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# Existence and Nonexistence of Weak Positive Solution for Classes of 3 × 3 P-Laplacian Elliptic Systems

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**Abstract** In this work, we are interested to obtain some result of existence and nonexistence of large positive weak solution for the following p-laplacian system $\begin{cases} -\Delta_p u = \lambda\alpha(x) f(u,v,w) \text{ in } \Omega, \\ -\Delta_q u = \mu\beta(x) g(u,v,w) \text{ in } \Omega, \\ -\Delta_r w = \nu\gamma(x) h(u,v,w) \text{ in } \Omega, \\ u = v = w = 0 \text{ on } \partial\Omega, \end{cases}$ where $\Delta_\sigma z = div\left(|\nabla z|^{\sigma-2}\nabla z\right), \sigma \geq 1$, λ, μ and ν are a positive parameter, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The proof of the main results is based to the sub-supersolutions method.

***Keywords:*** *Positive solutions, Sub-supersolutions, p-Laplacian systems*

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## 1. Introduction

Problems involving the p-Laplacian arise from many branches of pure mathe-matics as in the theory of quasiregular and quasiconformal mapping as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Hai, Shivaji [9] studied the existence of positive solution for the p-Laplacian system

$$\begin{cases} -\Delta_p u = \lambda f(v) \text{ in } \Omega, \\ -\Delta_p v = \lambda g(u) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

which f(s); g(s) are the increasing functions in $[0,\infty)$ and satisfy

$$\lim_{s\to+\infty} f\left(\frac{M\ (g(s))^{\frac{1}{p-1}}}{S^{p-1}}\right) = 0, M > 0$$

the authors showed that the problem (1.2) has at least one positive solution provided that $\lambda > 0$ is large enough.

In [6], the author studied the existence and nonexistence of positive weak solution to the following quasilinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda u^\alpha v^\gamma \text{ in } \Omega, \\ -\Delta_q u = \lambda u^\delta v^\beta \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

The first eigenfunction is used to construct the subsolution of problem (1.3), the main results are as follows:

(i) If $\alpha, \beta \geq 0, \gamma, \delta > 0, \theta = (p-1-\alpha)((q-1-\beta) - \gamma\delta > 0,$ then problem (1.3) has a positive weak solution for each $k > 0$;

(ii) If $\lambda = 0$ and $p\gamma = q(p-1-\alpha)$ then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, then problem (1.3) has no nontrivial nonnegative weak solution.

In this paper, we are concerned with the existence and nonexistence of positive weak solution to the quasilinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda\alpha(x) f(u,v,w) \text{ in } \Omega, \\ -\Delta_q v = \mu\beta(x) g(u,v,w) \text{ in } \Omega, \\ -\Delta_r w = \nu\gamma(x) h(u,v,w) \text{ in } \Omega, \\ u = v = w = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_\sigma z = div\left(|\nabla z|^{\sigma-2}\nabla z\right), \sigma \geq 1$, λ, μ and ν are a positive parameter, and Ω is a bounded domain in

$\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We prove the existence of a large positive weak solution for λ, μ and ν large when

$$\lim_{t\to+\infty}\frac{f(t,t,t)}{t^{p-1}}=\frac{g(t,t,t)}{t^{q-1}}=\frac{h(t,t,t)}{t^{r-1}}=0$$

## 2. Definitions and Notations

**Definition 1**. *We called positive weak solution (u; v; w) of (1.3) such that satisfies*

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u.\nabla\omega dx=\lambda\int_\Omega \alpha(x)f(u,v,w)\omega dx,$$
$$\int_\Omega |\nabla u_2|^{q-2}\nabla v.\nabla\omega dx=\mu\int_\Omega \beta(x)g(u,v,w)\omega dx,$$
$$\int_\Omega |\nabla u_3|^{r-2}\nabla w.\nabla\omega dx=\nu\int_\Omega \gamma(x)h(u,v,w)\omega dx,$$

*for all* $\omega\in H_0^1(\Omega)$ *with* $\omega\geq 0$

**Definition 2.** *We called positive weak subsolution ($\psi_1$, $\psi_2$, $\psi_3$) and supersolution ($z_1$, $z_2$, $z_3$) of (1.3) such that,* $\psi_i\leq z_i$, $i=1,2,3,$ *satisfies*

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla\omega dx\leq\lambda\int_\Omega \alpha(x)f(\psi_1,\psi_2,\psi_3)\omega dx,$$
$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla\omega dx\leq\mu\int_\Omega \beta(x)g(\psi_1,\psi_2,\psi_3)\omega dx,$$
$$\int_\Omega |\nabla\psi_3|^{r-2}\nabla\psi_3.\nabla\omega dx\leq\nu\int_\Omega \gamma(x)h(\psi_1,\psi_2,\psi_3)\omega dx,$$

*and*

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1.\nabla\omega dx\geq\lambda\int_\Omega \alpha(x)f(z_1,z_2,z_3)\omega dx,$$
$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2.\nabla\omega dx\geq\mu\int_\Omega \beta(x)g(z_1,z_2,z_3)\omega dx,$$
$$\int_\Omega |z\psi_3|^{r-2}\nabla z_3.\nabla\omega dx\geq\nu\int_\Omega \gamma(x)h(z_1,z_2,z_3)\omega dx,$$

*for all* $q\in H_0^1(\Omega)$ *with* $q\geq 0$.

We suppose that α, β, γ, f, g and h verify the following assumptions;

(H1) $f,g,h:([0,\infty[)^2\to\mathbb{R}$ are $C^1$, monotone functions such that

$$\lim_{t_1,t_2,t_3\to+\infty}f(t_1,t_2,t_3)=\lim_{t_1,t_2,t_3\to+\infty}g(t_1,t_2,t_3)=\lim_{t_1,t_2,t_3\to+\infty}h(t_1,t_2,t_3)=+\infty$$

$$\exists k_0>0:\ f(y,z,w),g(y,z,w),h(y,z,w)\geq -k_0$$

for all $y,z,w\geq 0$

$$\exists\alpha_0,\beta_0,\gamma_0,\alpha_1,\beta_1,\gamma_1>0:\begin{cases}\alpha_0\leq\alpha(x)\leq\alpha_1\\ \beta_0\leq\beta(x)\leq\beta_1\\ \gamma_0\leq\gamma(x)\leq\gamma_1\end{cases}$$

(H2) $$\lim_{t\to+\infty}\frac{f(t,t,t)}{t^{p-1}}=\frac{g(t,t,t)}{t^{q-1}}=\frac{h(t,t,t)}{t^{r-1}}=0$$

$$\exists\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3,\nu_1,\nu_2,\nu_3>0:$$

(H3) $$\begin{cases}f(t_1,t_2,t_3)\leq\xi_1t_1^{p-1}+\eta_1t_2^{q\left(\frac{p-1}{p}\right)}+\zeta_1t_3^{r\left(\frac{p-1}{p}\right)}\\ g(t_1,t_2,t_3)\leq\xi_2t_1^{p\left(\frac{q-1}{q}\right)}+\eta_2t_2^{q-1}+\zeta_2t_3^{r\left(\frac{q-1}{q}\right)}\\ h(t_1,t_2,t_3)\leq\xi_3t_1^{p\left(\frac{r-1}{r}\right)}+\eta_3t_2^{q\left(\frac{r-1}{r}\right)}+\zeta_3t_3^{r-1}\end{cases}$$

Let $\lambda_1$, $\mu_1$ and $\nu_1$ be the first eigenvalue of $-\Delta_p$, $-\Delta_q$ and $-\Delta_r$ with Dirichlet boundary conditions and $\varphi_p$, $\varphi_q$ and $\varphi_r$ the corresponding positive eigenfunction with $\|\varphi_p\|_\infty=\|\varphi_q\|_\infty=\|\varphi_r\|_\infty=1$, , and m, δ > 0, such that

$$\begin{cases}|\nabla\varphi_p|^p-\lambda_1\varphi_p^p\geq m\\ |\nabla\varphi_q|^q-\mu_1\varphi_q^q\geq m\ \ on\ \overline{\Omega_\delta}=\{x\in\Omega:d(x,\partial\Omega)\leq\delta\}\\ |\nabla\varphi_r|^r-\nu_1\varphi_r^r\geq m\end{cases}$$

We denote by

$$\lambda_0=\frac{\lambda}{p}(p\xi_1+\eta_1+\zeta_1)+\frac{\mu(q-1)}{q}\xi_2+\frac{\nu(r-1)}{r}\xi_3,$$
$$\mu_0=\frac{\lambda(p-1)}{p}\eta_1+\frac{\mu}{q}(\xi_2+q\eta_2+\zeta_2)+\frac{\nu(r-1)}{r}\eta_3,$$
$$\nu_0=\frac{\lambda(p-1)}{p}\zeta_1+\frac{\mu(q-1)}{q}\zeta_2+\frac{\nu}{r}(\xi_3+\eta_3+r\zeta_3),$$

## 3. Existence Results

**Theorem 1**. *Let (H1) and (H2) hold. Then for λ large, the system (1.3) has a large positive solution (u, v, w):*

*Proof.* We shall verify that

$$\psi_1=\left(\frac{\lambda\alpha_0k_0}{m}\right)^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\varphi^{\frac{p}{p-1}},$$
$$\psi_2=\left(\frac{\mu\beta_0k_0}{m}\right)^{\frac{1}{q-1}}\left(\frac{q-1}{q}\right)\varphi^{\frac{q}{q-1}},$$
$$\psi_3=\left(\frac{\nu\gamma_0k_0}{m}\right)^{\frac{1}{r-1}}\left(\frac{r-1}{r}\right)\varphi^{\frac{r}{r-1}},$$

is a subsolution of (1.3) for λ large: Let $\omega\in H_0^1(\Omega)$ with $\omega\geq 0$ .A calculation shows that

$$\begin{aligned}\int_\Omega -\Delta_p\psi_1\omega dx&=\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla\omega dx\\&=\frac{\lambda k_0}{m}\int_\Omega\varphi_\sigma|\nabla\varphi_\sigma|^{p-2}\nabla\varphi_\sigma.\nabla\omega dx\\&=\frac{\lambda k_0}{m}\left\{\begin{array}{l}\int_\Omega|\nabla\varphi_\sigma|^{p-2}\nabla\varphi_\sigma.\nabla_{(\varphi_\sigma\omega)}dx\\ -\int_\Omega|\nabla\varphi_\sigma|^p\omega dx\end{array}\right\}\\&=\frac{\lambda k_0}{m}\int_\Omega\left(\lambda_1\varphi_\sigma^p-|\nabla\varphi_\sigma|^p\right)\omega dx.\end{aligned}$$

Now, on $\overline{\Omega_\sigma}$ we have

$$\frac{\alpha_0 k_0}{m}\left(\lambda_1\varphi_p^p-\left|\nabla\varphi_p\right|^p\right)-\alpha(x)\,f(\psi_1,\psi_2,\psi_3)$$
$$\le k_0\left(\alpha_0-\alpha(x)\right)\le 0$$
$$\frac{\beta_0 k_0}{m}\left(\mu_1\varphi_q^q-\left|\nabla\varphi_q\right|^q\right)-\beta(x)\,f(\psi_1,\psi_2,\psi_3)$$
$$\le k_0\left(\beta_0-\beta(x)\right)\le 0$$
$$\frac{\gamma_0 k_0}{m}\left(v_1\varphi_r^r-\left|\nabla\varphi_r\right|^r\right)-\gamma(x)\,f(\psi_1,\psi_2,\psi_3)$$
$$\le k_0\left(\gamma_0-\gamma(x)\right)\le 0$$

Next, on $\Omega/\Omega_\sigma$ we have $\varphi_p,\varphi_q,\varphi_r\ge\rho$ for some $\rho\ge 0$, and therefore for λ, μ and ν large

$$f(\psi_1,\psi_2,\psi_3)\ge\frac{k_0}{m}\lambda_1\ge\frac{k_0}{m}\left(\lambda_1\varphi_p^p-\left|\nabla\varphi_p\right|^p\right)$$
$$g(\psi_1,\psi_2,\psi_3)\ge\frac{k_0}{m}\mu_1\ge\frac{k_0}{m}\left(\mu_1\varphi_q^q-\left|\nabla\varphi_q\right|^q\right)$$
$$h(\psi_1,\psi_2,\psi_3)\ge\frac{k_0}{m}v_1\ge\frac{k_0}{m}\left(v_1\varphi_r^r-\left|\nabla\varphi_r\right|^r\right)$$

Hence

$$\int_\Omega\left|\nabla\psi_1\right|^{p-2}\nabla\psi_1.\nabla\omega dx\le\lambda\int_\Omega f(\psi_1,\psi_2,\psi_3)\,\omega dx,$$
$$\int_\Omega\left|\nabla\psi_2\right|^{p-2}\nabla\psi_2.\nabla\omega dx\le\mu\int_\Omega g(\psi_1,\psi_2,\psi_3)\,\omega dx,$$
$$\int_\Omega\left|\nabla\psi_3\right|^{p-2}\nabla\psi_3.\nabla\omega dx\le v\int_\Omega h(\psi_1,\psi_2,\psi_3)\,\omega dx,$$

i.e. ($\psi_1$, $\psi_2$, $\psi_3$) is a subsolution of (1.3). Next, let $\varphi_p$, $\varphi_q$ and $\varphi_r$ are the solution of

$$\begin{cases}-\Delta_p\varphi_p=1\ in\ \Omega,\\ \varphi_p=0\ on\ \partial\Omega.\end{cases},\begin{cases}-\Delta_p\varphi_p=1\ in\ \Omega,\\ \varphi_q=0\ on\ \partial\Omega.\end{cases}\ and\ \begin{cases}-\Delta_p\varphi_r=1\ in\ \Omega,\\ \varphi_r=0\ on\ \partial\Omega.\end{cases}$$

Let

$$z_1=\frac{C}{\left\|\varphi_p\right\|_\infty}\lambda^{\frac{1}{p-1}}\varphi_p$$
$$z_2=\mu^{\frac{1}{q-1}}\left(g\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)\right)^{\frac{1}{q-1}}\varphi_q$$
$$z_3=v^{\frac{1}{r-1}}\left(h\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)\right)^{\frac{1}{r-1}}\varphi_r$$

Where $C>0$ is a large number to be chosen later: We shall verify that ($z_1$, $z_2$, $z_3$) is a supersolution of (1.3) for λ, μ and ν large.

To this end, let $\omega\in H_0^1(\Omega)$ with $\omega\ge 0$.

By (H1) and (H2), we can choose C large enough so that

$$\left(C\lambda^{\frac{1}{p-1}}\right)^{q-1}$$
$$\ge\left\|\varphi_p\right\|_\infty^{q-1}\mu g\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)$$
$$\ge\mu g\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)\varphi_q^{q-1}=z_2^{q-1}$$

then

$$C\lambda^{\frac{1}{p-1}}\ge z_2$$

and

$$\left(C\lambda^{\frac{1}{p-1}}\right)^{r-1}$$
$$\ge\left\|\varphi_r\right\|_\infty^{r-1}vh\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)$$
$$\ge vh\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)\varphi_r^{r-1}=z_3^{r-1}$$

then

$$C\lambda^{\frac{1}{p-1}}\ge z_3$$

which imply that

$$\left(C\lambda^{\frac{1}{p-1}}\right)^{p-1}$$
$$\ge\left\|\varphi_p\right\|_\infty^{p-1}\lambda f\left(C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)$$
$$\ge\left\|\varphi_p\right\|_\infty^{p-1}\lambda f\left(\frac{C}{\left\|\varphi_p\right\|_\infty}\lambda^{\frac{1}{p-1}}\left\|\varphi_p\right\|_\infty,C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)$$
$$\ge\left\|\varphi_p\right\|_\infty^{p-1}\lambda f\left(\frac{C}{\left\|\varphi_p\right\|_\infty}\lambda^{\frac{1}{p-1}}\varphi_p,C\lambda^{\frac{1}{p-1}},C\lambda^{\frac{1}{p-1}}\right)$$
$$=\left\|\varphi_p\right\|_\infty^{p-1}\lambda f(z_1,z_2,z_3)$$

Then we have

$$\int_\Omega\left|\nabla z_1\right|^{p-2}\nabla z_1.\nabla\omega dx$$
$$=\lambda\left(\frac{C}{\left\|\varphi_p\right\|_\infty}\right)\int_\Omega\left|\nabla\varphi_p\right|^{p-2}\nabla\varphi_p.\nabla\omega dx$$
$$=\lambda\left(\frac{C}{\left\|\varphi_p\right\|_\infty}\right)^{p-1}\int_\Omega\omega dx\ge\lambda\int_\Omega f(z_1,z_2,z_3)\,\omega dx$$

in another hand

$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2.\nabla\omega dx$$
$$= \mu g\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\int_\Omega |\nabla\varphi_q|^{q-2}\nabla\varphi_q.\nabla\omega dx$$
$$\geq \mu\int_\Omega g\left(\frac{C}{\|\varphi_p\|_\infty}\lambda^{\frac{1}{p-1}}\|\varphi_p\|_\infty, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\omega dx$$
$$\geq \mu\int_\Omega g(z_1,z_2,z_3)\omega dx$$

similar

$$\int_\Omega |\nabla z_3|^{r-2}\nabla z_3.\nabla\omega dx$$
$$= vh\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\int_\Omega |\nabla\varphi_r|^{r-2}\nabla\varphi_r.\nabla\omega dx$$
$$\geq v\int_\Omega h\left(\frac{C}{\|\varphi_p\|_\infty}\lambda^{\frac{1}{p-1}}\|\varphi_p\|_\infty, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\omega dx$$
$$\geq v\int_\Omega h(z_1,z_2,z_3)\omega dx$$

i.e ($z_1$, $z_2$ , $z_3$) is a supersolution of (1.3) with $z_i \geq \psi_i$ for $C$ large $i = 1,2,3.$ Thus, there exists a solution (u, v, w) of (1.3) with $\psi_1 \leq u \leq z_1, \psi_2 \leq v \leq z_2, \psi_3 \leq \omega \leq z_3.$ □

## 4. Nonexistence Results

**Theorem 2.** *If f, g and h verify (H 3) ,the system (1.3) has not nontrivial positive solutions for*

$$0 < \lambda_0 < \lambda_1,\ 0 < \mu_0 < \mu_1 \text{ and } 0 < v_0 < v_1. \qquad (4.1)$$

*Proof.* Multiplying the first equation by u; we have from Young inequality that

$$\|\nabla u\|_p^p$$
$$= \int_\Omega \lambda f(u,v,w)udx \leq \lambda\int_\Omega\left(\xi_1 u^{p-1} + \eta_1 v^{q\left(\frac{p-1}{p}\right)} + \zeta_1 w^{r\left(\frac{p-1}{p}\right)}\right)udx$$
$$\leq \lambda\int_\Omega\left(\xi_1 u^p + \frac{\eta_1}{p}\left(u^p + (p-1)v^q\right) + \frac{\zeta_1}{p}\left(u^p + (p-1)v^r\right)\right)dx$$
$$\leq \lambda\int_\Omega\left(\left(\xi_1 + \frac{\eta_1+\zeta_1}{p}\right)u^p + \left(\frac{p-1}{p}\right)\eta_1 v^q + \left(\frac{p-1}{p}\right)\zeta_1 v^r\right)dx$$
$$= \frac{\lambda}{p}(p\xi_1+\eta_1+\zeta_1)\|u\|_p^p + \frac{\lambda(p-1)}{p}\eta_1\|v\|_p^p + \frac{\lambda(p-1)}{p}\zeta_1\|w\|_r^r$$

then, we have

$$\|\nabla u\|_p^p \leq \frac{\lambda}{p}(p\xi_1+\eta_1+\zeta_1)\|u\|_p^p + \frac{\lambda(p-1)}{p}\eta_1\|v\|_p^p + \frac{\lambda(p-1)}{p}\zeta_1\|w\|_r^r$$

$$\|\nabla v\|_p^p \leq \frac{\mu(q-1)}{q}\xi_2\|u\|_p^p + \frac{\mu}{q}(\xi_2+q\eta_2+\zeta_2)\|v\|_q^q + \frac{\mu(q-1)}{q}\zeta_2\|w\|_r^r \qquad (4.2)$$
$$\|\nabla w\|_p^p \leq \frac{v(r-1)}{r}\xi_3\|u\|_p^p + \frac{v(r-1)}{r}\eta_3\|v\|_q^q + \frac{v}{r}(\xi_3+\eta_3+r\zeta_3)\|w\|_r^r$$

Note that

$$\lambda_1 = \inf\frac{\|\nabla u\|_p^p}{\|u\|_p^p},\ \mu_{1=}\inf\frac{\|\nabla v\|_q^q}{\|v\|_q^q} \qquad (4.3)$$
$$\text{and } v_{1=}\inf\frac{\|\nabla w\|_r^r}{\|w\|_r^r}$$

Combining (4.2) and (4.3), we obtain

$$(\lambda_1-\lambda_0)\|u\|_p^p + (\mu_1-\mu_0)\|v\|_q^q + (v_1-v_0)\|w\|_r^r \leq 0$$

which is a contradiction if (4.2) hold.

Thus (1.3) has no nontrivial nonnegative weak solution.□

## 5. Applications

**Theorem 3.** *consider the following system in* $W_0^{1,P}(\Omega)$

$$\begin{cases} -\Delta_p u = \lambda u^{m1}v^{n1}w^{l1} \text{ in } \Omega, \\ -\Delta_q v = \mu u^{m2}v^{n2}w^{l2} \text{ in } \Omega, \\ -\Delta_r w = vu^{m3}v^{n3}w^{l3} \text{ in } \Omega, \\ u = v = w = 0 \text{ on } \partial\Omega, \end{cases} \qquad (5.1)$$

*1) the system has a large positive solutions if*

$$\begin{aligned} m_1+n_1+l_1 &< p-1, \\ m_2+n_2+l_2 &< q-1 \\ m_3+n_3+l_3 &< r-1 \end{aligned} \qquad (5.2)$$

*2) In the case where*

$$\begin{aligned} qrm_1+prn_1+pql_1 &= qr(p-1), \\ qrm_2+prn_2+pql_2 &= pr(q-1) \\ qrm_3+prn_3+pql_3 &= pq(r-1) \end{aligned} \qquad (5.3)$$

*the system has not non trivial positive solutions if*

$$0 < \lambda < \lambda_1,\ 0 < \mu < \mu_1 \text{ and } 0 < v < v_1. \qquad (5.4)$$

*Proof.* 1) (5.2) imply that (H2). By using theorem 1, the system has a large positive solutions.
2) The first equation in (5.3) imply that

$$\frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{1}{\theta_3} = \frac{1}{\left(\frac{p-1}{m_1}\right)} + \frac{1}{\frac{q}{p}\left(\frac{p-1}{n_1}\right)} + \frac{1}{\frac{r}{p}\left(\frac{p-1}{l_1}\right)} = 1 \qquad (5.5)$$

then, the generalized Young inequality gives

$$f_1(u,v,w) = u^{m1}v^{n1}w^{l1} \le \frac{1}{\theta_1}u^{m1\theta 1} + \frac{1}{\theta_2}v^{n1\theta 2} + \frac{1}{\theta 3}w^{l1\theta 3}$$

$$= \frac{1}{\theta_1}u^{p-1} + \frac{1}{\theta_2}v^{q\left(\frac{p-1}{p}\right)} + \frac{1}{\theta 3}w^{r\left(\frac{p-1}{p}\right)}$$

By the same manner, we conclude that, the assumption (H3) holds. Then he system (5.1) has not nontrivial positive solutions if

$$\lambda_0 = \frac{1}{p}\left(\lambda(m_1+1) + \mu m_2 + \nu m_3\right) < \lambda_1,$$

$$\mu_0 = \frac{1}{q}\left(\lambda n_1 + \mu(n_2+1) + \nu n_3\right) < \mu_1$$

$$\nu_0 = \frac{1}{r}\left(\lambda l_1 + \mu l_2 + \nu(l_3+1)\right) < \nu_1.$$

which imply that

$$p(\lambda-\lambda_1) + q(\mu-\mu_1) + r(\nu-\nu_1) < 0 \qquad (5.6)$$

this inequality hold if (5.4) hold. □

**Theorem 4.** *The following problem has a large positive solution if $\lambda$ large*

$$\left\{\begin{array}{l} -\Delta_p^3 u = \lambda^3\gamma(x)H\left(u,\Delta_p u,\Delta_p^2 u\right) \text{ in } \Omega, \\ u = \Delta_p u = \Delta_p^2 u = 0 \text{ on } \partial\Omega, \end{array}\right\} \qquad (5.7)$$

*where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\lambda$ is a positive parameter and $\gamma$ is a function of class $L^\infty(\Omega)$ and*

$H:([0,\infty[)^3 \to \mathbb{R}$ is of class $C^1$,

$H(t_1,t_2,t_3)$ is increasing with respect to $t_1$ and $t_3$

$H(t_1,t_2,t_3)$ is decreasing with respect to $t_2$

$$\lim_{t\to+\infty}\frac{H\left(t,-\lambda t,\lambda^2 t\right)}{t^{p-1}} = 0,\ p>2$$

$$\exists k_0 > 0 : H(t_1,t_2,t_3) \ge -k_0,\ \forall(t_1,t_2,t_3) \in ([0,\infty[)^3$$

*Proof.* The problem (5.7) can be written under the following system form

$$\left\{\begin{array}{l} -\Delta_p u = \lambda v \text{ in } \Omega, \\ -\Delta_p v = \lambda w \text{ in } \Omega, \\ -\Delta_p w = \lambda\gamma(x)H\left(u,-\lambda v,\lambda^2 w\right) \text{ in } \Omega, \\ u = v = w = 0 \text{ on } \partial\Omega, \end{array}\right\}$$

In this case, the assumptions of theorem (3.1) holds.□

# References


[1] S. Ala, G. A. Afrouzi, Q. Zhang & A. Niknam, Existence of positive solutions for variable exponent elliptic systems, Boundary Value Problems 2012, 2012:37.

[2] G. A. Afrouzi & J. Vahidi, On critical exponent for the existence and stability properties of positive weak solutions for some nonlinear elliptic systems involving the (p, q)-Laplacian and indefinite weight function, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 1, February 2011, pp. 83-91.

[3] G. A. Afrouzi & Z. Valinejad, Nonexistence of result for some p-Laplacian Systems, The Journal of Mathematics and Computer Science Vol. 3 No. 2 (2011) 112-116.

[4] J. Ali, R. Shivaji, Existence results for classes of Laplacian systems with sign-changing weight, Applied Mathematics Letters 20 (2007) 558-562.

[5] J. Ali, R. Shivaji, Positive solutions for a class of p-Laplacian systems with multiple parameters, J. Math. Anal. Appl. 335 (2007) 1013-1019.s

[6] C. Chen, On positive weak solutions for a class of quasilinear elliptic systems, Nonlinear Analysis 62 (2005) 751-756.

[7] R. Dalmasso, Existence and uniqueness of positive solutions of semilinear elliptic systems, Nonlinear Analysis 39 (2000) 559-568.

[8] H. Dang, S. Oruganti & R. Shivaji, nonexistence of non positive solutions for a class of semilinear elliptic systems, Rocky Mountain Journal of mathematics, Volume 36, Number 6, 2006.

[9] D. D. Hai, R. Shivaji, An existence result on positive solutions for a class of p-Laplacian systems, Nonlinear Analysis 56 (2004) 1007-1010.

[10] P. Drˆabek & J. Hern´andez, Existence and uniqueness of positive solutions for some quasilinear elliptic problems, Nonlinear Analysis 44 (2001).

[11] S. Haghaieghi & G. A. Afrouzi, Sub-super solutions for (p-q) Laplacian systems, Boundary Value Problems 2011, 2011:52.

[12] R. Shivaji a & J. Ye, Nonexistence results for classes of 3 × 3 elliptic systems, Nonlinear Analysis 74 (2011) 1485-1494. 189-204.