}{2\sqrt{\tau+\beta}}\right)\frac{(\tau+\beta)^{\frac{j}{2}}}{j!}u_j \tag{14}$$

where

$$u_j=\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{4(\tau+\beta)}}}{2\sqrt{\pi\beta}}H_j\left(\frac{\xi}{2\sqrt{\beta}}\right)u(\tau,\xi)d\xi .$$

Finally, we will prove the third new formula for the inverse Cauchy problem. We use (12) which can be written as

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}e^{\lambda^2(\tau+\beta)}\left(\int_{-\infty}^{\infty}e^{i\lambda(x-\xi)}u(\tau,\xi)d\xi\right)d\lambda ,$$

where $\beta > 0$. Because of formula (13) we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}(\tau+\beta)^{\frac{j}{2}}\int_{-\infty}^{\infty}H_j\left(\frac{x-\xi}{2\sqrt{\tau+\beta}}\right)u(\tau,\xi)d\xi d\lambda .$$

If we use formula (4), then the initial distribution of the thermal field takes the form

$$f(x)=\frac{1}{2\sqrt{\pi\beta}}\sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\beta})^j}H_j(0)\frac{(\tau+\beta)^{\frac{j}{2}}}{j!}u_j ,$$

where

$$u_j=\int_{-\infty}^{\infty}H_j\left(\frac{x-\xi}{2\sqrt{\tau+\beta}}\right)u(\tau,\xi)d\xi .$$

Because of formula (8) for $H_j(0)$ as a result we get

$$f(x)=\frac{1}{\sqrt{x}}\sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\beta})^{2j+1}}\frac{(-1)^j(\tau+\beta)^j}{2^j(2j)!}u_{2j} \tag{15}$$

Where

$$u_j=\int_{-\infty}^{\infty}H_j\left(\frac{x-\xi}{2\sqrt{\tau+\beta}}\right)u(\tau,\xi)d\xi .$$

# 3. Cauchy Problem for the Heat Equation at Polar Coordinates

## 3.1. Auxiliary Statements

We define polynomials $W_j(z)$ with the help of the generating function $e^{-t^2}I_0(2tz)$

$$e^{-t^2}I_0(2tz)=\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j)!}W_j(z) \tag{16}$$

where $I_0(x)$- the zero-order modified Bessel function of the first kind. It follows from (16) that the polynomials $W_j(z)$ have the form

$$W_j(z)=\frac{d^{2j}}{dt^{2j}}\left[e^{-t^2}I_0(2tz)\right]_{t=0} \tag{17}$$

We get another view of polynomials $W_j(z)$. From the definition of the Bessel operator, [10],

$$B=\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}$$

it follows that

$$B^j\left[I_0(2tz)\right]=(2t)^{2j}I_0(2tz),$$

therefore

$$\exp\left(-\frac{B}{4}\right)\left[I_0(2tz)\right]=e^{-t^2}I_0(2tz),\exp\left(-\frac{B}{4}\right)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{B}{4}\right)^j.$$

In this formula we equate the coefficients of $\xi^{2j}$- degree at the left and right sides. We get

$$\exp\left(-\frac{B}{4}\right)\left[\frac{2^{2j}z^{2j}}{2^{2j}j!^2}\right]=\frac{W_j(z)}{(2j)!},$$

hence the polynomials $W_j(z)$ have the form

$$W_j(z)=\frac{(2j)!}{j!^2}\exp\left(-\frac{B}{4}\right)\left[z^{2j}\right].$$

Using polynomials $W_j(z)$ we will obtain the new formulas for solutions of direct and inverse Cauchy problems at polar coordinates.

## 3.2. New Formulas for Solution of the Cauchy Problem at Polar Coordi-Nates

We deduce the new formulas for the solution of the Cauchy problem for the heat equation at polar coordinates $(r, \phi)$ if the thermal regime depends on only the variable $r$. We use the explicit formula for solution of the Cauchy problem at polar coordinates [6]

$$u(\tau,r)=\int_0^{\infty}\lambda e^{-\lambda^2 t}J_0(\lambda r)\left(\int_0^{\infty}\xi J_0(\lambda\xi)f(\xi)d\xi\right)d\lambda,$$

where $J_0(\lambda\xi)$ - the zero-order Bessel function of the first kind [1]. We write the last equation in the form

$$u(\tau,r)=\int_0^{\infty}\lambda e^{-\lambda^2(t+\beta)}e^{\lambda^2\beta}J_0(\lambda r)\left(\int_0^{\infty}\xi J_0(\lambda\xi)f(\xi)d\xi\right)d\lambda \tag{18}$$

where $\beta>0$. In (16) we make the substitution: $t=i\lambda\sqrt{\beta}$, $z=\frac{x}{2\sqrt{\beta}}$ Then we obtain

$$e^{\lambda^2\beta}J_0(\lambda x)=\sum_{j=0}^{\infty}(-1)^j\frac{\lambda^{2j}\beta^j}{(2j)!}W_j\left(\frac{x}{2\sqrt{\beta}}\right) \tag{19}$$

Formula (18) according to the relation (19) takes the form

$$u(\tau,r)=\int_{-\infty}^{\infty}e^{-\lambda^2(\tau+\beta)}J_0(\lambda x)\sum_{j=0}^{\infty}(-1)^j\frac{\lambda^{2j}}{(2j)!}\beta^j\int_{-\infty}^{\infty}W_j\left(\frac{\xi}{2\sqrt{\beta}}\right)f(\xi)d\xi d\lambda \tag{20}$$

In last formula we change the order of integration and compute the inner integral with respect to $\lambda$. The Weber integral [6] is used in calculation.

$$\int_{-\infty}^{\infty}\lambda e^{-\lambda^2(\tau+\beta)}J_0(\lambda x)J_0(\lambda\xi)d\lambda=\frac{e^{-\frac{r^2+\xi^2}{4(r+\beta)}}}{2(\tau+\beta)}I_0\left(\frac{r\xi}{2(\tau+\beta)}\right).$$

Equating the coefficients at $\xi^{2j}$- degree we get the value of an integral

$$\frac{1}{j!^2\,2^{2j}}\int_{-\infty}^{\infty}(-1)^j e^{-\lambda^2(\tau+\beta)}J_0(\lambda r)\lambda^{2j}d\lambda=\frac{1}{(2j)!}\frac{d^{2j}}{d\xi^{2j}}\left[\frac{e^{-\frac{r^2+\xi^2}{4(r+\beta)}}}{2(\tau+\beta)}I_0\left(\frac{r\xi}{2(\tau+\beta)}\right)\right]_{\xi=0} \tag{21}$$

Due to (17) the last formula can be written:

$$\frac{e^{-\frac{r^2}{4(r+\beta)}}}{2(\tau+\beta)}\frac{1}{2^{2j}(\tau+\beta)^j(2j)!}W_j\left(\frac{r}{2\sqrt{\tau+\beta}}\right)=\frac{1}{(2j)!}\frac{d^{2j}}{d\xi^{2j}}\left[\frac{e^{-\frac{r^2+\xi^2}{4(r+\beta)}}}{2(\tau+\beta)}I_0\left(\frac{r\xi}{2(\tau+\beta)}\right)\right]_{\xi=0}.$$

Then, (21) becomes

$$\int_{-\infty}^{\infty}(-1)^j e^{-\lambda^2(\tau+\beta)}J_0(\lambda r)\lambda^{2j}d\lambda=\frac{e^{-\frac{r^2}{4(r+\beta)}}}{2(\tau+\beta)}\frac{j!^2}{(\tau+\beta)^j(2j)!}W_j\left(\frac{r}{2\sqrt{\tau+\beta}}\right) \tag{22}$$

Finally, formula (20) for the thermal field $u(\tau, r)$ with (22) takes the form

$$u(\tau,x)=\frac{e^{-\frac{r^2}{4(r+\beta)}}}{2(\tau+\beta)}\sum_{j=0}^{\infty}\frac{1}{(\tau+\beta)^j}W_j\left(\frac{r}{2\sqrt{\tau+\beta}}\right)\frac{j!^2\beta^j}{(2j)!^2}f_j \tag{23}$$

where

$$f_j=\int_{-\infty}^{\infty}W_j\left(\frac{\xi}{2\sqrt{\beta}}\right)f(\xi)d\xi.$$

**Remark**. From the last equality for $u(\tau, r)$ at $\tau=0$ the expansion theorem with respect to eigenfunctions $\left\{e^{-\frac{r^2}{4\beta}}W_j\left(\frac{r}{2\sqrt{\beta}}\right)\right\}$ can be obtained

$$f(r)=\frac{e^{-\frac{r^2}{4\beta}}}{2\beta}\sum_{j=0}^{\infty}W_j\left(\frac{r}{2\sqrt{\beta}}\right)\frac{j!^2}{(2j)!^2}f_j,$$

where

$$f_j = \int_{-\infty}^{\infty} W_j\left(\frac{\xi}{2\sqrt{\beta}}\right) f(\xi)\, d\xi .$$

Similarly section-1 we can get two new formulas for the thermal field. If we replace in (23): $\beta \leftrightarrow \tau + \beta$, then the new formula takes the form

$$u(\tau, x) = \frac{e^{-\frac{r^2}{4\beta}}}{2\beta} \sum_{j=0}^{\infty} \frac{1}{\beta^j} W_j\left(\frac{r}{2\sqrt{\beta}}\right) \frac{j!^2 (\tau+\beta)^j}{(2j)!^2} f_j \quad (24)$$

where

$$f_j = \int_{-\infty}^{\infty} W_j\left(\frac{\xi}{2\sqrt{\tau+\beta}}\right) f(\xi) d\xi .$$

The third new formula is proved similarly section-1. We apply formula from [10]

$$J_0(\lambda x) J_0(\lambda y) = \frac{1}{\pi} \int_0^{\pi} J_0\left(\lambda \sqrt{x^2 + y^2 - 2xy\cos\phi d\phi}\right)$$

to (18). Then we get

$$e^{\lambda^2 \tau} J_0(\lambda x) J_0(\lambda y) = \frac{1}{\pi} \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{2j} \tau^j}{(2j)!} \int_0^{\pi} W_j\left(\frac{\sqrt{x^2 + y^2 - 2xy\cos\phi}}{2\sqrt{\tau}}\right) d\phi \quad (25)$$

Write equation (18) as

$$u(\tau, r) = \int_0^{\infty} \lambda e^{-\lambda^2 (t+\beta)} \left(\int_0^{\infty} e^{\lambda^2 \beta} J_0(\lambda\xi) \xi f(\xi) d\xi\right) d\lambda ,$$

where $\beta > 0$. Next, on the basis of (25) we have

$$u(\tau, r) = \frac{1}{\pi} \int_0^{\infty} e^{-\lambda^2 (\tau+\beta)} \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{2j}}{(2j)!} \beta^j \cdot \int_0^{\infty} \int_0^{\pi} W_j\left(\frac{\sqrt{x^2 + y^2 - 2xy\cos\phi}}{2\sqrt{\beta}}\right) d\phi \xi f(\xi) d\xi d\lambda .$$

We change the order of integration and use the definition of the Gamma-function [10] to compute the inner integral with respect to $\lambda$.

$$\int_0^{\infty} (-1)^j e^{-\lambda^2 (\tau+\beta)} \lambda^{2j} d\lambda = \frac{(-1)^j \Gamma(j + \frac{1}{2})}{2(\tau+\beta)^{j+\frac{1}{2}}} .$$

Finally, we obtain

$$u(\tau, r) = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \frac{\beta^j}{(\tau+\beta)^{j+\frac{1}{2}}} \frac{\Gamma\left(j+\frac{1}{2}\right)}{(2j)!} f_j \quad (26)$$

where

$$f_j = \int_0^{\infty} \int_0^{\pi} W_j\left(\frac{\sqrt{r^2 + \xi^2 - 2r\xi\cos\phi}}{2\sqrt{\beta}}\right) d\phi \xi f(\xi) d\xi .$$

## 4. The Inverse Cauchy Problem for the Heat Equation at Polar Coordinates

The inverse Cauchy problem [3,4,5,7,8] at polar coordinates leads to the solution of the first- kind Fredholm integral equation

$$\int_0^{\infty} \frac{e^{-\frac{r^2+\xi^2}{4\tau}}}{2\tau} I_0\left(\frac{r\xi}{2\tau}\right) f(\xi) d\xi = u(\tau, \xi) .$$

To find three previously unknown expressions for the solution of inverse Cauchy problem as the series with respect to polynomials $W_j(z)$ (17) we will do the same section-1.

The first expression is

$$f(r) = \frac{e^{-\frac{r^2}{4\beta}}}{2\beta} \sum_{j=0}^{\infty} \frac{1}{\beta^j} W_j\left(\frac{r}{2\sqrt{\beta}}\right) \frac{j!^2 (\beta+\tau)^j}{(2j)!^2} u_j \quad (27)$$

where

$$u_j = \int_0^{\infty} W_j\left(\frac{\xi}{2\sqrt{\tau+\beta}}\right) u(\tau, \xi) d\xi .$$

The second expression is

$$f(r) = \frac{e^{-\frac{r^2}{4(\beta+\tau)}}}{2(\beta+\tau)} \sum_{j=0}^{\infty} \frac{1}{(\beta+\tau)^j} W_j\left(\frac{r}{2\sqrt{(\beta+\tau)}}\right) \frac{j!^2 \tau^j}{(2j)!^2} u_j \quad (28)$$

where

$$u_j = \int_0^{\infty} W_j\left(\frac{\xi}{2\sqrt{\beta}}\right) u(\tau, \xi) d\xi .$$

The third expression is

$$f(r) = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \frac{(\tau+\beta)^j}{\tau^{j+\frac{1}{2}}} \frac{\Gamma\left(j+\frac{1}{2}\right)}{(2j)!} u_j \quad (29)$$

where

$$u_j = \int_0^{\infty} \int_0^{\pi} W_j\left(\frac{\sqrt{r^2 + \xi^2 - 2r\xi\cos\phi}}{2\sqrt{\tau+\beta}}\right) d\phi \xi u(\tau, \xi) d\xi .$$

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