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An Alternative Approach for Quantum Forces in Toroidal Topology

A.P.C. Malbouisson[*]

Centro Brasileiro de Pesquisas Fisicas - CBPF/MCTI, Brazil
*Corresponding author: adolfo@cbpf.br

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Abstract We investigate the Casimir effect in the context of a nontrivial topology by means of a generalized Matsubara formalism. This is performed in the context of a scalar field in D Euclidean spatial dimensions with d compactified dimensions. The procedure gives us the advantage of considering simultaneously spatial constraints and thermal effects. In this sense, the Casimir pressure in a heated system between two infinite planes is obtained and the results are compared with those found in the literature.

Keywords: *radiation in cavities, dressed states approach*

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1. Introduction

The Casimir effect is a quantum phenomenon, having a macroscopic manifestation. It has been originally described as the attraction of two conducting, neutral plates in vacuum, induced by changes in the zero-point energy of the electromagnetic field [1]. The first observation of the Casimir force was made by Sparnaay in 1956 [30]. A few decades later, a large number of precise experimental evidences of Casimir physics was found [31]. However, this is not an exclusive feature of electromagnetic fields. It has been shown that any relativistic field under the influence of external conditions is able to exhibit an analogous kind of phenomenon [2]. This quantum vacuum effect is strongly dependent on the material properties of the medium where the macroscopic objects interact, on the nature of the quantum field, and on the boundary conditions under investigation. It has been related to many different physical situations ranging from cosmology, condensed matter, atomic and molecular physics to more recent developments in micro and nanoelectricmechanical devices as discussed in the reviews found in Refs. [3-11]. Moreover, it is a well-known fact that thermal fluctuations also produce Casimir forces. Pioneering works devoted to explain this thermodynam- ical behavior are [12,13]. General theoretical works on the subject are [14-21]. Controversial results in realistic situations [22-29] were also explored.

On the other hand, the analysis of quantum field theory models defined on toroidal spaces has been the focus of a large number of investigations in theoretical foundations and applications of the formalism: spontaneous symmetry breaking drive by both temperature and spatial boundaries [37], second-order phase transitions in superconducting films, wires and grains [38,39,40], finite-size effects in the presence of magnetic fields, finite chemical potential in first-order phase transitions [41], and also the Casimir effect [42-47]. It is well-known that one way to obtain thermal effects in quantum field theories is to consider the Matsubara formalism, in which a fourth dimension (mathematically analogous to imaginary time) has a finite extension equal to the inverse of temperature, β, with a periodic boundary condition. The application of this procedure also to spatial dimensions has been introduced by Birrell and Ford [48] in order to describe field theories in spaces with finite geometries and has been generalized to what came to be known as quantum field theories on toroidal topologies [37,49,50,52,53]. This procedure can also be called a generalized Matsubara formalism. In general, this technique basically consists in considering quantum fields defined on spaces with topologies of the type $\left(\mathbb{S}^1\right)^d \times \mathbb{R}^{D-d}$, with $1 \le d \le D$, where D represents the total number of Euclidean dimensions and d the number of compactified ones through the imposition of periodic boundary conditions on the components of the fields along them. One of these dimensions is compactified in a circumference of length β, whereas each of the spatial ones ($i = 1, \ldots, d-1$) in a circumference of length L_i and can be interpreted as boundaries of the Euclidean space [50,52]. In short, this corresponds to impose periodic (antiperiodic, for fermions) boundary conditions for fields in D Euclidean dimensions with d compactified ones.

In the present paper we revisit the Casimir effect, as an application of the above mentioned generalized Matsubara formalism. We investigate the pressure experienced by the boundary in a compactified space when a scalar field is heated. The starting point is the so-called "local formulation", introduced in [14], in which the pressure is associated with the 33 component of the energymomentum tensor. Then, we follow the zeta-function regularization method originally employed by Elizalde and Romeo [51] for the computation of the Casimir energy. However, here we employ a general formalism of field theories on toroidal spaces as in Ref. [52], which allows to apply the method for several simultaneously compactified dimensions. This is the case, for instance, of thermal field theories with a finite spatial extension, which needs the compactification of both the imaginarytime dimension and a spatial one for a unified approach for heated Casimir cavities.

We stress that in our computation with the toroidal formalism periodic boundary conditions are implemented both in imaginary time (circumference of length β) and the third spatial coordinate (circumference of length L), by construction. Moreover, as stated in [52], results for other boundary conditions may be obtained from theperiodic ones. For instance, the pressure for Dirichlet boundary conditions (much studied in the literature) can be determined by putting $L = 2a$ in the expression from the toroidal computation, where a is the distance separating the parallel plates in Ref. [14].

The paper is organized as follows. In section II the Casimir pressure is linked to the vacuum expectation value of the energy-momentum tensor for a scalar field in D dimensions of the Euclidean space. The point-splitting technique is used to write it in terms of the free scalar propagator in Fourier space. In section III a corresponding expression for the pressure is obtained when one of the spatial dimensions is compactified with a finite extension. The computation of the Casimir pressure follows a path similar to that of the Elizalde–Romeo method [51], leading to a well-known result from the literature. In section IV, we compute the Casimir pressure in the configuration of a compactified spatial dimension in the presence of a thermal bath. This can also be compared with results found in the literature obtained from other techniques. In section V we present our final comments. Throughout this paper, we consider $\hbar = c = k_B = 1$.

2. Energy-momentum Tensor Forscalar Fields

We start by writing the Euclidean Lagrangian of the free scalar field in a D-dimensional space,

$$\mathcal{L}_E = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2, \tag{1}$$

where m is the mass of the quanta of the scalar field. With the help of the point-splitting technique, the vacuum expectation value of the canonical energy-momentum tensor $T_{\mu\nu}$ can be written as [52]

$$\mathcal{T}_{\mu\nu} = \left\langle 0\left|T_{\mu\nu}\right|0\right\rangle = \lim_{x'\to x}\mathcal{O}_{\mu\nu}\left(x,x'\right)\left\langle 0\left|T\phi(x)\phi\left(x'\right)\right|0\right\rangle, \tag{2}$$

where T denotes the time-ordered product of field operators and $\mathcal{O}_{\mu\nu}\left(x,x'\right)$ is a differential operator given by [52]

$$\mathcal{O}_{\mu\nu}\left(x,x'\right) = \partial_\mu\partial'_\nu - \frac{1}{2}\delta_{\mu\nu}\left[\partial_\sigma\partial'_\sigma + m^2\right], \tag{3}$$

where ∂_μ and ∂'_μ are derivatives acting on x^μ and x'^μ, respectively, and $\delta_{\mu\nu}$ represents the components of the metric tensor of the Euclidean space (Kronecker delta). Defining the Euclidean Green function of the scalar field as $G\left(x-x'\right) = i\left\langle 0\left|T\left\{\phi(x)\phi\left(x'\right)\right\}\right|0\right\rangle$, we obtain

$$\mathcal{T}_{\mu\nu} = \lim_{x'\to x}\mathcal{O}_{\mu\nu}\left(x,x'\right)\left[G\left(x-x'\right)\right]. \tag{4}$$

Considering the Fourier integral of the Euclidean Green function in momentum space,

$$G\left(x-x'\right) = \int_{-\infty}^{\infty}\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+m^2}e^{ik\left(x-x'\right)}, \tag{5}$$

where k and x are D-dimensional vectors, we are able to rewrite the vacuum expectation value of the energy-momentum tensor in Eq. (4) as

$$\mathcal{T}_{\mu\nu} = \int_{-\infty}^{\infty}\frac{d^Dk}{(2\pi)^D}\left[\frac{k_\mu k_\nu}{k^2+m^2} - \frac{1}{2}\delta_{\mu\nu}\right]. \tag{6}$$

3. Casimir Pressure in a Compactified Space

In this section, we investigate the Casimir pressure for the particular case of just one compactified spatial dimension ($d = 1$). It is sufficient to consider the 33 component of the energy-momentum tensor to obtain the Casimir pressure resulting from a topological constraint imposed by periodic boundary conditions on the field at the parallel plates (taken as infinite planes) separated by a fixed distance L in the x_3-direction.

From Eq. (6), it is straightforward to write the bulk expression

$$\mathcal{T}_{33} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d^Dk}{(2\pi)^D}\left[\frac{k_3^2-\left(k_\perp^2+m^2\right)}{k_3^2+k_\perp^2+m^2}\right], \tag{7}$$

where $k^2 = k_3^2 + k_\perp^2$, and $k_\perp$ refers to the ($D-1$)-dimensional vector orthogonal to the 3-direction in Fourier space.

Let us call $\mathcal{T}_{33}^c$ the response of vacuum fluctuations on the plates, viewed as a topological constraint. We perform this by means of the compactification of just one spatial dimension. In order to obtain the Casimir pressure that acts on the boundary of the compactified space, we shall use the generalized Matsubara procedure, which is the original contribution of the present manuscript. Basically, in the general case, the technique consists in the replacement of integrals in momentum space by sums, namely,

$$\int \frac{dk_j}{2\pi} \to \frac{1}{L_j} \sum_{n_j=-\infty}^{+\infty}$$

where the index j assumes the values $j = 1, 2, \ldots, D-1$, the momentum coordinate k_j exhibits discrete values,

$$k_j = k_{n_j} = \frac{2\pi n_j}{L_j},$$

and L_j refer to the finite extension of each of the j spatial dimensions (compactification of $D-1$ spatial coordinates). For practical purposes, let us compactify just the x_3-component of the vector x. With these ideas in mind, the generalized Matsubara formalism enables us to substitute the bulk expression of Eq.(7) by the following one:

$$\mathcal{T}_{33}^{c} = \frac{1}{2L} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{d^{D-1}k_{\perp}}{(2\pi)^{D-1}} \left[\frac{k_n^2 - \left(k_{\perp}^2 + m^2\right)}{k_n^2 + k_{\perp}^2 + m^2} \right]. \tag{8}$$

Using well-known results from dimensional regularization techniques, we get,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 + b^2\right]^s} \\ &= \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(s - \frac{D}{2}\right)}{\Gamma(s)} \times \left(\frac{1}{b^2}\right)^{s-\frac{D}{2}}, \end{aligned} \tag{9}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{k^2}{\left[k^2 + b^2\right]^s} \\ &= \frac{D}{2} \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(s - \frac{D}{2} - 1\right)}{\Gamma(s)} \times \left(\frac{1}{b^2}\right)^{s-\frac{D}{2}-1}, \end{aligned} \tag{10}$$

we obtain

$$\begin{aligned} \mathcal{T}_{33}^{c} = \Bigg\{ f_s(\nu, L) \sum_{n=-\infty}^{+\infty} \Bigg[&\frac{\left(an^2 - c^2\right)\Gamma(\nu)}{\left(an^2 + c^2\right)^{\nu}} \\ &- (s-\nu)\Gamma(\nu-1) \frac{1}{\left(an^2 + c^2\right)^{\nu-1}} \Bigg] \Bigg\}_{s=1}, \end{aligned} \tag{11}$$

where $a = L^{-2}$, $c = m/2\pi$, $\nu = s - (D-1)/2$, and $f_s(\nu, L)$ a function given by

$$f_s(\nu, L) = \frac{1}{2L} \frac{1}{(4\pi)^{s-\nu} (2\pi)^{2(\nu-1)} \Gamma(s)}. \tag{12}$$

Adding and subtracting the term $c^2\Gamma(\nu)$ to the numerator of the first term on the right-hand side of Eq. (11), we obtain

$$\begin{aligned} \mathcal{T}_{33}^{c} = \Bigg\{ f_s(\nu, L) \Bigg[&(2\nu - s - 1) \sum_{n=-\infty}^{+\infty} \frac{1}{\left(an^2 + c^2\right)^{\nu-1}} \\ &- 2c^2(\nu-1) \sum_{n=-\infty}^{+\infty} \frac{1}{\left(an^2 + c^2\right)^{\nu}} \Bigg] \Bigg\}_{s=1}, \end{aligned} \tag{13}$$

where we have used the relation $\Gamma(\nu) = (\nu-1)\Gamma(\nu-1)$. Recalling the general definition of themultidimensional Epstein--Hurwitz zeta function [51,54,55,56],

$$\begin{aligned} &Z_d^{c^2}(\nu; a_1, \ldots, a_d) \\ &= \sum_{n_1, \ldots, n_d = -\infty}^{+\infty} \left(a_1 n_1^2 + \cdots + a_d n_d^2 + c^2\right)^{-\nu}, \end{aligned} \tag{14}$$

In the particular case of one-dimensional compactification ($d = 1$), the above formula simplifies to

$$Z_1^{c^2}(\nu; a) = \sum_{n=-\infty}^{+\infty} \left(an^2 + c^2\right)^{-\nu}. \tag{15}$$

Substituting the previous expression into Eq. (13), the pressure can then be rewritten as

$$\begin{aligned} \mathcal{T}_{33}^{c} = \Bigg\{ f_s(\nu, L) \Big[&(2\nu - s - 1) Z_1^{c^2}(\nu - 1; a) \\ &- 2c^2(\nu - 1) Z_1^{c^2}(\nu; a) \Big] \Bigg\}_{s=1}. \end{aligned} \tag{16}$$

Following Ref [37], these zeta functions can be evaluated on the whole complex plane by means of an analytic continuation along lines similar to those described in Refs. [51,54,55,56]

$$\begin{aligned} &Z_d^{c^2}(\nu; a_1, \ldots, a_d) \\ &= \frac{2\pi^{\frac{d}{2}}}{\sqrt{a_1 \cdots a_d}\,\Gamma(\nu)} \Bigg[\frac{1}{2c^{2\nu-d}} \Gamma\left(\nu - \frac{d}{2}\right) \\ &+ 2 \sum_{j=1}^{d} \sum_{n_j=1}^{\infty} \left(\frac{\pi n_j}{c\sqrt{a_j}}\right)^{\nu-\frac{d}{2}} K_{\nu-\frac{d}{2}}\left(2\pi c \frac{n_j}{\sqrt{a_j}}\right) + \ldots \\ &+ 2^d \sum_{n_1, \ldots, n_d = 1}^{\infty} \left(\frac{\pi}{c}\sqrt{\frac{n_1^2}{a_1} + \cdots + \frac{n_d^2}{a_d}}\right)^{\nu-\frac{d}{2}} \\ &\times K_{\nu-\frac{d}{2}}\left(2\pi c \sqrt{\frac{n_1^2}{a_1} + \cdots + \frac{n_d^2}{a_d}}\right) \end{aligned} \tag{17}$$

where $K_{\nu}(z)$ denotes modified Bessel functions of the second kind. For $d=1$, the analytical continuation above is reduced to

$$Z_1^{c^2}(\nu;a)=\frac{2\pi^{\frac{1}{2}}}{\sqrt{a}\,\Gamma(\nu)}\left[\frac{1}{2c^{2\nu-1}}\Gamma\left(\nu-\frac{1}{2}\right)\right.$$
$$\left.+2\sum_{n=1}^{\infty}\left(\frac{\pi n}{c\sqrt{a}}\right)^{\nu-\frac{1}{2}}K_{\nu-\frac{1}{2}}\left(2\pi c\frac{n}{\sqrt{a}}\right)\right]. \tag{18}$$

After some algebraic manipulations, we notice the presence of terms which are independent of the variable L, and for this reason are considered unphysical. Neglecting these terms, we can show that

$$\mathcal{T}_{33}^{c}=2\left(\frac{m}{2\pi L}\right)^{\frac{D}{2}}\left[(1-D)\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{D}{2}}K_{\frac{D}{2}}(mnL)\right.$$
$$\left.-mL\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{D}{2}-1}K_{\frac{D}{2}-1}(mnL)\right]. \tag{19}$$

The formula above corresponds to a general expression for the Casimir pressure exerted by the vacuum fluctuations on the boundaries of the compactified manifold formed of two parallel planes separated by a length L. The result presented in Eq. (19) is the Casimir vacuum radiation pressure for a massive scalar field submitted to periodic boundary conditions in D dimensions and is in agreement with Refs. [7,57,58].

For a 4-dimensional Euclidean space, we obtain [58]

$$\mathcal{T}_{33}^{c}(L,m)=-\frac{m^2}{2\pi^2L^2}\left[3\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(mnL)\right.$$
$$\left.+mL\sum_{n=1}^{\infty}\frac{1}{n}K_1(mnL)\right]. \tag{20}$$

From the following asymptotic formula of the Bessel function,

$$K_{\nu}(z)\approx 2^{\nu-1}z^{-\nu}\Gamma(\nu), \tag{21}$$

evaluated for small values of its argument $(z\sim 0)$ and $\mathcal{R}\mathrm{e}\,(\nu)>0$, we obtain the small-mass limit Casimir pressure ($mL\ll 1$)

$$\mathcal{T}_{33}^{c}(L,0)=-\frac{\pi^2}{30L^4}, \tag{22}$$

where we have neglected terms of $\mathcal{O}\left(m^2\right)$. The vacuum fluctuation Casimir force per unit area is a finite negative expression which suggests that the radiation pressure tends to diminish the distance L between the planes.

An interesting comment we present to finalize this section is that the corresponding negative Casimir pressure between two infinitely parallel planes, when one imposes to the massless scalar field Dirichlet boundary conditions, that is, $\phi(x_3=0)=\phi(x_3=L)=0$, is immediately recovered when the plane separation distance a is equal to the half circumference length L of the space dimension under compactification.

4. Thermal Effects

In this section, thermal and boundary effects are treated simultaneously through the generalized Matsubara prescription. We then consider a D -dimensional space with a double compactification ($d=2$) of the Euclidean space corresponding to a compactified spatial dimension with length L and a compactification of the imaginary-time dimension with length β. In other words, we take the simultaneous compactification of both the x_0 and x_3 coordinates of the vector x.

Following the same steps as in the previous sections, the stress tensor component $\mathcal{T}_{33}^{c}$ given by Eq. (7) now becomes

$$\mathcal{T}_{33}^{c}=\frac{1}{2\beta L}\sum_{n_1,n_2=-\infty}^{+\infty}\int_{-\infty}^{\infty}\frac{d^{D-2}k_{\perp}}{(2\pi)^{D-2}}$$
$$\times\left[\frac{k_{n_1}^2-k_{n_2}^2-\left(k_{\perp}^2+m^2\right)}{k_{n_1}^2+k_{n_2}^2+k_{\perp}^2+m^2}\right] \tag{23}$$

Using dimensional regularization, Eqs. (9) and (10), the previous formula is rewritten as

$$\mathcal{T}_{33}^{c}=\left\{f_s(\nu,\beta,L)\left[\sum_{n_1,n_2=-\infty}^{+\infty}\frac{\left[a_1n_1^2-a_2n_2^2-c^2\right]\Gamma(\nu)}{\left[a_1n_1^2+a_2n_2^2+c^2\right]^{\nu}}\right.\right.$$
$$\left.\left.-\sum_{n_1,n_2=-\infty}^{+\infty}\frac{(s-\nu)\Gamma(\nu-1)}{\left[a_1n_1^2+a_2n_2^2+c^2\right]^{\nu-1}}\right]\right\}_{s=1}, \tag{24}$$

where $a_1=L^{-2}$, $a_2=\beta^{-2}$, $c=m/2\pi$, $\nu=s-(D-2)/2$, and $f_s(\nu,\beta,L)$ is a function given by

$$f_s(\nu,\beta,L)=\frac{1}{2\beta L}\frac{1}{(4\pi)^{s-\nu}(2\pi)^{2(\nu-1)}\Gamma(s)}. \tag{25}$$

Adding and subtracting the term $\left(a_2n_2^2+c^2\right)\Gamma(\nu)$ in the numerator of the first term on the right-handside of Eq. (24), we obtain

$$\mathcal{T}_{33}^{c}=\left\{f_s(\nu,\beta,L)\Gamma(\nu-1)\left[(2\nu-s-1)\right.\right.$$
$$\times Z_2^{c^2}(\nu-1;a_1,a_2)-2c^2(\nu-1)Z_2^{c^2}(\nu;a_1,a_2)$$
$$\left.\left.+2a_2\frac{\partial}{\partial a_2}Z_2^{c^2}(\nu-1;a_1,a_2)\right]\right\}_{s=1}, \tag{26}$$

where we have used the definition of the two-dimensional Epstein-Hurwitz zeta function, $Z_2^{c^2}(\nu;a_1,a_2)$, obtained from Eq. (14) for $d=2$. From Eq. (17), we get for $d=2$

$$\begin{aligned}Z_2^{c^2}(\nu;a_1,a_2)=&\frac{2\pi}{\sqrt{a_1a_2}\,\Gamma(\nu)}\Bigg[\frac{1}{2c^{2(\nu-1)}}\Gamma(\nu-1)\\&+2\sum_{n_1=1}^{\infty}\left(\frac{\pi n_1}{c\sqrt{a_1}}\right)^{\nu-1}K_{\nu-1}\left(2\pi c\frac{n_1}{\sqrt{a_1}}\right)\\&+2\sum_{n_2=1}^{\infty}\left(\frac{\pi n_2}{c\sqrt{a_2}}\right)^{\nu-1}K_{\nu-1}\left(2\pi c\frac{n_2}{\sqrt{a_2}}\right)\\&+2^2\sum_{n_1,n_2=1}^{\infty}\left(\frac{\pi}{c}\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)^{\nu-1}\\&\times K_{\nu-1}\left(2\pi c\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)\Bigg].\end{aligned}\tag{27}$$

Substituting Eq. (27) in Eq. (26), splitting $\mathcal{T}_{33}^c$ into three terms, $\mathcal{T}_{33}^c=\mathcal{T}_{\mathrm{n}_1}^c+\mathcal{T}_{\mathrm{n}_2}^c+\mathcal{T}_{\mathrm{n}_1\mathrm{n}_2}^c$, after removing removing nonphysical terms, we have

$$\begin{aligned}\mathcal{T}_{\mathrm{n}_1}^c=&\frac{4\pi}{\sqrt{a_1a_2}}f_s(\nu,\beta,L)(2\nu-s-2)\sum_{n_1=1}^{\infty}\left(\frac{\pi n_1}{c\sqrt{a_1}}\right)^{\nu-2}\\&\times K_{\nu-2}\left(2\pi c\frac{n_1}{\sqrt{a_1}}\right)-2c^2\sum_{n_1=1}^{\infty}\left(\frac{\pi n_1}{c\sqrt{a_1}}\right)^{\nu-1}\\&\times K_{\nu-1}\left(2\pi c\frac{n_1}{\sqrt{a_1}}\right)\Bigg]\Bigg|_{s=1},\end{aligned}\tag{28}$$

which corresponds to the contribution to the Casimir pressure due to vacuum fluctuations only. Using the definition (25), for $a_1=L^{-2}$, $a_2=\beta^{-2}$, $c=m/2\pi$, $\nu=s-(D-2)/2$, Eq. (19) the result shown in the previous section is recovered.

Also,

$$\begin{aligned}\mathcal{T}_{\mathrm{n}_2}^c=&\frac{4\pi}{\sqrt{a_1a_2}}f_s(\nu,\beta,L)\Big[(2\nu-s-2)\\&\times\sum_{n_2=1}^{\infty}\left(\frac{\pi n_2}{c\sqrt{a_2}}\right)^{\nu-2}K_{\nu-2}\left(2\pi c\frac{n_2}{\sqrt{a_2}}\right)\\&-2c^2\sum_{n_2=1}^{\infty}\left(\frac{\pi n_2}{c\sqrt{a_2}}\right)^{\nu-1}K_{\nu-1}\left(2\pi c\frac{n_2}{\sqrt{a_2}}\right)\\&+2a_2\frac{\partial}{\partial a_2}\sum_{n_2=1}^{\infty}\left(\frac{\pi n_2}{c\sqrt{a_2}}\right)^{\nu-2}K_{\nu-2}\left(2\pi c\frac{n_2}{\sqrt{a_2}}\right)\Big]\Bigg|_{s=1},\end{aligned}\tag{29}$$

yields

$$\mathcal{T}_{\mathrm{n}_2}^c(\beta,m)=2\left(\frac{m}{2\pi\beta}\right)^{\frac{D}{2}}\sum_{n_2=1}^{\infty}\left(\frac{1}{n_2}\right)^{\frac{D}{2}}K_{\frac{D}{2}}(m\beta n_2),\tag{30}$$

which is the Casimir force formula due exclusively to the thermal fluctuations. The final form of Eq. (30) was obtained by means of the useful recurrence formula for Bessel functions,

$$K_{\alpha-1}(z)-K_{\alpha+1}(z)=-\frac{2\alpha}{z}K_\alpha(z).\tag{31}$$

For $D=4$, we find

$$\mathcal{T}_{n_2}^c(\beta,m)=\left(\frac{m^2}{2\pi^2\beta^2}\right)\sum_{n_2=1}^{\infty}\left(\frac{1}{n_2}\right)^2K_2(m\beta n_2).\tag{32}$$

Using Eq. (21), we obtain the small-mass limit for the purely thermal Casimir pressure ($m\beta\ll1$)

$$\mathcal{T}_{n_2}^c c(\beta,0)=\frac{\pi^2}{90\beta^4},\tag{33}$$

which is in accordance with the well-known Stefan-Bolztmann thermal radiation pressure result. This is a finite positive force per unit area which is more intense than vacuum radiation Casimir pressure for low values of β (high-temperature or classical limit).

If we plot the ratio between the thermal radiation pressure for the massive scalar field (Eq. (32)) and the massless one (Eq. (33), as a function of the dimensionless parameter $m\beta$, the normalized thermal Casimir force per unit area $\mathcal{T}_{\mathrm{n}_2}^c(\beta,m)/\mathcal{T}_{\mathrm{n}_2}^c(\beta,0)$ presents the typical monotonically decreasing shape for increasing values of the parameter $m\beta$.

Finally, the formula

$$\begin{aligned}\mathcal{T}_{\mathrm{n}_1\mathrm{n}_2}^c=&\frac{8\pi}{\sqrt{a_1a_2}}f_s(\nu,\beta,L)\\&\times\Bigg\{(2\nu-s-2)\sum_{n_1,n_2=1}^{\infty}\left(\frac{\pi}{c}\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)^{\nu-2}\\&\times K_{\nu-2}\left(2\pi c\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)\\&-2c^2\sum_{n_1,n_2=1}^{\infty}\left(\frac{\pi}{c}\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)^{\nu-1}\\&\times K_{\nu-1}\left(2\pi c\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)\\&+2a_2\frac{\partial}{\partial a_2}\sum_{n_1,n_2=1}^{\infty}\Bigg[\left(\frac{\pi}{c}\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)^{\nu-2}\\&\times K_{\nu-2}\left(2\pi c\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)\Bigg]\Bigg\}\Bigg|_{s=1},\end{aligned}\tag{34}$$

or

$$\begin{aligned}\mathcal{T}^{c}_{\mathrm{n_1n_2}}(L,\beta,m)=4\left(\frac{m}{2\pi}\right)^{\frac{D}{2}}\Bigg[&\sum_{n_1,n_2=1}^{\infty}\left(\frac{1}{\sqrt{n_1^2L^2+n_2^2\beta^2}}\right)^{\frac{D}{2}}\\ &\times\left(\frac{(1-D)n_1^2L^2+n_2^2\beta^2}{n_1^2L^2+n_2^2\beta^2}\right)\\ &\times K_{\frac{D}{2}}\left(m\sqrt{n_1^2L^2+n_2^2\beta^2}\right)\\ &-m\sum_{n_1,n_2=1}^{\infty}n_1^2L^2\left(\frac{1}{\sqrt{n_1^2L^2+n_2^2\beta^2}}\right)^{\frac{D}{2}+1}\\ &\times K_{\frac{D}{2}-1}\left(m\sqrt{n_1^2L^2+n_2^2\beta^2}\right)\Bigg],\end{aligned}\qquad(35)$$

gives the corrections to the Casimir pressure in a compact space in the presence of a massive scalar field heated at temperature $1/\beta$. In order to obtain the final form of the above expression, we have used the recurrence formula given by Eq. (31). Considering $D=4$, we get

$$\begin{aligned}\mathcal{T}^{c}_{\mathrm{n_1n_2}}(L,\beta,m)=-\left(\frac{m}{\pi}\right)^{2}\Bigg[&\sum_{n_1,n_2=1}^{\infty}\frac{3n_1^2L^2-n_2^2\beta^2}{\left(n_1^2L^2+n_2^2\beta^2\right)^2}\\ &\times K_2\left(m\sqrt{n_1^2L^2+n_2^2\beta^2}\right)\\ &+m\sum_{n_1,n_2=1}^{\infty}\frac{n_1^2L^2}{\left(n_1^2L^2+n_2^2\beta^2\right)^{\frac{3}{2}}}\\ &\times K_1\left(m\sqrt{n_1^2L^2+n_2^2\beta^2}\right)\Bigg],\end{aligned}\qquad(36)$$

which is valid for arbitrary values of m, L and β. Using Eq. (21), we can show that in the small-mass case it reduces to

$$\mathcal{T}^{c}_{\mathrm{n_1n_2}}(L,\beta,0)=-\frac{2}{\pi^2}\sum_{n_1,n_2=1}^{\infty}\frac{3n_1^2L^2-n_2^2\beta^2}{\left(n_1^2L^2+n_2^2\beta^2\right)^3},\qquad(37)$$

To clarify our results, we can show that the small-mass limit given by Eq. (37) can be written as

$$\mathcal{T}^{c}_{\mathrm{n_1n_2}}(L,\beta,0)=\frac{1}{L^4}\left[3f(\xi)+\xi s(\xi)\right],\qquad(38)$$

where $\xi=L/\beta$ and

$$f(\xi)=-\frac{1}{8\pi^2}\sum_{n_1,n_2=1}^{\infty}\frac{(2\xi)^4}{\left[(\xi n_1)^2+(n_2)^2\right]^2},\qquad(39)$$

$$s(\xi)=-f'(\xi)=\frac{1}{\pi^2}\sum_{n_1,n_2=1}^{\infty}\frac{(2\xi)^3n_2^2}{\left[(\xi n_1)^2+(n_2)^2\right]^3}.\qquad(40)$$

The function $f(\xi)$ obeys the inversion symmetry formula,

$$f(\xi)=\xi^4\,f\left(\frac{1}{\xi}\right).\qquad(41)$$

This is an intriguing expression, known as temperature inversion symmetry, that enables us to obtain the low and high-temperature limits after simple algebraic manipulations, (see Refs. [14,18,59-65] for more details). Following [14], the particular low-temperature limit ($\beta\gg1$) can be more easily performed after we compute the sum over index n_1 in Eq. (39)

$$\begin{aligned}f(\xi)=&\frac{\xi^4}{\pi^2}\sum_{n_2=1}^{\infty}\frac{1}{n_2^4}-\frac{\xi^3}{2\pi}\sum_{n_2=1}^{\infty}\frac{\coth(\pi n_2/\xi)}{n_2^3}\\ &-\frac{\xi^2}{2}\sum_{n_2=1}^{\infty}\frac{1}{n_2^2}\frac{1}{\sinh^2(\pi n_2/\xi)}.\end{aligned}\qquad(42)$$

In the limit $\xi\gg1$, the approximations

$$\coth(\pi n_2/\xi)\approx1,\qquad(43)$$

$$\sinh(\pi n_2/\xi)\approx\frac{1}{2}e^{\pi n_2/\xi},\qquad(44)$$

are valid. Substituting Eqs. (43) and (44) into Eq. (42), and performing the sum over index n_2, we find, for $\xi\ll1$,

$$f(\xi)=\frac{\pi^2}{90}\xi^4-\frac{\zeta(3)}{2\pi}\xi^3-2\xi^2\left(1+\frac{\xi}{\pi}\right)e^{-2\pi/\xi}+\mathcal{O}\left(e^{-4\pi/\xi}\right).\qquad(45)$$

Inserting the above formula into Eq. (38), we can show that

$$\mathcal{T}^{c}_{\mathrm{n_1n_2}}(L,\beta,0)=-\frac{\pi^2}{90\beta^4}+\frac{4\pi}{\beta L^3}\left(1+\frac{L}{2\pi\beta}\right)e^{-2\pi\beta/L}.\qquad(46)$$

In this sense, in the low-temperature limit ($L\ll\beta$), collecting all the contributions, the final form of Casimir pressure in the massless case reads

$$\mathcal{T}^{c}_{33}(L,\beta,0)=-\frac{\pi^2}{30L^4}+\frac{4\pi}{\beta L^3}e^{-2\pi\beta/L}.\qquad(47)$$

If we neglect the exponential factor, the Casimir pressure due exclusively to the vacuum fluctuations is dominant in this regime.

The high-temperature limit is also easily found by means of the inversion symmetry relation given by Eq. (41). Applying this formula in Eq. (45), we get

$$\begin{aligned}f(\xi)=&\frac{\pi^2}{90}-\frac{\zeta(3)}{2\pi}\xi-2\xi^2\left(1+\frac{1}{\pi\xi}\right)e^{-2\pi\xi}\\ &+\mathcal{O}\left(e^{-4\pi\xi}\right).\end{aligned}\qquad(48)$$

Substituting Eq. (48) into Eq. (38), we find

$$\mathcal{T}^{c}_{n_1 n_2}(L,\beta,0) = \frac{\pi^2}{30L^4} - \frac{\zeta(3)}{\pi\beta L^3} - \frac{1}{\beta L^3}\left(\frac{4\pi L^2}{\beta^2} + \frac{6L}{\beta} + \frac{4}{\pi}\right) e^{-2\pi L/\beta}. \tag{49}$$

Finally, in the high-temperature limit ($L \gg \beta$), computing all terms,the final form of Casimir pressure is written as follows:

$$\mathcal{T}^{c}_{33}(L,\beta,0) = \frac{\pi^2}{90\beta^4} - \frac{\zeta(3)}{\pi\beta L^3} - \frac{1}{\beta L^3}\left(\frac{4\pi L^2}{\beta^2} + \frac{6L}{\beta} + \frac{4}{\pi}\right) e^{-2\pi L/\beta}. \tag{50}$$

Notice that if we neglect the exponential factor, the Casimir pressure for large temperature is given by the classical thermal radiation pressure $\pi^2/\left(90\beta^4\right)$ plus a negative linear correction factor proportional to β^{-1}.

5. Final Remarks

In the present work we investigate some aspects of the Casimir effect in the context of field theories in nontrivial topologies. In particular, we revisited the Casimir effect for a massive scalar field in a heated compact space by means of the generalized Matsubara formalism. The usual attractive response of quantum and thermal fluctuations are obtained and our results are in accordance with those found in the literature. One may notice that all thermal contributions to the Casimir pressure, given by $\mathcal{T}^{c}_{n_2}$ and $\mathcal{T}^{c}_{n_1 n_2}$, vanish in the zero-temperature ($\beta \to \infty$) limit, remaining the pure dependence on the distance L between plates, which has a well-known L^{-4} dependence in the small- L limit for a four-dimensional space. Also, the bulk limit $L \to \infty$ reduces all expressions in $D = 4$ to the Stefan--Boltzmann law β^{-4}.

A rather peculiar aspect of the generalized Matsubara formalism is related to the renormalization of the expressions. Usually, in the Casimir context, the divergent terms are taken care of by subtraction of the bulk integral, without compactifications (see [52]). Here, there is no need to do so, as was also remarked by Elizalde and Romeo [51]. It is sufficient to obtain correct physical expressions to renormalize by subtracting the divergent term of the expansion of the Epstein--Hurwitz zeta functions $Z_d^{c^2}$, as it does not depend on the physical parameters L or β.

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Data Availability

The data that supports the findings of this study are available within the article and its quoted references.

References

[1] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).
[2] C. Farina, Braz. J. Phys. 36, 1137 (2006).
[3] P.W. Milonni, *The Quantum Vacuum* (Academic, San Diego, 1994).
[4] A.A. Actor, Fortschr. Phys. 43, 141 (1995).
[5] V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect and its Applications* (Clarendon Press, Oxford, 1997).
[6] M. Bordag, U. Mohideen, and V.M. Mostepanenko, Phys. Rep. 353, 1 (2001).
[7] K.A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2001).
[8] K.A. Milton, J. Phys. A 37, R209 (2004).
[9] S.K. Lamoreaux, Rep. Prog. Phys. 68, 201 (2005).
[10] A.W. Rodriguez, F. Capasso, and S.G. Johnson, Nat. Photonics 5, 211 (2011).
[11] D.A.R. Dalvit, P.A.M. Neto, and F.D. Mazzitelli, in *Casimir Physics*, edited by D.A.R. Dalvit, P. Milonni, D. Roberts, and F. da Rosa, Lecture Notes in Physics Vol. 834 (Springer, New York, 2011.
[12] M. Fierz, Helv. Phys. Acta 33, 855 (1960).
[13] J. Mehra, Physica 37, 145 (1967).
[14] L.S. Brown and G.J. Maclay, Phys. Rev. 184, 1272 (1969).
[15] R. Balian and B. Duplantier, Ann. of Phys. 104, 300 (1977); 112, 165 (1978).
[16] J.S. Dowker and R. Critchley, J. Phys. A 11, 895 (1978).
[17] G. Kennedy, R. Critchley, and J.S. Dowker, Ann. Phys. 125, 346 (1980).
[18] S. Tadaki and S. Takagi, Prog. Theor. Phys. 75, 262 (1986).
[19] G. Plunien, B. Muller, and W. Greiner, Phys. Rep. 134, 87 (1986).
[20] G. Plunien, B. Muller, and W. Greiner, Physica A 145, 202 (1987).
[21] D. Robaschik, K. Scharnhorst, and E. Wieczorek, Ann. Phys. 174 401 (1987).
[22] M. Bostr¨om and B.E. Sernelius, Phys. Rev. Lett. 84, 4757 (2000).
[23] V.B. Bezerra, G.L. Klimchitskaya, and V.M. Mostepa- nenko, Phys. Rev. A 66, 062112 (2002).
[24] J.S. Høye, I. Brevik, J.B. Aarseth, and K.A. Milton, Phys. Rev. E 67, 056116 (2003).
[25] R.S. Decca, D. L´opez, E. Fischbach, G.L. Klimchitskaya, D.E. Krause, V.M. Mostepanenko, Ann. Phys. 318, 37 (2005).
[26] I. Brevik, S.A. Ellingsen, and K.A. Milton, New J. Phys. 8, 236 (2006).
[27] F. Chen, G.L. Klimchitskaya, V.M. Mostepanenko, and U. Mohideen, Phys. Rev. B 76, 035338 (2007).
[28] G.L. Klimchitskaya, U. Mohideen, and V.M. Mostepa- nenko, Rev. Mod. Phys. 81, 1827 (2009).
[29] I. Brevik and J.S. Høye, Eur. J. Phys. 35, 015012 (2014).
[30] M.J. Sparnaay, Physica 24, 751 (1958).
[31] S.K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1997).
[32] U. Mo- hideen and A. Roy, Phys. Rev. Lett. 81, 4549 (1998).
[33] A. Roy and U. Mohideen, Phys. Rev. Lett. 82, 4380 (1999).
[34] A. Roy, C.-Y. Lin, and U. Mohideen, Phys. Rev. D 60, 111101(R) (1999).
[35] B.W. Harris, F. Chen and U. Mohideen, Phys. Rev. A 62, 052109 (2000).
[36] H.B. Chan,V.A. Aksyuk, R.N. Kleiman, D.J. Bishop, and F. Capasso, Science 291, 1941 (2001).
[37] A.P.C. Malbouisson, J.M.C. Malbouisson, and A.E. San- tana, Nucl. Phys. B 631, 83 (2002).
[38] A.P.C. Malbouisson, J.M.C. Malbouisson, A.E. Santana, and F.C. Khanna, Mod. Phys. Lett. A 20, 965 (2005).
[39] L.M. Abreu, C. de Calan, A.P.C. Malbouisson, J.M.C. Malbouisson, and A.E. Santana, J. Math. Phys. (N.Y.) 46, 012304 (2005).

[40] A.P.C. Malbouisson, J.M.C. Malbouisson, and R.C. Pereira, J. Math. Phys. (N.Y.) 50, 083304 (2009).

[41] E.B.S. Corrˆea, C.A. Linhares and A.P.C. Malbouisson, Phys. Lett. A 377, 1984 (2013).

[42] L.H. Ford, Phys. Rev. D 21, 933 (1980).

[43] H. Kleinert and A. Zhuk, Theor. Math. Phys. 108, 1236 (1996).

[44] J.C. da Silva, F.C. Khanna, A. Matos Neto, and A.E. Santana, Phys. Rev. A 66, 052101 (2002).

[45] H. Queiroz, J.C. da Silva, F.C. Khanna, J.M.C. Malbouisson, M. Revzen, and A.E. Santana, Ann. Phys.(N.Y.) 317, 220 (2005); 321, 1274 (2006).

[46] N. Ahmadi and M. Nouri-Zonoz, Phys. Rev. D 71, 104012 (2005).

[47] J.L. Tomazelli and L.C. Costa, Int. J. Theor. Phys. 45, 499 (2006).

[48] N.D. Birrell and L.H. Ford, Phys. Rev. D 22, 330 (1980).

[49] F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, and A.E. Santana, Ann. Phys. (N.Y.) 324, 1931 (2009).

[50] F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, and A. E. Santana, *Thermal Quantum Field Theory: Al- gebraic Aspects and Applications* (World Scientific, Sin- gapore, 2009).

[51] E. Elizalde and E. Romeo, J. Math. Phys. 30, 1133 (1989); J. Math. Phys. 31, 771(E) (1990).

[52] F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, and A.E. Santana, Phys. Rep. 539, 134 (2014).

[53] F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, and A.E. Santana, Ann. Phys. (N.Y.) 326, 2634 (2011).

[54] K. Kirsten, J. Math. Phys. 35, 459 (1994).

[55] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bitsenko, and S. Zerbini, *Zeta Regularization Techniques with Ap- plications* (World Scientific, Singapore, 1994).

[56] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions* (Springer, Berlin, 1995).

[57] J. Ambjørn and S. Wolfram, Ann. Phys. (N.Y.) 147, 1 (1983).

[58] A.C. Aguiar Pinto, T.M. Britto, R. Bunchaft, F. Pascoal, and F.S.S. da Rosa, Braz. J. Phys. 33, 860 (2003).

[59] S.A. Gundersen and F. Ravndal, Ann. Phys. (N.Y.) 182, 90 (1988).

[60] C. Lu¨tken and F. Ravndal, J. Math.Phys. A: Math. Gen. 21, L793 (1988).

[61] F. Ravndal and D. Tollefsen, Phys. Rev. D 40, 4191 (1989).

[62] C. Wotzasek, J. Math.Phys. A: Math. Gen. 23, 1627 (1990).

[63] F.C. Santos, A. Ten´orio, and A.C. Tort, Phys. Rev. D. 60, 105022 (1999).

[64] F.C. Santos and A.C. Tort, Phys. Lett. B, 482, 323 (2000).

[65] A.C. Aguiar Pinto, T.M Britto, F. Pascoal, and F.S.S. da Rosa, Phys. Rev. D 67, 107701 (2003).