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Maxwell Equations Derived from Coulomb' Law vs. Maxwell-type Gravity Derived from Newton's Law

Hui Peng*

*Corresponding author: davidpeng1749@gmail.com

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Abstract A Universal Mathematical Field Theory (UMFT) is established and states that the combination of the operations of both gradient and divergence of vector fields, such as electric field and velocity field, create the curl of an axial vector field, such as magnetic field. Utilizing UMFT, Extended-Maxwell equations and the equation of Lorentz force are derived from the combination of Coulomb's law and velocity of source mathematically, and new effects are predicted. For a source moving with non-spatially-varying velocity Extended-Maxwell equations reduce to Maxwell equations. This derivation mathematically explains how a moving electric charge creates magnetic field, and shows that there is no magnetic monopole charge. The duality between the Newton's law and the Coulomb's law leads us to derive Maxwell-type gravitational equations and Lorentz-type gravitational force by combining UMFT, the Newton's law and velocity of gravitational source, denoted as Gravito-electromagnetic, which is dual of Electromagnetics. The benefits of the duality are that the effects and phenomena of Electromagnetics may be directly converted to that of gravity. The Gravito-Electromagnetics are employed to study the accelerating expansion of the universe, rotation curve, virial theorem, and gravitation radiation.

Keywords: *Maxwell equation, Coulomb law, Ampere law, Faraday law, Gauss law for magnetism, Lorentz force, Gravito-electromagnetics, Newton law, rotation curve, acceleration of universe, gravitation radiation*

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1. Introduction

Historically, Maxwell equations were established based on series experiments, except the introduction of the term of the displacement current. Consider a situation of a stationary e-charge. The Coulomb's law states that a stationary e-charge Q_e induces a static vector electric field **E.** The created electric field is determined by only one quantity, e-charge.

A heuristic phenomenon is that the observer moving relative to the same e-charge and its electric field (or the e-charge moves relative to the observer) will observe not only the electric field but also a magnetic field. The magnetic field is determined by three quantities, e-charge, electric field and velocity (Table 1.1), which will be demonstrated in Section 3.2.

Table 1.1. Static electric charge vs. moving e-charge

	Quantity	Generation of Field
Static Electric Field	Q_e	$Q_e \to \mathbf{E}$ via Coulomb's law
Magnetic Field	Q_e, **E, v**	$(Q_e, \mathbf{E}, \mathbf{v}) \to \mathbf{B}$

Therefore, since magnetic phenomena are related with the motion of electric charge (e-charge) and are inevitable, we suggest that the Ampere's law, Faraday's law and Gauss's law of magnetism, which are all related with the motion of e-charges, should be derivable mathematically from Coulomb' law and velocity of source.

Let's first review the fundamental differences between a static electric field **E** and a magnetic field **B,** as shown in Table 1.2:

Table 1.2. Differences between static electric field and magnetic field

	Static electric field	Magnetic field
1.Source	Q_e	$Q_e\mathbf{v}$
2. Created via	$\nabla \cdot \mathbf{E}$	$\nabla \times \mathbf{B}$
3. Nature	vector field	axial vector field
4. Force	$q_e\mathbf{E}$	$q_e\mathbf{v} \times \mathbf{B}$

A physics student may ask questions: Why the motion of e-charge creates magnetic fields? Does the generation of magnetic fields relate with the Coulomb's law? The regular answer is that magnetism is the combination of electric field with Special Relativity (SR) and does not relate with the Coulomb's law. We argue that the answer is not sufficient to explain the above four fundamental differences.

Now we ask a further question: Is the generation of magnetic fields by a moving e-charge *inevitable*? The answer is yes. Thus, we argue that:

1. The Coulomb's law can be established only by experiments and thus, is a primary law;
2. Maxwell equations related with magnetic field should be mathematically derivable from velocity

of e-charge and Coulomb's law, and thus, are secondary laws.

To mathematically derive Maxwell equations from the Coulomb's law and velocity of an e-charge, we establish Universal Mathematical Field Theory (UMFT). Extended-Maxwell equations are derived from UMFT and Coulomb's law, which demonstrates the validity of UMFT. Recently, Faraday's law has been derived by curling the Lienard-Wiechert potential [1]. Ampere's law was derived from the Biot-Savart law [2].

Based on the similarity between the Coulomb's law and Newton's law, naturally, the same UMFT can be employed to derive Maxwell-type gravitational equations from Newton's law and the motion of a source, denoted as Gravito-Electromagnetics or Gravito-EM, which provides a clear physical picture of gravity. We show that both theories, Maxwell theory (denoted as EM) and Gravito-EM are convertible to each other by converting the e-charge to gravitational charge (g-charge), and vice versa, which we denote as "Ultra-symmetry".

Historically, Maxwell-type vector field theories of gravity were studied, which described gravity as a physical field. However, physicists gave up this direction before Einstein's General Relativity (GR), because they realized that such vector theories faced the issue: the potential energy of Newton field is negative. After SR was established, second issue raised that those vector theories of gravity did not comply with SR.

The benefits of the Gravito-EM are: (1) Address the long-standing issues of negative potential energy; (2) comply with SR; (3) explain the accelerating expansion of the universe and to predict a jerking expansion of the universe.

Therefore, we argue that the Gravito-EM provides, at least, a "bridge" between the Newton's theory and an ultimate theory of gravity. To understand and describe all of forces in one framework, a possible and simple way is to treat gravity as a physical field.

Recently, a gauge theory of gravity has been proposed, quantized and unified with other forces [3]. The Gravito-EM is consistent with this gauge theory of gravity.

2. Universal Mathematical Field Theory (UMFT)

2.1. Motivation

Motivation: To address above differences between a static electric field **E** and a magnetic field **B** inspires us to establish UMFT [4]. The UMFT is equally applicable to derive Electromagnetics (denoted as EM) and Gravito-EM. Thus, there is duality between EM and Gravito-EM, so that many concepts and effects of the well-established EM can be transferred directly to gravity. Most important, gravity and EM are linked together clearly and closely. Gravito-EM is powerful and fruitful.

The significances of UMFT are the following.

A) combining UMFT and the Coulomb's law re-derives *mathematically* Maxwell's equations, which addresses the above-mentioned differences related with EM (Table 1.2), shows that the experiments-based Maxwell equations have their mathematical origin, and justifies UMFT.
B) combining UMFT and the Newton's law derives *mathematically* Gravito-EM.
C) Thirdly, UMFT provides the mathematic origin of dualities between different physic fields derived from it, such as duality between electricity and magnetism, and duality between EM and Gravito-EM. Duality is a powerful tool to find intrinsic similarities between apparently different phenomena, and predict new effects.

"It turns out that most of the important concepts and theories of physics can be unified and understood by their common attribute of duality" (Damian P Hampshire).

2.2. General UMFT

To establish UMFT, we need to find a mathematical identity that connecting divergences of either a vector or an axial vector with curls of induced axial vectors. For this aim, the following vector analysis identity is the most significant foundation of UMFT,

$$\nabla\times(\mathbf{S}\times\mathbf{T}) = \mathbf{S}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{S})+(\mathbf{T}\cdot\nabla)\mathbf{S}-(\mathbf{S}\cdot\nabla)\mathbf{T} \quad (2.1)$$

which indicates that the combination of gradient and divergence of two arbitrary vectors induces inevitably an axial vector ($\mathbf{S}\times\mathbf{T}$). One of two terms, ($\nabla\cdot\mathbf{T}$) and ($\nabla\cdot\mathbf{S}$), represents fundamental inverse-square laws, and introduce a "charge". Now there are three quantities, S, T and "charge" in Eq. (2.1).

It is useful to write Eq. (2.1) in a different but equivalent form. By using another mathematical identity,

$$(\mathbf{T}\cdot\nabla)\mathbf{S} = \nabla(\mathbf{S}\cdot\mathbf{T})-(\mathbf{S}\cdot\nabla)\mathbf{T}-\mathbf{S}\times(\nabla\times\mathbf{T})-\mathbf{T}\times(\nabla\times\mathbf{S})$$

Eq. (2.1) can be rewritten as an identity,

$$\nabla\times(\mathbf{S}\times\mathbf{T}) = \mathbf{S}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{S})-\nabla(\mathbf{S}\cdot\mathbf{T}) ++2(\mathbf{T}\cdot\nabla)\mathbf{S}+\mathbf{S}\times(\nabla\times\mathbf{T})+\mathbf{T}\times(\nabla\times\mathbf{S}). \quad (2.2)$$

Eq. (2.1) and Eq. (2.2) are mathematical equivalent. When apply UMFT to describe physical fields, the "S" and "T" in Eq. (2.1) and Eq. (2.2) represent different physical quantities respectively.

2.3. UMFE Related with Velocity of Charges

The basic concept is that the combination of the inverse-square laws and the motion of charges must induce axial vector fields. To show it, we set the "**S**" being motion parameters, velocity **v**,

$$\mathbf{S} = \mathbf{v} \text{ (vector)} \quad (2.3)$$

Note: the physical quantities **S** representing are not limited to velocity.

Substituting Eq. (2.3) into Eq. (2.1) and Eq. (2.2) respectively, we obtain,

$$\nabla\times(\mathbf{v}\times\mathbf{T}) = \mathbf{v}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{v})+(\mathbf{T}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{T} \quad (2.4)$$

$$\nabla\times(\mathbf{v}\times\mathbf{T})=\mathbf{v}(\nabla\cdot\mathbf{T})-\mathbf{T}(\nabla\cdot\mathbf{v})-\nabla(\mathbf{v}\cdot\mathbf{T}) +2(\mathbf{T}\cdot\nabla)\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{T})+\mathbf{T}\times(\nabla\times\mathbf{v}). \quad (2.5)$$

Fortunately, Eq. (2.4) links both the motion of source and the inverse-square law to an axial vector field. In this section we start with Eq. (2.4) to derive the Maxwell-type equations for the fields induced by the velocity of sources.

Note in Eq. (2.4), the velocity is spatially varying, e.g., $\mathbf{T}(\nabla\cdot\mathbf{v})\neq 0$, $(\mathbf{T}\cdot\nabla)\mathbf{v}\neq 0$, and is instantaneous velocity at a given space point. Eq. (2.4) implies that a velocity and its spatial variations induce the axial vector $(\mathbf{v}\times\mathbf{T})$ field equally.

2.3.1. Ampere-Maxwell-type UMFT

Firstly, we derive the Ampere-Maxwell-type UMFE. Let's assume the "**T**" is an arbitrary vector field **G** and,

$$\nabla\cdot\mathbf{G}\neq 0. \quad (2.6)$$

Since

$$-(\mathbf{v}\cdot\nabla)\mathbf{G}=-\left[v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right]\mathbf{G}=\frac{\partial\mathbf{G}}{\partial t}-\frac{d\mathbf{G}}{dt},$$

substituting it into Eq. (2.4), we obtain the Ampere-Maxwell-type UMFE,

$$\nabla\times(\mathbf{v}\times\mathbf{G})+\frac{d\mathbf{G}}{dt} =\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial t}-\mathbf{G}(\nabla\cdot\mathbf{v})+(\mathbf{G}\cdot\nabla)\mathbf{v}. \quad (2.7)$$

Defining a First level axial vector field **M**,

$$\mathbf{M}\equiv\mathbf{v}\times\mathbf{G}. \quad (2.8)$$

For the axial vector field **M**, by mathematical definition, Eq. (2.8), we have,

$$\nabla\cdot\mathbf{M}\equiv 0. \quad (2.9)$$

Substituting Eq. (2.8) into Eq. (2.4) and Eq. (2.7) respectively, we obtain

$$\nabla\times\mathbf{M}=\mathbf{v}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{v}) +(\mathbf{G}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{G}, \quad (2.10)$$

$$\nabla\times\mathbf{M}+\frac{d\mathbf{G}}{dt}=\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial t}-\mathbf{G}(\nabla\cdot\mathbf{v})+(\mathbf{G}\cdot\nabla)\mathbf{v}. \quad (2.11)$$

All terms on the right-hand side of Eq. (2.11) induce equally the axial vector field **M**. The interpretations for those terms are,

1) The term, $\mathbf{v}(\nabla\cdot\mathbf{G})$, plays the role of the "current";
2) The term, $\frac{\partial\mathbf{G}}{\partial t}$, plays the role of the "displacement current";
3) The term, $\mathbf{G}(\nabla\cdot\mathbf{v})$, describes stretching of the G field due to source velocity compressibility;
4) The term, $(\mathbf{G}\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the G field due to the velocity gradients;
5) The terms, $\frac{\partial\mathbf{G}}{\partial t}$ and $\mathbf{G}(\nabla\cdot\mathbf{v})$, have the same direction; while the terms, $\mathbf{v}(\nabla\cdot\mathbf{G})$ and $(\mathbf{G}\cdot\nabla)\mathbf{v}$, have the same direction.

The Ampere-Maxwell-type UMFE, Eq. (2.7), Eq. (2.10), Eq. (2.11), can be written respectively in the integral form as,

$$\oint(\mathbf{v}\times\mathbf{G})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{G}\cdot ds =\iint\left[\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial t}-\mathbf{G}(\nabla\cdot\mathbf{v})+(\mathbf{G}\cdot\nabla)\mathbf{v}\right]\cdot ds, \quad (2.12)$$

$$\oint\mathbf{M}\cdot d\mathbf{l}=\int\left[\begin{array}{l}\mathbf{v}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{v})\\ +(\mathbf{G}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{G}\end{array}\right]\cdot ds, \quad (2.13)$$

$$\oint\mathbf{M}\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{G}\cdot ds ==\int\left[\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial t}-\mathbf{G}(\nabla\cdot\mathbf{v})+(\mathbf{G}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (2.14)$$

2.3.2. Faraday-type UMFT

For deriving the Faraday-type UMFE, taking the **T** field as the First level axial vector field **M** defined by Eq. (2.8), $\mathbf{T}=\mathbf{M}$. For the **M** field, we have,

$$-(\mathbf{v}\cdot\nabla)\mathbf{M}=-\left[v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right]\mathbf{M} =\frac{\partial\mathbf{M}}{\partial t}-\frac{d\mathbf{M}}{dt}. \quad (2.15)$$

Substituting Eq. (2.15) into Eq. (2.4), we obtain Faraday-type UMFE,

$$\nabla\times(\mathbf{v}\times\mathbf{M})+\frac{d\mathbf{M}}{d\mathbf{t}} =\mathbf{v}(\nabla\cdot\mathbf{M})+\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}. \quad (2.16)$$

Let's define a Second level axial vector field **N**,

$$\mathbf{N}\equiv-\mathbf{v}\times\mathbf{M}, \quad (2.17)$$

which satisfies mathematically,

$$\nabla\cdot\mathbf{N}=0. \quad (2.18)$$

Substituting Eq. (2.17) into Eq. (2.16), we obtain Faraday-type UMFE,

$$\nabla\times\mathbf{N}-\frac{d\mathbf{M}}{dt} =-\mathbf{v}(\nabla\cdot\mathbf{M})-\frac{\partial\mathbf{M}}{\partial t}+\mathbf{M}(\nabla\cdot\mathbf{v})-(\mathbf{M}\cdot\nabla)\mathbf{v}. \quad (2.19)$$

The Faraday-type UMFE, Eq. (2.16) and Eq. (2.19), can be written respectively in the integral form as,

$$\oint(\mathbf{v}\times\mathbf{M})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{M}\cdot ds =\iint\left[\mathbf{v}(\nabla\cdot\mathbf{M})+\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (2.20)$$

$$\oint\mathbf{N}\cdot d\mathbf{l}-\frac{d}{dt}\iint\mathbf{M}\cdot ds =-\iint\left[\mathbf{v}(\nabla\cdot\mathbf{M})+\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (2.21)$$

The term $\mathbf{v}(\nabla\cdot\mathbf{M})$ is the source term. We still keep the source term in Eq. (2.16), (2.19), (2.20) and (2.21), because this source term leaves a door open for a possible existence of a monopole physically, although, mathematically, the monopole of the **M** field does not exist, i.e., $\nabla\cdot\mathbf{M}\equiv 0$ (Eq. 2.9).

The interpretations of those right-hand side terms of Eq. (2.19) are,

1) The term, $\mathbf{v}(\nabla\cdot\mathbf{M})$, plays the role of the "current", which is mathematically zero;
2) The term, $\frac{\partial\mathbf{M}}{\partial t}$, describes the time change of the **M** fields as the source;
3) The term, $\mathbf{M}(\nabla\cdot\mathbf{v})$, describes stretching of the **M** field due to source velocity compressibility;
4) The term, $(\mathbf{M}\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the M field due to the velocity gradients;
5) The terms, $\frac{\partial\mathbf{M}}{\partial t}$ and $\mathbf{M}(\nabla\cdot\mathbf{v})$, have the same direction; while the terms, $\mathbf{v}(\nabla\cdot\mathbf{M})$ and $(\mathbf{M}\cdot\nabla)\mathbf{v}$, have the same direction.

Next, substituting Eq. (2.9), $\nabla\cdot\mathbf{M}=0$, into Eq. (2.16), Eq. (2.19), Eq. (2.20) and Eq. (2.21) respectively, we obtain source-free Faraday-type UMFE,

$$\nabla\times(\mathbf{v}\times\mathbf{M})+\frac{\mathbf{dM}}{\mathbf{dt}}=\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}, \quad (2.22)$$

$$\nabla\times\mathbf{N}-\frac{\mathbf{dM}}{\mathbf{dt}}=-\frac{\partial\mathbf{M}}{\partial t}+\mathbf{M}(\nabla\cdot\mathbf{v})-(\mathbf{M}\cdot\nabla)\mathbf{v}. \quad (2.23)$$

$$\begin{aligned}&\oint(\mathbf{v}\times\mathbf{M})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{M}\cdot ds\\&=\iint\left[\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot ds.\end{aligned} \quad (2.24)$$

$$\begin{aligned}&\oint\mathbf{N}\cdot d\mathbf{l}-\frac{d}{dt}\iint\mathbf{M}\cdot ds\\&=-\iint\left[\frac{\partial\mathbf{M}}{\partial\mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot ds.\end{aligned} \quad (2.25)$$

2.3.3. Type-2 Duality

The **M** field is a First level axial vector field, while the **N** field is a Second level axial vector field. The **M** field and the **N** field are determined respectively by Eq. (2.11) and Eq. (2.19), which have the same form. Thus, there is type-2 duality between the First level axial vector field **M** and the Second level axial vector field **N**, which is pre-determined mathematically. For different type of duality, different level of axial vector, see Appendix A.

We have derived the basic UMFT for the fields induced by velocity of sources.

2.3.4. UMFT Related with Non-Spatially-Varying Velocity

For non-spatially-varying velocity, we have

$$\mathbf{M}(\nabla\cdot\mathbf{v})=(\mathbf{M}\cdot\nabla)\mathbf{v}=\nabla\times\mathbf{v}=0, \quad (2.26)$$

Substituting Eq. (2.26) into Eq. (2.11), we obtain Ampere-Maxwell-type UMFE for the **M** field,

$$\nabla\times\mathbf{M}+\frac{d\mathbf{G}}{dt}=\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial t}. \quad (2.27)$$

Or in the integral forms,

$$\oint\mathbf{M}\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{G}\cdot ds=\iint\left[\mathbf{v}(\nabla\cdot\mathbf{G})+\frac{\partial\mathbf{G}}{\partial\mathbf{t}}\right]\cdot ds. \quad (2.28)$$

Substituting Eq. (2.26) into Eq. (2.23), we obtain Faraday-type UMFE for the N field,

$$\nabla\times\mathbf{N}-\frac{\mathbf{dM}}{\mathbf{dt}}=-\frac{\partial\mathbf{M}}{\partial\mathbf{t}}. \quad (2.29)$$

Or in the integral forms,

$$\oint\mathbf{N}\cdot d\mathbf{l}-\frac{d}{dt}\iint\mathbf{M}\cdot ds=-\iint\frac{\partial\mathbf{M}}{\partial t}\cdot ds. \quad (2.30)$$

3. Electromagnetics Derived from UMFT and Coulomb's Law

In Section 3, the extended-Maxwell equations are derived mathematically from the combination of UMFT and the Coulomb's law. Under condition that electric charges (e-charge) move with non-spatially-varying velocity, Extended-Maxwell equations reduce to regular Maxwell equations, which demonstrates that UMFT is a powerful tool and valid for studying physic fields and, thus leads us to apply UMFE to study other physical fields [4].

3.1. Extended-Faraday's Law

Starting from the Faraday-type UMFT, Eq. (2.20). Let the **M** field is a magnetic field **B**, $\mathbf{M}\equiv\mathbf{B}$, Eq. (2.20) gives,

$$\begin{aligned}&\oint-(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}-\frac{d}{dt}\iint\mathbf{B}\cdot ds\\&=-\iint\left[\mathbf{v}(\nabla\cdot\mathbf{B})+\frac{\partial\mathbf{B}}{\partial\mathbf{t}}-\mathbf{B}(\nabla\cdot\mathbf{v})+(\mathbf{B}\cdot\nabla)\mathbf{v}\right]\cdot ds.\end{aligned} \quad (3.1)$$

And the definition of **M**, Eq. (2.9) gives mathematically Gauss' s law of magnetism,

$$\nabla\cdot\mathbf{B}\equiv 0. \quad (3.2)$$

Let's show that Eq. (3.1) is consistent with Faraday's law.

The Faraday's law gives

$$\frac{d}{dt}\iint\mathbf{B}\cdot ds=-\oint\mathbf{E}'\cdot dl \quad (3.3)$$

where $\mathbf{E}'$ is the electric field at the circuit d**l** in a reference frame in which d**l** is at rest. The **B** is a magnetic field at the neighborhood of the circuit.

Applying Eq. (3.3), Eq. (3.1) becomes the Extended-Faraday law,

$$\begin{aligned}&\oint(\mathbf{E}'-\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}\\&=-\iint\left[\mathbf{v}(\nabla\cdot\mathbf{B})+\frac{\partial\mathbf{B}}{\partial\mathbf{t}}-\mathbf{B}(\nabla\cdot\mathbf{v})+(\mathbf{B}\cdot\nabla)\mathbf{v}\right]\cdot ds\end{aligned} \quad (3.4)$$

where the $\mathbf{v}$ is the velocity of the circuit $d\mathbf{l}$ relative to a laboratory frame.

Let's define an electric field $\mathbf{E}$ in the laboratory frame,

$$\mathbf{E} \equiv \mathbf{E}' - \mathbf{v} \times \mathbf{B}. \tag{3.5}$$

Applying Eq. (3.5), Eq. (3.4) gives the integral and differential forms of Extended-Faraday's law,

$$\oint \mathbf{E} \cdot dl = -\iint \begin{bmatrix} \mathbf{v}(\nabla \cdot \mathbf{B}) + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} \\ -\mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v} \end{bmatrix} \cdot \mathrm{ds}, \tag{3.6}$$

$$\nabla \times \mathbf{E} = -\mathbf{v}(\nabla \cdot \mathbf{B}) - \frac{\partial \mathbf{B}}{\partial \mathrm{t}} + \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{B} \cdot \nabla)\mathbf{v}. \tag{3.7}$$

In order to show the duality between the Extended-Faraday's and Extended-Ampere-Maxwell's law, we still keep the term $\nabla \cdot \mathbf{B}$ in Eq. (3.7), although it is equal to zero.

The interpretations of terms of the right-hand side of Eq. (3.7) are,

1) The term, $\mathbf{B}(\nabla \cdot \mathbf{v})$, describes stretching of the B field due to source velocity compressibility;
2) The term, $(\mathbf{B} \cdot \nabla)\mathbf{v}$, describes the stretching or tilting of the B field due to the velocity gradients.

Let's define a "current" generating induced electric field,

$$\mathbf{j}_{\mathbf{v}-\mathbf{E}} \equiv \mathbf{v}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}. \tag{3.8}$$

Where the subscripts "v-E" represent the quantities related with velocity and electric field respectively.

Then Extended-Faraday's law, Eq. (3.7), may be rewritten as,

$$\nabla \times \mathbf{E} = -\mathbf{j}_{\mathbf{v}-\mathbf{E}} - \frac{\partial \mathbf{B}}{\partial \mathrm{t}}. \tag{3.9}$$

Eq. (3.9) shows the equation of continuity as

$$\nabla \cdot \left(\mathbf{j}_{\mathbf{v}-\mathbf{E}} + \frac{\partial \mathbf{B}}{\partial \mathrm{t}} \right) = 0. \tag{3.10}$$

For the situation in which, (1) $\nabla \cdot \mathbf{B} = 0$; and (2) the velocity is non-spatially-varying, i.e., $\mathbf{B}(\nabla \cdot \mathbf{v}) = (\mathbf{B} \cdot \nabla)\mathbf{v} = 0$, Extended-Faraday's law, Eq. (3.7), gives the Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}}, \tag{3.11}$$

where the $\mathbf{E}$ field is an axial vector field and

$$\mathbf{j}_{\mathbf{v}-\mathbf{E}} = 0.$$

Therefore, Extended-Faraday's law, Eq. (3.1), is consistent with Faraday's law. Extended-Faraday's law predicts that the spatially-varying velocity, via terms $\mathbf{B}(\nabla \cdot \mathbf{v})$ and $(\mathbf{B} \cdot \nabla)\mathbf{v}$, create axial vector $\mathbf{E}$ fields.

To detect the effects of the terms, $\mathbf{B}(\nabla \cdot \mathbf{v})$ and $(\mathbf{B} \cdot \nabla)\mathbf{v}$, will confirm whether Extended-Faraday's law and thus, UMFT are correct.

Indeed, by the definition of the axial vector field $\mathbf{B}$, its divergence is zero mathematically, namely, from the perspective of the UMFT and Coulomb's law, there is no "magnetic monopole charge".

3.2. Extended-Ampere-Maxwell's Law (1)

The combination of UMFE and Coulomb's law leads us to let $\mathbf{G} \equiv \mathbf{E}$. Then Eq. (2.12) of UMFE provides,

$$\begin{aligned} &\oint (\mathbf{v} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{l} + \frac{\mathrm{d}}{\mathrm{dt}} \iint \mathbf{E} \cdot \mathrm{ds} \\ &= \iint \left[\mathbf{v}(\nabla \cdot \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial \mathrm{t}} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v} \right] \cdot \mathrm{ds}. \end{aligned} \tag{3.12a}$$

Substituting the Coulomb's law, Eq. 3.12a becomes,

$$\begin{aligned} &\oint (\mathbf{v} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{l} + \frac{\mathrm{d}}{\mathrm{dt}} \iint \mathbf{E} \cdot \mathrm{ds} \\ &= \iint \left[\mathrm{Q_e} \mathbf{v} + \frac{\partial \mathbf{E}}{\partial \mathrm{t}} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v} \right] \cdot \mathrm{ds}. \end{aligned} \tag{3.12b}$$

Eq. (3.12b) shows that it is the three quantities, $\mathrm{Q_e}, \mathbf{E}, \mathbf{v}$, that create magnetic field as we mentioned in Table 1.1

3.3. Type-2 Duality between Electric and Magnetic Fields

The First level axial vector field $(\mathbf{v} \times \mathbf{E})$ and Eq. (3.12) are the type-2 dual to the Second level axial vector field $(\mathbf{v} \times \mathbf{B})$ and Eq. (3.1) respectively. Moreover, Eq. (3.6) is equivalent to Eq. (3.1). Based on the Transfer Rules between dualities of Appendix A, there is a dual of Eq. (3.6), i.e., under transformation,

$$\mathbf{E} \leftrightarrow \mathbf{B} \text{ and } \mathbf{B} \leftrightarrow -\mathbf{E},$$

we have a type-2 dual of Eq. (3.5) and Eq. (3.6), which are,

$$\mathbf{E} \equiv \mathbf{E}' - \mathbf{v} \times \mathbf{B} \leftrightarrow \mathbf{B} \equiv \mathbf{B}' + \mathbf{v} \times \mathbf{E},$$

$$\oint \mathbf{B} \cdot dl = \iint \begin{bmatrix} \mathbf{v}(\nabla \cdot \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial \mathbf{t}} \\ -\mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v} \end{bmatrix} \cdot \mathrm{ds}, \tag{3.13}$$

which should be equivalent to Eq. (3.6) and thus, to Eq. (3.1), and finally, to Eq. (3.12a) and Eq. (3.12b).

With distinguishable feature of "type-1 duality" and "type-2 duality", the duality between induced axial electric field determined by the Faraday's law and magnetic field determined by Ampere-Maxwell's equation is actually a type-2 duality. UMFE provides the mathematical originations of the type-2 duality between induced axial electric field and magnetic field.

3.4. Extended-Ampere-Maxwell's Law (2)

Eq. (3.13) gives Extended-Ampere-Maxwell law,

$$\nabla \times \mathbf{B} = \mathbf{v}(\nabla \cdot \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial \mathbf{t}} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v}. \tag{3.14}$$

The magnetic field $\mathbf{B}$ is, in the laboratory frame,

$$\mathbf{B} = \mathbf{B}' + \mathbf{v} \times \mathbf{E}. \tag{3.15}$$

Where $\mathbf{B}'$ is the magnetic field at the circuit $d\mathbf{l}$ in a reference frame in which $d\mathbf{l}$ is at rest. The $\mathbf{E}$ is an electric field at the neighborhood of the circuit. The $\mathbf{v}$ is the velocity of the circuit relative to a laboratory frame.

Note there is no negative sign in front of the $\frac{\partial \mathbf{E}}{\partial \mathbf{t}}$, because that the time change of the $\mathbf{E}$ field through the circuit $d\mathbf{l}$ purely induces a magnetic field $\mathbf{B}'$ that does not accumulate e-charges to against the time change of the **E** field.

3.5. Equation of Continuity

Utilizing Coulomb's law,

$$\nabla \cdot \mathbf{E} = \rho_e, \tag{3.16}$$

let's define a "current" generating magnetic field, denote it as

$$\mathbf{j_{v-B}} = \rho_e \mathbf{v} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v}. \tag{3.17}$$

Where the subscripts "v-B" represent the quantity related with velocity and magnetic field respectively. Then Eq. (3.14) becomes,

$$\nabla \times \mathbf{B} = \mathbf{j_{v-B}} + \frac{\partial \mathbf{E}}{\partial \mathbf{t}}. \tag{3.18}$$

The current $\mathbf{j_{v-B}}$ satisfies the equation of continuity,

$$\nabla \cdot \mathbf{j_{v-B}} + \frac{\partial \rho_e}{\partial \mathbf{t}} = 0. \tag{3.19}$$

For the situation of the non-spatially-varying velocity, i.e.,

$$\mathbf{E}(\nabla \cdot \mathbf{v}) = (\mathbf{E} \cdot \nabla)\mathbf{v} = 0,$$

we have $\mathbf{j_{v-B}} = \rho_e \mathbf{v}$, Extended-Ampere-Maxwell equation, Eq. (3.14), becomes the Ampere-Maxwell law,

$$\nabla \times \mathbf{B} = \mathbf{v}(\nabla \cdot \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial t}. \tag{3.20}$$

Extended-Ampere-Maxwell's law Eq. (3.14), predicts that the products, $\mathbf{E}(\nabla \cdot \mathbf{v})$ and $(\mathbf{E} \cdot \nabla)\mathbf{v}$, create respectively axial vector **B** field; provides a mathematical origin of why and how e-current, $\rho_e \mathbf{v}$, and displacement-current, $\frac{\partial \mathbf{E}}{\partial \mathbf{t}}$, create inevitably magnetic fields.

3.6. Extended-Maxwell Equations

Now, from UMFT and Coulomb's law, we obtain Extended-Maxwell equations,

$$\nabla \cdot \mathbf{E} = \rho_e, \tag{3.16}$$

$$\nabla \times \mathbf{B} = \mathbf{v}(\nabla \cdot \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial \mathbf{t}} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v}, \tag{3.14}$$

$$\nabla \times \mathbf{E} = -\mathbf{v}(\nabla \cdot \mathbf{B}) - \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{B} \cdot \nabla)\mathbf{v}, \tag{3.7}$$

$$\nabla \cdot \mathbf{B} \equiv 0. \tag{3.2}$$

Where the **E** field is a combination of vector field (Coulomb's field) and induced axial vector field,

$$\mathbf{E} = \mathbf{E}_{\text{vect or}} + \mathbf{E}_{\text{axial -vector}},$$

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}_{\text{vector}},$$

$$\nabla \times \mathbf{E} = \nabla \times \mathbf{E}_{\text{axial -vector}}.$$

$$\nabla \cdot \mathbf{E}_{\text{axial -vector}} \equiv 0,$$

$$\nabla \times \mathbf{E}_{\text{vector}} \equiv 0.$$

According to above derivation of Extended-Maxwell equations,

$$\mathbf{B} = \mathbf{B}_{\text{vector}} + \mathbf{B}_{\text{axial -vector}} \equiv \mathbf{B}_{\text{axial -vector}},$$

we have,

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{B}_{\text{axial -ve ctor}} \equiv 0.$$

Therefore, it is the induced $\mathbf{E}_{\text{axial -vector}}$ that is type-2 dual to $\mathbf{B}_{\text{axial -vector}}$. There is no magnetic dual of $\mathbf{E}_{\text{vector}}$.

3.7. Extended-Maxwell Equations Reducing to Maxwell Equations for Uniform Motion

Note the Extended-Maxwell equations are not compatible with SR, the main reason is that SR was derive from and applied for inertial frames. So, when we ignore the non-inertial terms, $(\nabla \cdot \mathbf{v})$, $(\mathbf{E} \cdot \nabla)\mathbf{v}$, $(\mathbf{B} \cdot \nabla)\mathbf{v}$, the extended-Maxwell equations reduce to standard Maxwell equations that is compatible with SR,

$$\nabla \cdot \mathbf{E} = \rho_e, \tag{3.16}$$

$$\nabla \times \mathbf{B} = \mathbf{v}(\nabla \cdot \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial \mathbf{t}}, \tag{3.20}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.11}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{3.2}$$

Writing Eq. (3.16), Eq. (3.20), Eq. (3.11) and Eq. (3.2) in tensor form,

$$\frac{\partial F_e^{\alpha\beta}}{\partial x^\beta} = J_e^\alpha, \tag{3.21}$$

$$\frac{\partial F_e^{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_e^{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_e^{\gamma\alpha}}{\partial x^\beta} = 0, \tag{3.22}$$

$$J_e^\alpha = J_{e+}^\alpha + J_{e-}^\alpha = (\rho_e, \mathbf{J}_e),$$

Where $F_e^{\alpha\beta} = \partial^\alpha A_e^\beta - \partial^\beta A_e^\alpha$ is the field strength tensor, A_e^α is four-vector potential, J_e^α is four-current. Eq. (3.21) and Eq. (3.22) satisfy Lorentz transformation and comply with Special Relativity.

3.8. Lorentz Force Derived from Coulomb Force

In a reference frame S' in which a test e-charge q_e is at rest, an electric field is denote as E'. The force acting on the test e-charge is the Coulomb force,

$$\mathbf{F} = q_e \mathbf{E}'. \tag{3.23}$$

Transferring to a laboratory frame in which the test e-charge is moving with velocity **v**, Eq. (3.5) gives,

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{3.24}$$

where the electric field **E** is measured in the laboratory frame. The magnetic field **B** is measure in the Laboratory frame at the neighborhood of the test e-charge. Substituting Eq. (3.24) into Eq. (3.23), the Coulomb force extended to the Lorentz force measured in the laboratory frame,

$$\mathbf{F} = q_e\mathbf{E} + q_e\mathbf{v} \times \mathbf{B}. \tag{3.25}$$

3.9. Positive and Negative Energy of Electromagnetic Field

To form an assemble of e-charges with the same signs, an external force is needed to bring those same sign e-charges together, namely the external force done the work $W_{External}$ and transfer the energy to the electric field of the assemble. The work $W_{External}$ done by an external force to assemble a volume configuration with a e-charge density ρ_e is

$$W_{External} = \frac{1}{2}\int \rho_e V_e d\tau. \tag{3.26}$$

Thus, the change of the energy of an electric field is defined as positive,

$$\Delta U_e = W_{External} > 0, \tag{3.27}$$

Which represents the energy stored in the configuration, which is also considered conventionally as the potential energy of the electric field.

On the contrary, for bring e-charges with opposite signs together, no external force is needed, and the electric field of e-charges done the work, i.e., the electric field losses energy. The change of the potential energy is defined as the negative of the work done by the electric field, i.e., the electric field loss energy,

$$\Delta U_e = -W_{field} < 0. \tag{3.28}$$

Where the potentials of positive and negative e-charges are respectively,

$$V_{e+} = \frac{Q_{e+}}{r} > 0, \tag{3.29}$$

$$V_{e-} = \frac{Q_{e-}}{r} < 0. \tag{3.30}$$

Therefore, the definition of the positive or negative potential energy of an electric field is a matter of bookkeeping [5].

Using Coulomb's law and vector formula, Eq. (3.26) gives

$$\Delta U_e = W_{External} = \frac{1}{2}\int E^2 d\tau. \tag{3.31}$$

Similarly, the work done to retain the e-current going is

$$W_B = \frac{1}{2}\int B^2 d\tau, \tag{3.32}$$

Which is also considered as the energy stored in a magnetic field.

The total energy, W_{EM}, stored in electromagnetic field is the sum of energies of both electric field and magnetic field,

$$W_{EM} = \frac{1}{2}\int (E^2 + B^2) d\tau. \tag{3.33}$$

For simplicity, we have set $\epsilon_0 = \mu_0 = 1$.

3.10. Extended-Poynting Theorem

Now let's calculate the rate at which work "W" done to move e-changes under the influence of electric and magnetic fields,

$$dW = \mathbf{F} \cdot d\mathbf{l}.$$

Substituting Lorentz-force, Eq. (3.25), then

$$\frac{dW}{dt} = \int (\mathbf{E} \cdot \mathbf{J}) d\tau. \tag{3.34}$$

Substituting Eq. (3.14), $\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial \mathbf{t}} - \mathbf{E}(\nabla \cdot \mathbf{v}) + (\mathbf{E} \cdot \nabla)\mathbf{v}$, into Eq. (3.34), we obtain

$$\begin{aligned}\mathbf{E} \cdot \mathbf{J} &= \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial \mathbf{t}} \\ &+ E^2 (\nabla \cdot \mathbf{v}) - \mathbf{E} \cdot [(\mathbf{E} \cdot \nabla)\mathbf{v}].\end{aligned} \tag{3.35}$$

The formula of vector analysis gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) \equiv \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Substituting Eq. (3.7), $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{B} \cdot \nabla)\mathbf{v}$, into it, then

$$\begin{aligned}&\mathbf{E} \cdot (\nabla \times \mathbf{B}) \\ &= \mathbf{B} \cdot \left[-\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{B} \cdot \nabla)\mathbf{v} \right] - \nabla \cdot (\mathbf{E} \times \mathbf{B}),\end{aligned}$$

or

$$\begin{aligned}&\mathbf{E} \cdot (\nabla \times \mathbf{B}) \\ &= -\frac{1}{2}\frac{\partial B^2}{\partial t} + B^2 (\nabla \cdot \mathbf{v}) - \mathbf{B} \cdot [(\mathbf{B} \cdot \nabla)\mathbf{v}] - \nabla \cdot (\mathbf{E} \times \mathbf{B}).\end{aligned} \tag{3.36}$$

Substituting Eq. (3.36) into Eq. (3.35), we obtain

$$\begin{aligned}&\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}(B^2 + E^2) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \\ &+ (B^2 + E^2)(\nabla \cdot \mathbf{v}) - \mathbf{B} \cdot [(\mathbf{B} \cdot \nabla)\mathbf{v}] - \mathbf{E} \cdot [(\mathbf{E} \cdot \nabla)\mathbf{v}].\end{aligned} \tag{3.37}$$

Substituting Eq. (3.33) and Eq. (3.37) into Eq. (3.34), we derived the *extended-Poynting's theorem*,

$$\frac{dW}{dt} = -\frac{dW_{EM}}{dt} - \oint\!\!\oint \mathbf{S} \cdot d\mathbf{a} + \int \left\{ \begin{aligned} &(B^2 + E^2)(\nabla \cdot \mathbf{v}) \\ &-\mathbf{B} \cdot [(\mathbf{B} \cdot \nabla)\mathbf{v}] \\ &-\mathbf{E} \cdot [(\mathbf{E} \cdot \nabla)\mathbf{v}] \end{aligned} \right\} d\tau, \tag{3.38}$$

Where the Poynting's vector S is defined as,

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{B}. \tag{3.39}$$

For the special situation of non-spatially-varying velocity, $\nabla \cdot \mathbf{v} = (\mathbf{B} \cdot \nabla)\mathbf{v} = (\mathbf{E} \cdot \nabla)\mathbf{v} = 0$, the extended-Poynting theorem deduces to the Poynting's theorem.

3.11. Origin of Differences between Static Electric Field and Magnetic Field

In the Section of **Motivation**, we have mentioned several fundamental differences: "A magnetic field is completely different from a static electric field in the following senses: (1) "e-charge" vs. "e-current"; (2) "$\nabla \cdot \mathbf{E}$" vs. "$\nabla \times \mathbf{B}$"; (3) "vector field $\mathbf{E}$" vs. "axial vector field $\mathbf{B}$"; (4) "$q_e\mathbf{E}$" vs. "$q_e\mathbf{v} \times \mathbf{B}$".

In this Section we explain those differences below:

Comparison of Eq. (3.6) and Eq. (3.14) explains the first, second and third differences as the following: e-charge ρ_e induces $\mathbf{E}$ via $\nabla \cdot \mathbf{E} = 4\pi\rho_e$; it is UMF E combining with the Coulomb law that makes the term, $\mathbf{v}(\nabla \cdot \mathbf{E}) = 4\pi\rho_e\mathbf{v}$, induces $\mathbf{B}$ field via $\nabla \times \mathbf{B}$; Coulomb e-field $\mathbf{E}$ is a vector field, and $\mathbf{B} \sim \mathbf{v} \times \mathbf{E}$ is a first level axial vector field. Eq. (3.20) to Eq. (3.22) show that due to the fact that motion of e-charge creates magnetic field, thus, transferring the Coulomb's force in an e-charge-rest frame to the Lorentz force in an e-charge-moving frame.

All of above-mentioned differences between static electric field and magnetic field have a mathematical origin, Eq. (2.1), which shows how a divergence field creates a curl field.

3.12. No Magnetic Monopole Charge Mathematically

In the derivation of Extended-Ampere-Maxwell equation, magnetic field is defined in Eq. (3.1) as an axial vector field,

$$\mathbf{B} \equiv \mathbf{v} \times \mathbf{E}.$$

By the mathematical definition, the divergence of an axial vector is identically zero, which is represented as Eq. (3.2), $\nabla \cdot \mathbf{B} \equiv 0$.

Indeed, mathematically, there is no magnetic monopole charge in vector electromagnetic theory, as long as the magnetic field is an axial vector field.

3.13. Conclusion and Suggestion

We, in Section 3, have shown that magnetic field exists mathematically due to movement of e-charge. Namely its existence is inevitable. The effects of magnetic field were discovered experimentally started in 1860s. Historically, the vector analysis was established in the same era, 1860s [6]. It is reasonable to assume that physicists were not familiar with vector analysis in 1860s. We argue that this might be the reason why UMFT was not developed when Maxwell equation were established. Moreover, then Maxwell equations were well applied to variety of areas, so no mathematical derivation of Maxwell equations was focused.

4. Gravito-Electromagnetic Derived from UMFT and Newton's Law

We will apply the UMFT to gravity in this Section to derive a Maxwell-type vector theory, Gravito-EM, for gravity, which is one step further than Newton's theory to understand gravity, although may not be the ultimate theory of gravity. Indeed, since the Gravito-EM is derived mathematically from Newton's law, thus, any further theory of gravity should have not only Newton's law but Gravito-EM as approximations under certain condition. Moreover, we have shown that Gravito-EM has other fundamental benefits that it can be quantized and unified with other interaction without difficulty [3,7].

4.1. Positive and Negative Gravitational Charge (g-Charge)

Two paths of studying gravity are: First path is to treat gravity and other three forces as physical fields; Second path is to treat gravity and other forces as geometric phenomena. Along the first path, other three forces, except gravity, were physically understood, quantized and unified successfully. Along the second path, to geometrize gravity was quite successful, but not other three forces. Tremendous efforts have been devoted on to quantize geometric theory of gravity and unify it with theories of other three forces, so far, without commonly accepted approach. Facing the tremendous difficulties in quantizing geometric theory of gravity and unification, Extended-Maxwell equations have been derived mathematically in Section 3 from the combination of UMFT and the Coulomb's law. The fact that the Newton's law is similar to the Coulomb's law strongly suggests to study whether and under what condition, one can mathematically derived a counterpart of Extended-Maxwell equation from the UMFT and the Newton's law. For this aim, let's compare the Newton's law and Coulomb's law at the level of the classical field theory.

Table 4.1. Comparison between Newton's law and Coulomb's law

	Coulomb's law	Newton's law
Force	Long-range inverse-square-law $\mathbf{F} = q_e\mathbf{E}$ Attractive/repulsive	Long-range inverse-square-law $\mathbf{F} = m\mathbf{g}$ Attractive/?
Coulomb's law/ Newton's law	$\nabla \cdot \mathbf{E} = \rho_e$	$\nabla \cdot \mathbf{g} = -\rho_g$
Charge	Two kinds e-charge: Positive/negative	One kind g-charge: Positive/?

Table 4.1 suggests that it is the fact, two kind of e-charge vs. one kind of gravitational charge (denoted as g-charge), that leads to that attractive/repulsive electric force vs. attractive gravitation force. The significant differences are in three aspects.

Firstly, in forces: The Coulomb's forces are either attractive or repulsive, while conventional gravity force is only attractive. However, this understanding of gravitational force was changed. In 1998, scientists reported that the universe is expended with acceleration [8,9,10], even the acceleration is accelerating (to date, not confirmed yet), which indicates the existence of repulsive gravitational force. Astronomers invoke dark energy for this repulsive force that creates a constant acceleration.

An effective dynamic model for explaining the accelerated expansion of the universe was proposed [11]. The model is based on a hypothesis of the existence of the negative g-charges. The results of this model contain those of the dark energy models and consistent with the observations. Moreover, an alternative physical interpretation of the cosmological constant is that it is equivalent to the negative g-charges as a source of gravitational fields. This interpretation avoids the fine-tuning problem of the cosmological constant.

We suggest that the accelerated expansion of the normal universe is an observational evidence of the existence of the negative g-charges. On the other hand, this model predicts that a sub-universe filled with negative g-charges is accelerated collapse and thus, that the equation of state and the acceleration of the expansion of the universe are time dependent, namely the acceleration of the expansion of the universe is accelerating, called jerking.

Although GR is now a hundred-year-old theory, it remains a powerful and controversial idea in cosmology. It is one of the basic assumptions behind the current cosmological model: a model that is both very successful in matching observations but implies the existence of both dark matter and dark energy. These indicates that our understanding of gravity is incomplete. We will likely need a new profound idea to explain these mysteries and require more powerful observations and experiments to light the path toward our new insights.

Secondly, the difference in charges: There are positive and negative e-charges in EM, while in gravity only one positive gravitational mass. To establish a perfect duality between the Coulomb's law and Newton's law, we need the concept of g-charge. The positive g-charge, $Q_{g+} \equiv +\sqrt{G}m$, was introduced [12]. After the discovery of the accelerating universe, the repel force emerged. We proposed that the origin of the repulsive force is gravity, and introduced the concept of negative g-charge [13] to explain the universe expansion.

Before introducing negative g-charge into the theories of gravity, let's postulate two hypotheses:

Hypothesis 1: An object having rest mass, $m_0 > 0$ $(m_0 = \int \rho_{m0} d^3x)$, may carry either positive or negative g-charge defined as,

$$Q_{g+} \equiv +\sqrt{G}m_0,\ \rho_{g+} = +\sqrt{G}\rho_{m0}, \tag{4.1}$$

$$Q_{g-} \equiv -\sqrt{G}m_0,\ \rho_{g-} = -\sqrt{G}\rho_{m0} \tag{4.2}$$

Eq. (4.1) and Eq. (4.2) can be combined as,

$$Q_{g\pm} \equiv \pm\sqrt{G}m_0,\ \rho_{g\pm} = \pm\sqrt{G}\rho_{m0}. \tag{4.3}$$

$$Q_{gNet} = Q_{g+} + Q_{g-},\ \rho_{gNet} = \rho_{g+} + \rho_{g-}. \tag{4.4}$$

Where "G" is the Newton constant; the quantities with subscript "$+$", "$-$", "$\pm$" and "Net" are related with positive, negative, either positive or negative, and Net g-charges, respectively.

When one kind of g-charges is defined as positive, then g-charges that repels it is defined as negative. For convenience, let's define the g-charges carried by the ordinary matters in the universe as the positive g-charges.

The existence of negative g-charge has been justified at quantum level [3].

Thirdly, Duality: When one converts e-charge Q_e to negative g-charge $(-Q_g)$, and the electric field strength $\mathbf{E}$ to the Newton gravitational field strength $\mathbf{g}$, then the Coulomb's field equation converts to Newton's field equation, and vice versa. However, duality between force laws are different, one needs to converts e-charge Q_e to g-charge, Q_g, and the electric field strength $\mathbf{E}$ converted to the Newton gravitational field strength $\mathbf{g}$, then the Coulomb's force law converts to Newton's force law, and vice versa.

4.1.1. Interaction between Positive and Negative g-Charges

We propose a positive/negative g-charge conjugation, which leads to:

Hypothesis 2: the laws governing gravitational interaction generated respectively by positive and negative g-charges have the same form.

By analogy to e-charge conjugation, the g-charge conjugation is that under transformation of g-charge conjugation, gravitation laws are unchanged. The existence of repulsive gravity would imply the existence of g-charges with opposite sign.

According to Hypothesis 2, the Newton's law,

$$\nabla \cdot \mathbf{g}_+ = -Q_{g+} = -\sqrt{G}m_{0+}, \tag{4.5}$$

$$\mathbf{F}_+ = q_{g+}\mathbf{g}, \tag{4.6}$$

is extended to contain that of negative g-charge as,

$$\nabla \cdot \mathbf{g}_- = -Q_{g-} = \sqrt{G}m_{0-}, \tag{4.7}$$

$$\mathbf{F}_- = q_{g-}\mathbf{g}. \tag{4.8}$$

Combining Eq. (4.5) and Eq. (4.7) leads to the extended-Newton's law,

$$\nabla \cdot \mathbf{g} = -Q_g, \tag{4.9}$$

$$\mathbf{F}_\pm = q_{g\pm}\mathbf{g}. \tag{4.10}$$

where,

$$\mathbf{g} \equiv \mathbf{g}_+ + \mathbf{g}_-, \tag{4.11}$$

$$Q_g = Q_{g+} + Q_{g-}. \tag{4.12}$$

Q_{g+}/Q_{g-} and q_{g+}/q_{g-} are g-charges of source and test-object respectively. Accordingly, two "negative g-charges" attract each other in the same way that two "positive g-charge" do, while a negative g-charge and a positive g-charge repel each other.

Now the Extended-Newton's law, Eq. (4.9), is perfect dual to the Coulomb's law. Thus, following the same procedure, we can apply UMFT and the Extended-Newton's law to derive vector field equations for gravitational fields, with moving sources.

For a point g-charge as a source, the solutions of Eq. (4.5), Eq. (4.7) and Eq. (9) are respectively,

$$\mathbf{g}_+ = -\frac{Q_{g+}}{r^2}\hat{\mathbf{r}}, \tag{4.13}$$

$$\mathbf{g}_- = -\frac{Q_{g-}}{r^2}\hat{\mathbf{r}}, \tag{4.14}$$

$$\mathbf{g} = -\frac{Q_{g+} + Q_{g-}}{r^2}\hat{\mathbf{r}}, \tag{4.15}$$

where $\hat{\mathbf{r}}$ is the unit vector and points radially outward.

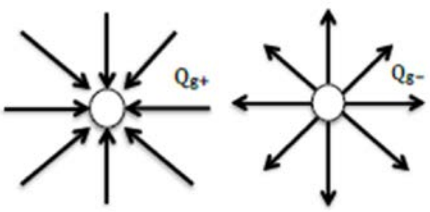

Figure 4.1. Field line of Q_{g+}. **Figure 4.2.** Field line of Q_{g-}

The field lines of gravitational fields, $\mathbf{g}_+$ and $\mathbf{g}_-$, are shown in Figure 4.1 and Figure 4.2, respectively [14].

Thus, two positive g-charges are attractive to each other (Figure 4.3):

$$\mathbf{F}_{+} = q_{g+}\mathbf{g}_{+} = q_{g+}\left(-\frac{Q_{g+}}{r^2}\hat{\mathbf{r}}\right). \qquad (4.16)$$

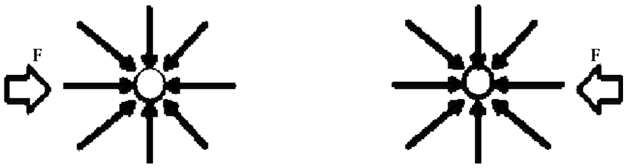

Figure 4.3. Two positive g-charges are attractive to each other

Two negative g-charges are attractive to each other (Figure 4.4):

$$\mathbf{F}_{-} = q_{g-}\mathbf{g}_{-} = q_{g-}\left(-\frac{Q_{g-}}{r^2}\hat{\mathbf{r}}\right). \qquad (4.17)$$

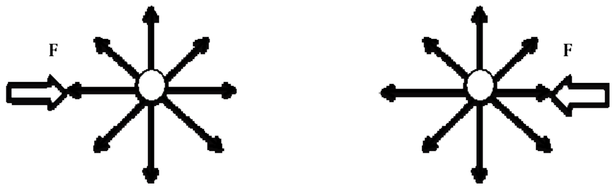

Figure 4.4. Two negative g-charges are attractive to each other

A positive and a negative g-charges repel to each other (Figure 4.5):

$$\mathbf{F}_{+} = q_{g+}\mathbf{g}_{-} = q_{g+}\left(-\frac{Q_{g-}}{r^2}\hat{\mathbf{r}}\right), \qquad (4.18)$$

$$\mathbf{F}_{-} = q_{g-}\mathbf{g}_{+} = q_{g-}\left(-\frac{Q_{g+}}{r^2}\hat{\mathbf{r}}\right). \qquad (4.19)$$

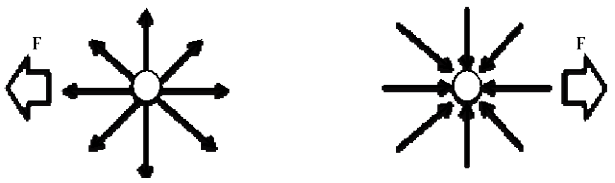

Figure 4.5. A positive and a negative g-charges repel each other

4.1.2. Indirect Evidence of Negative g-Charges: Accelerating and Jerking Expansion of Universe

4.1.2.1. Extended-Newton's Theory with Negative g-Charges

It has been shown that, by introducing negative g-charge, the Extended-Newton's theory and Extended-Einstein's equation can explain the accelerated expansion of the Universe and predict that the acceleration of universe is accelerating, which distinguishes negative g-charge model from that of dark energy model and cosmology constant model [13,14].

Let's assume that objects in our Universe: (1) have positive mass; (2) carry either positive or negative g-charges, and form positive or negative sub-Universe, respectively.

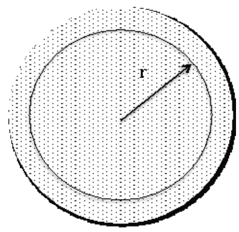

Figure 4.6. Accelerating Expansion of Universe

Let's consider gravitational fields of a spherical distribution (Figure 4.6) of mixed negative and positive g-charges. Eq. (4.9) gives

$$\nabla\cdot\mathbf{g} = -\sqrt{G}[\rho_{m0+}(r,t) - \rho_{m0-}(r,t)]. \qquad (4.20)$$

Where $\rho_{g-} \equiv -\sqrt{G}\,\rho_{m0-}$ and $\rho_{g+} \equiv \sqrt{G}\,\rho_{m0+}$ are negative g-charge density and positive g-charge density respectively. And ρ_{m0-} (>0) and ρ_{m0+} (>0) are the mass densities of objects with negative g-charge in negative sub-Universe and with positive g-charge in positive sub-Universe, respectively.

Eq. (4.19) gives the gravitational field at $\mathbf{r}$,

$$\mathbf{g} = -\frac{\sqrt{G}\int\left[\rho_{m0+}(r,t)-\rho_{m0-}(r,t)\right]d^3x}{r^2}\hat{\mathbf{r}}. \qquad (4.21)$$

For uniform distribution of positive and negative sub-Universes, the gravitational field $\mathbf{g}$ at r is,

$$\mathbf{g} = -\frac{4\pi\sqrt{G}}{3}[\rho_{m0+}(t)-\rho_{m0-}(t)]\mathbf{r}. \qquad (4.22)$$

The motion of a non-relativistic object carrying positive g-charge at a given r is described by Eq. (4.6). Substituting Eq. (4.15) into Eq. (4.6), we obtain

$$\frac{1}{r}\frac{d^2r}{dt^2} = -\frac{4\pi G}{3}\rho_{m0+}(t) + \frac{4\pi G}{3}\rho_{m0-}(t). \qquad (4.23)$$

To solve Eq. (4.23), we consider three different situations:

First situation: $\rho_{m0+} > \rho_{m0-}$, Eq. (4.23) yields,

$$\frac{1}{r}\frac{d^2r}{dt^2} = \frac{4\pi G}{3}\left[\rho_{m0-}(t)-\rho_{m0+}(t)\right] < 0, \qquad (4.24)$$

Which implies that the positive (negative) sub-Universe is in accelerated collapse (expansion).

Second situation: $\rho_{m0-} = \rho_{m0+}$, $\mathbf{g} = 0$, Eq. (4.23) yields,

$$\frac{1}{r}\frac{d^2r}{dt^2} = \frac{4\pi G}{3}\left[\rho_{m0-}(t)-\rho_{m0+}(t)\right] = 0. \qquad (4.25)$$

This case corresponds to static positive and negative sub-Universes.

Third situation: $\rho_{m0-} > \rho_{m0+}$, we obtain

$$\mathbf{g} = \frac{4\pi G}{3}\mathbf{r}\left[\rho_{m0-}(t)-\rho_{m0+}(t)\right]\mathbf{r} > 0. \qquad (4.26)$$

For a non-relativistic object of positive g-charge, Eq. (4.23) give,

$$\frac{1}{r}\frac{d^2r}{dt^2}=\frac{4\pi G}{3}\left[\rho_{0m-}(t)-\rho_{m0+}(t)\right]>0. \quad (4.27)$$

This result agrees with observation data of the accelerating expansion of the Universe, without needing of negative pressure and cosmological constant. Eq. (4.26) and Eq. (4.27) implies that an object carrying positive g-charge is pushed away from the distribution with acceleration, i.e., the negative g-charge provides physics mechanism for the accelerating expansion of positive sub-Universe.

For a non-relativistic object of negative g-charge, for the third situation, Eq. (4.23) and Eq. (4.26) give,

$$\frac{1}{r}\frac{d^2r}{dt^2}=-\frac{4\pi G}{3}\rho_{m-}(t)+\frac{4\pi G}{3}\rho_{m+}(t)<0. \quad (4.28)$$

Eq. (4.28) implies that an object carrying negative g-charge is attracted toward to the distribution, which leads to the accelerated collapse of the negative sub-Universe.

The mass density, ρ_{m0+}, of objects carrying positive g-charges in the ball of radius r decreases continuously, while the mass density, ρ_{m0-}, of objects carrying negative g-charges in the ball of radius r increase continuously. As the consequence, the gravitational field $\mathbf{g}_+$ decreases and the gravitational field $\mathbf{g}_-$ increases. The net effects of the combination of decreasing gravitational field $\mathbf{g}_+$ and increasing gravitational field $\mathbf{g}_-$, is that the acceleration of the expansion of objects with positive g-charge is increase, namely the acceleration of the expanded regular observed Universe is accelerating, so we have a jerking universe,

$$\frac{1}{r}\frac{d^3r}{dt^3}=\frac{4\pi G}{3}\left[\frac{d\rho_{m-}(t)}{dt}-\frac{d\rho_{m+}(t)}{dt}\right]>0. \quad (4.29)$$

Eq. (4.20) to Eq. (4.29) form a dynamical model that explains the accelerated expansion of our observed positive sub-Universe and predict that the acceleration is increase with time, which may distinguish negative g-charge model from dark energy/cosmological constant model.

We argue that this dynamical model plays a role of indirect evidence of existence of negative g-charge [13].

4.1.2.2. Extended-Einstein's Theory (GR) with Negative g-Charges

It has been shown that, similar to the dark energy or Einstein's cosmology constant Λ, the negative g-charges can naturally explain the accelerating expansion of the universe [13]. Moreover, the negative g-charges can resolve the fine-tuning problem encountered by Einstein's cosmology constant Λ when explaining the accelerating expansion of the universe.

Einstein's equation,

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=\frac{8\pi G}{C^4}T^{\mu\nu} \quad (4.30)$$

describes spacetime, $R^{\mu\nu}$, curved by sources, $T^{\mu\nu}$, of only positive gravitation mass.

Introducing the concept of g-charge, for a system containing positive g-charges only, Einstein's equation, Eq. (4.30) can be rewritten as,

$$R_+^{\mu\nu}-\frac{1}{2}g_+^{\mu\nu}R_+=\frac{8\pi\sqrt{G}}{C^4}T_{g+}^{\mu\nu}, \quad (4.31)$$

Writing Einstein's equation in this form, Eq. (4.31), gives us an indication how to generalize Einstein's theory to describe spacetime curved by negative g-charge. For this aim, based on Hypothesis 2, we generalize Einstein's theory to contain additional equations having the same form as that of Eq. (4.31) to describe spacetime curved either by negative g-charge alone or by a combination of positive and negative g-charges.

For source(s) carrying negative g-charge, $T_{g-}^{\mu\nu}$, we propose Einstein-type equation,

$$R_-^{\mu\nu}-\frac{1}{2}g_-^{\mu\nu}R_-=\frac{8\pi\sqrt{G}}{C^4}T_{g-}^{\mu\nu}. \quad (4.32)$$

It is obvious that the Einstein's equations, Eq. (4.31) and Einstein-type equation, Eq. (4.32), have g-Charge conjugation. Note objects having positive mass, $T_{m+}^{\mu\nu}$, might carry either positive $T_{g+}^{\mu\nu}$ or negative $T_{g-}^{\mu\nu}$.

We postulate that for a source of two-compound system, by analogy to electrodynamics, its gravitational field should be determined by net g-charges. In geometric term, even there are positive and negative g-charges, spacetime is still described by one metric. Next let's consider a two-compound system containing first and second kinds of object(s). The former has positive energy-momentum pseudo-tensor $T_1^{\mu\nu}$ and carrying positive g-charge $T_{g+}^{\mu\nu}$; the latter has positive energy-momentum pseudo-tensor $T_2^{\mu\nu}$ and carrying negative g-charge $T_{g-}^{\mu\nu}$. To describe this two-compound system, we propose Extended-Einstein equation,

$$R_{net}^{\mu\nu}-\frac{1}{2}g_{net}^{\mu\nu}R_{net}=\frac{8\pi\sqrt{G}}{C^4}T_{g,net}^{\mu\nu}=\frac{8\pi G}{C^4}(T_1^{\mu\nu}-T_2^{\mu\nu}). \quad (4.33)$$

Where

$$T_{g,net}^{\mu\nu}\equiv T_{g+}^{\mu\nu}+T_{g-}^{\mu\nu},\ T_{g+}^{\mu\nu}\equiv+\sqrt{G}T_1^{\mu\nu},\ T_{g-}^{\mu\nu}\equiv-\sqrt{G}T_2^{\mu\nu}.$$

There are also three situations:

1) $T_{g+}^{\mu\nu}>T_{g-}^{\mu\nu}$, spacetime $R_{net}^{\mu\nu}$ is the same as $R_+^{\mu\nu}$, thus Eq. (4.33) can be written as

$$R_+^{\mu\nu}-\frac{1}{2}g_+^{\mu\nu}R_+=\frac{8\pi G}{C^4}\left(T_1^{\mu\nu}-T_2^{\mu\nu}\right); \quad (4.34)$$

2) $T_{g+}^{\mu\nu}<T_{g-}^{\mu\nu}$, spacetime $R_{net}^{\mu\nu}$ is the same as $R_-^{\mu\nu}$, thus Eq. (4.33) can be written as

$$R_-^{\mu\nu}-\frac{1}{2}g_-^{\mu\nu}R_-=-\frac{8\pi G}{C^4}\left(T_2^{\mu\nu}-T_1^{\mu\nu}\right). \quad (4.35)$$

3) $T_{g+}^{\mu\nu}=T_{g-}^{\mu\nu}$, spacetime is flat Minkowski spacetime.

Now let's apply the Extended-Einstein equation to explain the accelerated expansion of Universe. Assume that there are negative g-charge and positive g-charges in Universe, the former forms negative sub-Universe, and the latter forms positive sub-Universe. Eq. (4.33) gives

$$R_{net}^{\mu\nu}-\frac{1}{2}g_{net}^{\mu\nu}R_{net}=8\pi\sqrt{G}\left(T_{g+}^{\mu\nu}+T_{g-}^{\mu\nu}\right). \quad (4.36)$$

Applying the FRW metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right], \quad (4.37)$$

where a(t) is the scale factor; K is a constant. Substituting Eq. (4.37) into Eq. (4.36), we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi\sqrt{G}}{3}\left[\left(\rho_{g+} + 3p_{g+}\right) - \left(\rho_{g-} + 3p_{g-}\right)\right], \quad (4.38)$$

and Hubble parameter,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\sqrt{G}}{3}\left[\rho_{g+} - \rho_{g-}\right] - \frac{K}{a^2}. \quad (4.39)$$

In the case of $3p_{g+} \ll \rho_{g+}$ and $3p_{g-} \ll \rho_{g-}$, then we obtain,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho_{m+}(t) - \rho_{m-}(t)\right]. \quad (4.40)$$

Where p_{g+} and p_{g-} are the pressures due to the positive and negative g-charges respectively. Eq. (4.40) implies that for $\rho_{g-} > \rho_{g+}$, the positive g-charges carried by regular objects will be repelled away. With negative g-charge, the Extended-Einstein equation can explain the accelerated expansion of the universe without the need of either "negative pressure" or "dark energy" or the cosmological constant that has issue of fine-turn.

As in Extended-Newton's law, the density of negative g-charge is increasing, $\frac{d\rho_{m-}(t)}{dt} > 0$, and the density of positive g-charge is decreasing, $\frac{d\rho_{m-}(t)}{dt} < 0$, therefore the acceleration of the expansion of the universe is increasing with time, i.e., the universe is a jerking universe,

$$\frac{1}{a}\frac{d^3a}{dt^3} = \frac{4\pi G}{3}\left[\frac{d\rho_{m-}(t)}{dt} - \frac{d\rho_{m+}(t)}{dt}\right] > 0. \quad (4.41)$$

To compare with Hubble's Constance, H, we calculate the velocity,

$$\dot{r} = \int \ddot{r}dt = \frac{G\,\rho_{m-}}{3}\int r(t)dt$$

then

$$H = \frac{\dot{r}}{r} = \frac{G\,\rho_{m-}}{3r}\int r(t)dt$$

$$\dot{H} = \frac{G\,\rho_{m-}}{3} - \frac{G\,\rho_{m-}}{3r^2}\dot{r}\int r(t)dt$$

$$= \frac{G\,\rho_{m-}}{3} - \frac{1}{r^2}\left[\frac{G\,\rho_{m-}}{3}\int r(t)dt\right]^2.$$

To summarize, by introducing the negative g-charge, both Extended-Newton's theory and Extended-Einstein's theory can explain the acceleration of the expansion of the universe and predict the jerking universe.

Thus, we argue that the accelerating expansion of the universe is an indirect evidence of the existence of negative g-charge.

4.1.3. Extended-Dirac Sea: Negative g-Charge Carrier

In order to explain the negative energy states, Dirac introduced the concept of Dirac Sea. An interpretation is that the positive operators add a positive energy particle and the negative operators annihilate a positive energy particle. Now we generalize this concept to include the g-charges, we denote it as the Extended-Dirac Sea, which states that the positive operator adds a positive rest mass particle and the negative operator annihilates a positive rest mass particle, i.e., Extended-Dirac Sea exchanges g-charges with particles during the processes of creation and annihilation.

An example: let's consider electron-positron annihilation,

$$e^- + e^+ \to \gamma + \gamma.$$

We divide this process into 3 separate processes relating with mass/energy, e-charge, and g-charge, correspondingly,

$$\left.\begin{array}{l} m_{electron} + m_{positron} \to \gamma + \gamma \\ Q_e^- + Q_e^+ \to \text{Extended} - \text{Dirac Sea} \\ Q_g^- + Q_g^+ \to \text{Extended} - \text{Dirac Sea} \end{array}\right\}, \quad (4.42a)$$

where $m_{electron}$, Q_e^-, Q_g^+, and $m_{positron}$, Q_e^+, Q_g^-, are the rest mass, e-charge, and g-charge of electron and positron, respectively. In this process,

1) Electron carries positive g-charge, while positron carries negative g-charge, such that the net g-charge is zero;
2) the electron's and positron's rest masses convert to radiation energy of gamma ray;
3) their negative and positive e-charges transport to and store in Extended-Dirac Sea;
4) their negative and positive g-charges transport to and store in Extended-Dirac Sea, which implies that the g-charge and inertial rest mass are two different entities.

Another example: the process of the creation of electron-positron pair, which is expressed as

$$\gamma \to e^- + e^+.$$

We divide this process into 3 separate processes,

$$\left.\begin{array}{l} \gamma \to m_{electron} + m_{positron} \\ \text{Extended} - \text{Dirac Sea} \to Q_e^- + Q_e^+ \\ \text{Extended} - \text{Dirac Sea} \to Q_g^- + Q_g^+ \end{array}\right\}. \quad (4.42b)$$

The gamma ray's energy converts to electron and positron's rest masses, respectively. The created electron and positron gain e-charge and g-charges from Extended-Dirac Sea, respectively.

4.2. Gravito-Electromagnetics (Gravito-EM): Vector Theory

4.2.1. Introduction

Historically, based on the similarity between the Coulomb's and Newton's laws and, by close analogy to electromagnetism (EM), Maxwell, Heaviside and others proposed the vector theory of gravity which has the form same to that of EM without either mathematical derivation or experimental confirmation, as earlier as year 1865 [15,16,17,18,19,20,21] after Maxwell's theory was proposed (1861). However, they stopped further investigation, because they concluded that there was an issue that the potential energy density of static Newtonian gravitational field was negative. In spite of the issue, many scientists still work along this line [22,23,24,25,26].

In 1905, SR was established, which predicted that the inertial mass is a function of velocity,

$$m_{\text{inertial}} = \gamma m_0.$$

Where m_0 is the rest mass, $\gamma = 1/\sqrt{1-v^2/c^2}$ is the Lorentz factor.

Now those vector theories faced another issue that is related with none-complying with SR. According to the Weak Equivalence Principle (WEP), the gravitational mass, m_g, is equal to the inertial mass, m_{inertial},

$$m_g = m_{\text{inertial}} = \gamma m_0.$$

The gravitational mass "current" of a point g-charge has the form,

$$\mathbf{j}_g = m_g v = \gamma m_0 \mathbf{v}.$$

Therefore, the gravitational mass density and the gravitational current density cannot construct a four-current. Namely, unlike Electromagnetic theory in which e-charge Q_e is a constant and $\mathbf{j}_e = Q_e\mathbf{v}$, those vector theories of gravity do not satisfy Lorentz transformation.

Based on the Original Weak Equivalent Principle (Original-WEP), the U (1) gauge theory of gravity is proposed, which is consistent with Gravito-EM [3,27]. As a vector field theory, Gravito-EM therefore needs to address:

1) the negative energy issue and
2) the lack of Lorentz invariance issue.

Indeed, it has been shown that the exchanged energy of gravitation fields is always positive, i.e., transported energy in and out of gravitation field is always positive [14]. Therefore, the total energy of gravitation fields being defined as either negative or positive is only a matter of bookkeeping. The energy issue of the vector field theory of gravity is addressed. There is no negative energy issue in Gravito-EM.

On the other hand, theory of gravity should inevitable contain magnetic-type gravitation. Actually, in 1983, Einstein's equation has been written in Maxwell-type form with second rank tensor fields [28],

$$\frac{\partial G^{\mu\nu\lambda}}{\partial x^{\lambda}} = -\left(T^{\mu\nu} + t^{\mu\nu}\right), \tag{4.43}$$

$$G^{\alpha\mu\nu,\lambda} + G^{\alpha\nu\lambda,\mu} + G^{\alpha\lambda\mu,\nu} = 0, \tag{4.44}$$

where $t^{\mu\nu}$ is non-linear term represent self-interaction of gravitation field.

Following EM, scalar potential, $V_{g/GR}$, vector potential, $\mathbf{A}_{g/GR}$, field strengths, $\mathbf{g}_{GR}$, and $\mathbf{B}_{g/GR}$ or G^{00i} and G^{0ij}, were introduced,

$$V_{g/GR} \equiv \frac{1}{4}\bar{h}^{00} \tag{4.45}$$

$$A^{i}_{g/GR} \equiv \frac{1}{4}\bar{h}^{0i}, \mathbf{A}_{g/GR} = \left(A^1_g, A^2_g, A^3_g\right), \tag{4.46}$$

$$G^{0ij} = A^{i,j}_{g/GR} - A^{j,i}_{g/GR}, \tag{4.47}$$

and

$$\mathbf{g}_{GR} = \left(g^1_{GR}, g^2_{GR}, g^3_{GR}\right), g^{i}_{GR} = G^{00i}, \tag{4.48}$$

$$\mathbf{B}_{g/GR} = (B^1_{g/GR}, B^2_{g/GR}, B^3_{g/GR}), \tag{4.49}$$

$$\mathbf{B}_{g/GR} = \nabla \times \mathbf{A}_{g/GR}, \tag{4.50}$$

$$B^1_{g/GR} = G^{023}, B^2_{g/GR} = G^{031}, B^3_{g/GR} = G^{012}. \tag{4.51}$$

For the situation of slow motion of a point source, ignore non-linear terms,

$$T^{00} \approx \rho_g, \tag{4.52}$$

$$T^{0i} \approx \rho_g v^i, \tag{4.53}$$

$$T^{ij} \approx 0.$$

Eqs. (4.43) and (4.44) give

$$\nabla \cdot \mathbf{g}_{GR} = -\rho_g, \tag{4.54}$$

$$\nabla \times \mathbf{B}_{g/GR} = \mathbf{v}(\nabla \cdot \mathbf{g}_{GR}) + \frac{\partial \mathbf{g}_{GR}}{\partial \mathbf{t}}, \tag{4.55}$$

$$\nabla \times \mathbf{g}_{GR} = -\mathbf{v}\left(\nabla \cdot \mathbf{B}_{g/GR}\right) - \frac{\partial \mathbf{B}_{g/GR}}{\partial t}, \tag{4.56}$$

$$\nabla \cdot \mathbf{B}_{g/GR} \equiv 0, \tag{4.57}$$

The subscript "GR" represents the quantities related with "General Relativity", which will distinguish above equations from the Maxwell-type Gravito-EM. Eq. (4.57) indicates that, mathematically, there is no Gravito-magnetic monopole charge in GR of gravity.

Many effects of magnetic-type gravitational field have been proposed [29,30,31,32,33,34,35]. The Lense-thirring effect [34] has been explained as the effect of magnetic-type g-field of the rotating Earth [28], and the positive results have been obtained experimentally [35]. Most significant result is that the gravitational wave, denoted as G-Wave, is detected [36,37]. We argue that, theoretically, the detection of G-Wave indicates the existence of time varying gravito-magnetic fields [38], just as "electromagnetic wave is equivalent to time varying magnetic field".

Experimentally, gravito-magnetic field exists.

4.2.2. Gravito-EM Field Equations

Let's derive Gravito-EM. Starting from UMFT, Eq. (2.12),

$$\oint(\mathbf{v}\times\mathbf{G})\cdot d\mathbf{l} + \frac{d}{dt}\iint \mathbf{G}\cdot d\mathbf{s} = \iint\left[\mathbf{v}(\nabla\cdot\mathbf{G}) + \frac{\partial \mathbf{G}}{\partial t} - \mathbf{G}(\nabla\cdot\mathbf{v}) + (\mathbf{G}\cdot\nabla)\mathbf{v}\right]\cdot d\mathbf{s}, \tag{2.12}$$

where $\mathbf{G}$ can be any vector or axial vector field, and is not Newton constant in Eq. (2.12). Following what we did in section 3, let $\mathbf{G} = \mathbf{g}$, then Eq. (2.12) gives Extended-Ampere-Maxwell-type Gravito-magnetic field equation,

$$\oint(\mathbf{v}\times\mathbf{g})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{g}\cdot ds =\iint\left[\mathbf{v}(\nabla\cdot\mathbf{g})+\frac{\partial\mathbf{g}}{\partial t}-\mathbf{g}(\nabla\cdot\mathbf{v})+(\mathbf{g}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (4.58)$$

The gravito-magnetic field equation, Eq. (4.58), is type-1 dual to the extended-Ampere-Maxwell equation,

$$\oint(\mathbf{v}\times\mathbf{E})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{E}\cdot ds =\iint\left[\mathbf{v}(\nabla\cdot\mathbf{E})+\frac{\partial\mathbf{E}}{\partial t}-\mathbf{E}(\nabla\cdot\mathbf{v})+(\mathbf{E}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (3.12)$$

Namely when we convert between $\mathbf{g}$ and $\mathbf{E}$, $\mathbf{g}\leftrightarrow\mathbf{E}$, gravitation Eq. (4.58) converts to EM Eq. (3.12), and vice versa.

Next let's start from the Faraday-type UMFT, Eq. (2.20),

$$\oint(\mathbf{v}\times\mathbf{M})\cdot d\mathbf{l}+\frac{d}{dt}\iint\mathbf{M}\cdot ds =\iint\left[\mathbf{v}(\nabla\cdot\mathbf{M})+\frac{\partial\mathbf{M}}{\partial \mathbf{t}}-\mathbf{M}(\nabla\cdot\mathbf{v})+(\mathbf{M}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (2.20)$$

Let the $\mathbf{M}$ field is a gravito-magnetic field $\mathbf{B}_g$, $\mathbf{M}=\mathbf{B}_g$, Eq. (2.20) gives,

$$\oint-(\mathbf{v}\times\mathbf{B}_g)\cdot d\mathbf{l}-\frac{d}{dt}\iint\mathbf{B}_g\cdot ds ==-\iint\left[\begin{matrix}\mathbf{v}(\nabla\cdot\mathbf{B}_g)+\frac{\partial\mathbf{B}_g}{\partial\mathbf{t}}\\ -\mathbf{B}_g(\nabla\cdot\mathbf{v})+(\mathbf{B}_g\cdot\nabla)\mathbf{v}\end{matrix}\right]\cdot ds. \quad (4.59)$$

The Faraday-type gravitation field equation, Eq. (4.59), is type-1 dual to the extended-Faraday EM equation, Eq. (3.1),

$$\oint-(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}-\frac{d}{dt}\iint\mathbf{B}\cdot ds =-\iint\left[\mathbf{v}(\nabla\cdot\mathbf{B})+\frac{\partial\mathbf{B}}{\partial \mathbf{t}}-\mathbf{B}(\nabla\cdot\mathbf{v})+(\mathbf{B}\cdot\nabla)\mathbf{v}\right]\cdot ds. \quad (3.1)$$

Namely when we convert between $\mathbf{B}_g$ and $\mathbf{B}$, $\mathbf{B}_g\leftrightarrow\mathbf{B}$, gravitation Eq. (4.59) converts to EM Eq. (3.1), and vice versa.

Eq. (2.9) gives mathematically,

$$\nabla\cdot\mathbf{B}_g\equiv 0, \quad (4.60)$$

Which is the type-1 dual to Eq. (3.2).

We have shown that the Extended-Newton's law, Newton's force law, Eq. (4.58), Eq. (4.59) and Eq. (4.60) are mathematical type-1 duals of Coulomb's law, Coulomb's force law, Eq. (3.12), Eq. (3.1) and Eq. (3.2) respectively. Also Eq. (3.12) and Eq. (3.1) are identical to Eq. (3.14) and Eq. (3.7) respectively.

By transferring law of dualities, mathematical duality can be transferred to physical duality, thus by transferring $\mathbf{E}\leftrightarrow\mathbf{g}$ and $\mathbf{B}\leftrightarrow\mathbf{B_g}$, the Extended-Maxwell equations, Eq. (3.14), Eq. (3.7) and Eq. (3.25), convert to Gravito-EM equations, Eq. (4.62) and Eq. (4.63) and Eq. (4.65) respectively, denoted as the Extended-Gravito-EM equations,

$$\nabla\cdot\mathbf{g}=-\rho_g, \quad (4.61)$$

$$\nabla\times\mathbf{B_g}=\mathbf{v}(\nabla\cdot\mathbf{g})+\frac{\partial\mathbf{g}}{\partial\mathbf{t}}-\mathbf{g}(\nabla\cdot\mathbf{v})+(\mathbf{g}\cdot\nabla)\mathbf{v}, \quad (4.62)$$

$$\nabla\times\mathbf{g}=-\mathbf{v}(\nabla\cdot\mathbf{B_g})-\frac{\partial\mathbf{g}}{\partial\mathbf{t}}+\mathbf{B_g}(\nabla\cdot\mathbf{v})-(\mathbf{B_g}\cdot\nabla)\mathbf{v}, \quad (4.63)$$

$$\nabla\cdot\mathbf{B_g}=0, \quad (4.64)$$

$$\mathbf{F}=q_g\mathbf{g}+q_g\mathbf{v}\times\mathbf{B}_g, \quad (4.65)$$

where $\mathbf{g}$ and $\mathbf{B_g}$ are the Extended-Newton gravitational field and magnetic-type gravitational field respectively, and

$$\mathbf{g}=\mathbf{g}_{\text{vector}}+\mathbf{g}_{\text{axial -vector}}, \quad (4.66)$$

$$\nabla\cdot\mathbf{g}=\nabla\cdot\mathbf{g}_{\text{vector}}, \quad (4.67)$$

$$\nabla\times\mathbf{g}=\nabla\times\mathbf{g}_{\text{axial -vector}}. \quad (4.68)$$

$$\mathbf{B_g}=\mathbf{B}_{g/\text{axial -vector}}, \quad (4.69)$$

$$\nabla\cdot\mathbf{B_g}=\nabla\cdot\mathbf{B}_{g/\text{axial -vector}}=0, \quad (4.70)$$

$$\nabla\times\mathbf{B_g}=\nabla\times\mathbf{B}_{g/\text{axial -vector}}. \quad (4.71)$$

Therefore, it is $\mathbf{g}_{\text{axial -vector}}$ that is type-2 dual to $\mathbf{B}_{g-\text{axial -vector}}$. The $\mathbf{g}_{\text{vector}}$ is the regular vector Newton gravitational field.

Note Gravito-EM field equations, Eq. (4.61) to Eq. (4.65), need empirical confirmation. The Extended Newton's field described by Eq. (4.61) contain two parts, vector field and axial vector field, Eq. (4.66).

The interpretations of terms of the right-hand side of Eq. (4.62) are,

1) The term, $\mathbf{g}(\nabla\cdot\mathbf{v})$, describes stretching of the $\mathbf{g}$ field due to source velocity compressibility;
2) The term, $(\mathbf{g}\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the $\mathbf{g}$ field due to the velocity gradients.
3) Extended-Ampare-Maxwell-type gravitational law, Eq. (4.62), predicts that the spatially-varying velocity, such as particles distributing and moving in space, terms $\mathbf{g}(\nabla\cdot\mathbf{v})$ and $(\mathbf{g}\cdot\nabla)\mathbf{v}$ create axial vector $\mathbf{B_g}$ fields.
4) To detect the effects of the terms, $\mathbf{g}(\nabla\cdot\mathbf{v})$ and $(\mathbf{g}\cdot\nabla)\mathbf{v}$, will confirm whether Extended-Maxwell-type gravitation equations are correct.

Similarly, the interpretations of terms of the right-hand side of Eq. (4.63) are,

1) The term, $\mathbf{B}_g(\nabla\cdot\mathbf{v})$, describes stretching of the $\mathbf{B}_g$ field due to source velocity compressibility;
2) The term, $(\mathbf{B}_g\cdot\nabla)\mathbf{v}$, describes the stretching or tilting of the $\mathbf{B}_g$ field due to the velocity gradients.
3) Extended-Faraday-type gravitational law, Eq. (4.63), predicts that the spatially-varying velocity, such as particles distributing and moving in space, terms $\mathbf{B}_g(\nabla\cdot\mathbf{v})$ and $(\mathbf{B}_g\cdot\nabla)\mathbf{v}$ induce axial vector $\mathbf{g}$ fields.
4) To detect the effects of the terms, $\mathbf{B}_g(\nabla\cdot\mathbf{v})$ and $(\mathbf{B}_g\cdot\nabla)\mathbf{v}$, will confirm whether Extended-Maxwell-type gravitation equations are correct.

For non-spatially-dependent velocity,

$$(\nabla\cdot\mathbf{v})=(\mathbf{g}\cdot\nabla)\mathbf{v}=(\mathbf{B_g}\cdot\nabla)\mathbf{v}=0,$$

the Extended-Gravito-EM (Eq. (4.61) to Eq. (4.64)) is simplified to Gravito-EM as

$$\nabla \cdot \mathbf{g} = -\rho_g, \tag{4.61}$$

$$\nabla \times \mathbf{B_g} = \mathbf{v}(\nabla \cdot \mathbf{g}) + \frac{\partial \mathbf{g}}{\partial \mathbf{t}}, \tag{4.72}$$

$$\nabla \times \mathbf{g} = -\mathbf{v}(\nabla \cdot \mathbf{B_g}) - \frac{\partial \mathbf{B_g}}{\partial t}, \tag{4.73}$$

$$\nabla \cdot \mathbf{B_g} = 0. \tag{4.64}$$

Due to the duality between EM equation and Gravito-EM equation, the Gravito-EM field strengths can be expressed in terms of scalar and vector potentials,

$$\mathbf{B}_g = \boldsymbol{\nabla} \times \mathbf{A_g}, \tag{4.74}$$

$$\mathbf{g} = -\nabla V_g - \frac{\partial \mathbf{A}_g}{\partial t}. \tag{4.75}$$

Table 4.2. Significant Different between Theories of Gravity

	Field equation	Source	Four-current
EM	$\frac{\partial F_e^{\alpha\beta}}{\partial x^\beta} = -J_e^\alpha$	Q_e: Q_e is invariant	$j_e^\mu = (\rho_e, \mathbf{j}_e)$
Gravito-EM	$\frac{\partial F_{g\pm}^{\alpha\beta}}{\partial x^\beta} = -J_{g\pm}^\alpha$	$Q_g = \sqrt{G}m_0$: m_0: rest mass Q_g is invariant	$j_{g\pm}^\mu = (\rho_g, \mathbf{j}_g)$
Linearized GR	$\frac{\partial G_g^{\alpha\beta\mu}}{\partial x^\mu} = -T_g^{\alpha\beta}$	$m_g = \gamma m_0$	No
Other vector theories	$\frac{\partial F_g^{\alpha\beta}}{\partial x^\beta} = -m_g U^\alpha$	$m_g = \gamma m_0$:	No

4.2.3. Type-1 Duality between Field Equations of Positive and Negative g-Charges

Due to the positive and negative g-charge conjugate, the Gravito-EM equations derived above can be applied to Gravito-EM fields created by either positive g-charge/g-current or negative g-charges/g-current with the same forms:

$$\nabla \cdot \mathbf{g}_\pm = -\rho_{g\pm}, \tag{4.76}$$

$$\nabla \times \mathbf{B_{g\pm}} = \mathbf{v}(\nabla \cdot \mathbf{g}_\pm) + \frac{\partial \mathbf{g}_\pm}{\partial t} - (\mathbf{g}_\pm)(\nabla \cdot \mathbf{v}) + (\mathbf{g}_\pm \cdot \nabla)\mathbf{v}, \tag{4.77}$$

$$\nabla \times \mathbf{g}_\pm = -\mathbf{v}(\nabla \cdot \mathbf{B_{g\pm}}) - \frac{\partial \mathbf{B_{g\pm}}}{\partial t} + (\mathbf{B_{g\pm}})(\nabla \cdot \mathbf{v}) - (\mathbf{B_{g\pm}} \cdot \nabla)\mathbf{v}, \tag{4.78}$$

$$\nabla \cdot \mathbf{B_{g\pm}} = 0, \tag{4.79}$$

$$\mathbf{F} = (q_{g+})\mathbf{g}_\pm + (q_{g+})(\mathbf{v} \times \mathbf{B}_{g\pm}), \tag{4.80}$$

$$\mathbf{F} = (q_{g-})\mathbf{g}_\pm + (q_{g-})(\mathbf{v} \times \mathbf{B}_{g\pm}), \tag{4.81}$$

where $\mathbf{g}_\pm$ and $\mathbf{B}_{g\pm}$ are the Extended-Newton gravitational field and magnetic-type gravitational field created by either positive or negative g-charge and either positive or negative g-current respectively. We have

$$\mathbf{g}_\pm = \mathbf{g}_{\pm/\text{vector}} + \mathbf{g}_{\pm/\text{axial}-\text{vector}}, \tag{4.82}$$

$$\nabla \cdot \mathbf{g}_\pm = \nabla \cdot \mathbf{g}_{\pm/\text{vector}}, \tag{4.83}$$

$$\nabla \times \mathbf{g}_\pm = \nabla \times \mathbf{g}_{\pm/\text{axial}-\text{vector}}. \tag{4.84}$$

$$\mathbf{B}_{g\pm} = \mathbf{B}_{g\pm/\text{axial}-\text{vector}}, \tag{4.85}$$

$$\nabla \cdot \mathbf{B_{g\pm}} = \nabla \cdot \mathbf{B}_{g\pm/\text{axial}-\text{vector}} = 0, \tag{4.86}$$

$$\nabla \times \mathbf{B_{g\pm}} = \nabla \times \mathbf{B}_{g\pm/\text{axial}-\text{vector}}. \tag{4.87}$$

$$\mathbf{B}_{g\pm} = \boldsymbol{\nabla} \times \mathbf{A_{g\pm}}, \tag{4.88}$$

$$\mathbf{g}_\pm = -\nabla V_{g\pm} - \frac{\partial \mathbf{A}_{g\pm}}{\partial t}. \tag{4.89}$$

Therefore, it is $\mathbf{g}_{\pm/\text{axial}-\text{vector}}$ that is type-2 dual to $\mathbf{B}_{g\pm}$. The $\mathbf{g}_{\pm/\text{vector}}$ is the vector Newton gravitational field.

For the situations of non-spatially-varying velocity, the extended-Gravito-EM equation become

$$\nabla \cdot \mathbf{g}_\pm = -\rho_{g\pm}, \tag{4.90}$$

$$\nabla \times \mathbf{B_{g\pm}} = \mathbf{v}(\nabla \cdot \mathbf{g}_\pm) + \frac{\partial \mathbf{g}_\pm}{\partial t}, \tag{4.91}$$

$$\nabla \times \mathbf{g}_\pm = -\mathbf{v}(\nabla \cdot \mathbf{B_{g\pm}}) - \frac{\partial \mathbf{B_{g\pm}}}{\partial t}, \tag{4.92}$$

$$\nabla \cdot \mathbf{B_{g\pm}} = 0. \tag{4.93}$$

For simplicity, hereafter, we will drop the subscript sign "±".

4.2.4. Type-2 Duality between Newton and Magnetic-type Gravitational Fields

Based on the Transfer Rules of Appendix A2, there is a duality between Eq. (4.91) and Eq. (4.92), i.e., under transformation $\mathbf{g} \leftrightarrow \mathbf{B}_g$ and $\mathbf{B}_g \leftrightarrow -\mathbf{g}$, Eq. (4.91) and Eq. (4.92) convert to each other. With distinguishable feature of "type-1 duality" and "type-2 duality", the duality between induced gravitational field determined by the Faraday-type gravitational law, Eq. (4.92), and magnetic-type gravitational field determined by Ampere-Maxwell-type gravitational equation, Eq. (4.91), is actually a type-2 duality, as well as Eq. (4.77) and Eq. (4.78). UMFT provides the mathematical origin of the type-2 duality between axial vector induced gravitational field and magnetic-type axial vector gravitational field.

4.2.5. Type-1 Duality between EM and Gravito-EM

So far, we have encountered and proposed several dualities:

1) positive and negative e-charges conjugate;
2) positive and negative g-charges conjugate (proposed);
3) duality between e-charge and g-charge;
4) duality between Coulomb's law and Newton's law, which is perfect after introducing duality (3);
5) duality between EM and Gravito-EM (proposed based on UMFT and duality (3).

Based on the above dualities, the phenomena in EM, such as wave, quantization, renormalization and unification, can be transferred into Gravito-EM directly without needing complex calculation.

4.2.6. Origin of Differences between Newton Field and Gravito-Magnetic Field

A magnetic-type gravitational field is completely different from a Newton field in the following senses:

(1) sources: “g-charge” vs. “g-current”;

(2) way the fields are generated: “$\nabla \cdot \mathbf{g}$” vs. “$\nabla \times \mathbf{B}_g$”;

(3) nature of fields: “vector field $\mathbf{g}$” vs. “axial vector field $\mathbf{B}_g$”;

(4) forces acting on a test g-charge: “$q_g \mathbf{g}$” vs. “$q_g \mathbf{v} \times \mathbf{B}_g$”.

The origin of those differences is the same as that of differences between Coulomb electric field and magnetic field. All of above-mentioned differences between Newton field and magnetic-type gravitation field have a mathematical origin that is Eq. (2.1).

Gravito-EM shows that gravity is a local physical field like the EM field and thus, is quantizable and renormalizable [3].

Moreover, Gravito-EM have the same form as that of linearized GR, Eq. (4.54) to Eq. (4.57), which implies that all of results of linearized GR can also be obtained from Gravito-EM, including gravitation wave (G-Wave). The existences of time-varying-gravito-magnetic field and G-Wave are equivalent. So now, there are either magnetic or magnetic-type fields in all of four interactions in nature.

4.2.7. Equation of Continuity

4.2.7.1. g-Current Creating Gravito-Magnetic Field

Utilizing Newton’s law, $\nabla \cdot \mathbf{g} = -\rho_g$, let’s define a “current”, $\mathbf{j}_{\mathbf{v}-\mathbf{B}_g}$, that generates magnetic-type gravitational field $\mathbf{B}_g$,

$$\mathbf{j}_{\mathbf{v}-\mathbf{B}_g} = -\rho_g \mathbf{v} - \mathbf{g}(\nabla \cdot \mathbf{v}) + (\mathbf{g} \cdot \nabla)\mathbf{v}. \quad (4.94)$$

Where the subscripts “$\mathbf{v} - \mathbf{B}_g$” represent the quantity generating the gravito-magnetic field $\mathbf{B}_g$. Then Eq. (4.62) becomes,

$$\nabla \times \mathbf{B}_g = \mathbf{j}_{\mathbf{v}-\mathbf{B}_g} + \frac{\partial \mathbf{g}}{\partial t}, \quad (4.95)$$

The current $\mathbf{j}_{\mathbf{v}-\mathbf{B}_g}$ satisfies the equation of continuity,

$$\nabla \cdot \mathbf{j}_{\mathbf{v}-\mathbf{B}_g} + \frac{\partial (\nabla \cdot \mathbf{g})}{\partial t} = 0. \quad (4.96)$$

For the situation of the non-spatially-varying velocity, $\mathbf{g}(\nabla \cdot \mathbf{v}) = (\mathbf{g} \cdot \nabla)\mathbf{v} = 0$, we have

$$\mathbf{j}_{\mathbf{v}-\mathbf{B}_g} = -\rho_g \mathbf{v}. \quad (4.97)$$

Extended-Ampere-Maxwell-type gravitational equation, Eq. (4.62), becomes the Ampere-Maxwell-type gravitational law,

$$\nabla \times \mathbf{B}_g = \mathbf{v}(\nabla \cdot \mathbf{g}) + \frac{\partial \mathbf{g}}{\partial t}. \quad (4.98)$$

The Ampere-Maxwell-type gravitation law Eq. (4.98) provides a mathematical origin of why and how e-current, $\rho_g \mathbf{v}$, and displacement-current, $\frac{\partial \mathbf{g}}{\partial \mathbf{t}}$, create inevitably gravito-magnetic fields.

4.2.7.2. G-Current Creating Induced Gravito-Electric Field

Let’s define a “current” generating induced gravitation field, as

$$\mathbf{j}_{\mathbf{v}-\mathbf{g}} \equiv \mathbf{v}(\nabla \cdot \mathbf{B}_g) - \mathbf{B}_g(\nabla \cdot \mathbf{v}) + (\mathbf{B}_g \cdot \nabla)\mathbf{v}. \quad (4.99)$$

Where the subscripts “v-g” represent the quantity generating the induced axial vector gravitational field.

Then Extended-Faraday-type gravitational law, Eq. (4.63), may be rewritten as,

$$\nabla \times \mathbf{g} = -\mathbf{j}_{\mathbf{v}-\mathbf{g}} - \frac{\partial \mathbf{B}_g}{\partial t}. \quad (4.100)$$

Eq. (4.100) shows the equation of continuity as

$$\nabla \cdot \mathbf{j}_{\mathbf{v}-\mathbf{g}} + \frac{\partial (\nabla \cdot \mathbf{B}_g)}{\partial t} = 0. \quad (4.101)$$

For the situation in which, the velocity is non-spatially-varying, i.e., $\mathbf{B}_g(\nabla \cdot \mathbf{v}) = (\mathbf{B}_g \cdot \nabla)\mathbf{v} = 0$, Extended-Faraday-type gravitation law, Eq. (4.100), becomes,

$$\nabla \times \mathbf{g} = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad (4.102)$$

where the $\mathbf{g}$ field is an axial vector field and

$$\nabla \cdot \mathbf{B}_g = 0,$$

$$\mathbf{j}_{\mathbf{v}-\mathbf{g}} = 0. \quad (4.103)$$

4.3. Gravito-Magnetic Field of Steady g-Current

4.3.1. Gravito-magnetic Field Line

The gravito-magnetic field lines of a steady g-current are determined by Eq. (4.72), which can be written as the following,

$$\mathbf{B_g} = \mathbf{B_{g+}} + \mathbf{B_{g-}},$$

$$\nabla \times \mathbf{B_{g+}} = -\rho_{g+}\mathbf{v} = -\rho_{m0+}\mathbf{v}, \quad (4.104)$$

$$\nabla \times \mathbf{B_{g-}} = -\rho_{g-}\mathbf{v} = \rho_{m0-}\mathbf{v}, \quad (4.105)$$

where $\rho_{m0-} > 0$, $\rho_{m0+} > 0$.

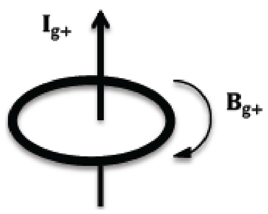

Figure 4.7. “The left-hand rule” for gravito-magnetic field $\mathbf{B}_{g+}$

To determine the directions of the steady positive g-current and its gravito-magnetic field, Eq. (4.104) indicates to use “the left-hand rule”. Similarly, Eq. (4. 105) requires to apply the “left-hand rule” on determining the gravito-magnetic field line of a negative g-current. The “left-hand rule” states: if one’s thumb of left-hand points to the direction of a g-current, for example a positive g-current, the fingers point to the direction of the gravito-magnetic field, say $\mathbf{B}_{g+}$ (Figure 4.7).

4.3.2. Interaction between Two Steady g-Currents

Two positive g-currents $\mathbf{I}_{g+}$ in same directions repel to each other (Figure 4.8). Two positive g-currents $\mathbf{I}_{g+}$ in opposite directions attract to each other (Figure 4.9).

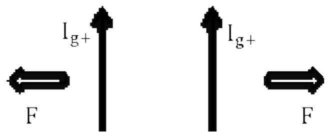

Figure 4.8.

Figure 4.9.

Two negative g-currents $\mathbf{I}_{g-}$ in same directions repel to each other (Figure 4.10). Two negative g-currents $\mathbf{I}_{g-}$ in opposite directions attract to each other (Figure 4.11).

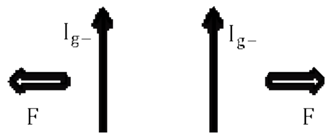

Figure 4.10.

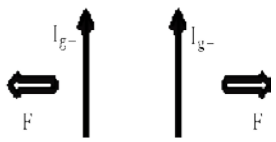

Figure 4.11.

A negative g-current $\mathbf{I}_{g-}$ and a positive g-current$\mathbf{I}_{g+}$ in same directions attract to each other (Figure 4.12). A negative g-current and a positive g-current in opposite directions repel to each other (Figure 4.13).

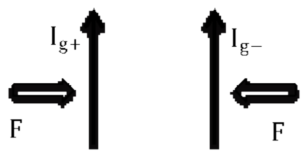

Figure 4.12.

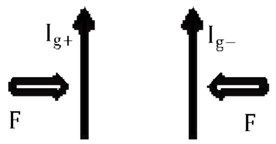

Figure 4.13.

4.4. Rotation Curve: Dark Matter

Considering a spiral galaxy with a rotation symmetry. The rotating motion of the galaxy generates a gravito-magnetic field $\mathbf{B}_g$. A moving star with g-charge, q_g , at a radius R to the center of the galaxy, will experience a Lorentz-type gravitational force,

$$\mathbf{F} = q_g \mathbf{g} + q_g \mathbf{v} \times \mathbf{B}_g,$$

which consists of two parts, the Newtonian force, $g = \frac{\sqrt{G}M}{R^2}$, and the gravito-magnetic force and $\mathbf{v} \perp \mathbf{B}_g$, both have the same direction.

The Lorentz-type force keeps the star moves around the center of the galaxy,

$$\frac{v^2}{R} = \frac{GM}{R^2} + \sqrt{G} v B_g, \qquad (4.106)$$

or

$$v^2 - \left(\sqrt{G} B_g R\right) v - \frac{GM}{R^2} = 0, \qquad (4.107)$$

where M(R) is the mass interior to the radius R, $\mathbf{B}_g$ is determined by the model of distribution of stars in the galaxy.

4.5. Extended-Virial Theorem in EM and Gravito-Magnetics Field

Let's review the concept and then extend the virial theorem to include Gravito-magnetic field. The scalar momentum of inertial I of a system containing "n" particles is,

$$I = \sum_{k=1}^{n} m_k r_k^2. \qquad (4.108)$$

The "G" is defined as (in Section 4.5, "G" is not Newton constant),

$$G = \sum_{k=1}^{n} \mathbf{p}_k \cdot \mathbf{r}_k. \qquad (4.109)$$

Taking time derivatives of Eq. (4.108) and Eq. (4.109), respectively,

$$\frac{dI}{dt} = 2\sum_{k=1}^{n} m_k \frac{d\mathbf{r}_k}{dt} \cdot \mathbf{r}_k = 2G, \qquad (4.110)$$

$$\frac{dG}{dt} = 2T + \sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{r}_k, \qquad (4.111)$$

$$T = \frac{1}{2}\sum_{k=1}^{n} m_k v_k^2, \qquad (4.112)$$

$$\mathbf{F}_k = \sum_{j=1}^{n} \mathbf{F}_{jk}. \qquad (4.113)$$

Where m_k, $\mathbf{r}_k$, $\mathbf{v}_k$ and $\mathbf{p}_k$ are the mass, position, velocity and momentum of the kth particle respectively. The $\mathbf{F}_{jk}$ is the force of the particle j acting on the particle k. The $\mathbf{F}_k$ is the total force acting on the kth particle. The "T" is the total kinetic energy of the system, which is the common notation in the virial theorem, and is represented as the K_g.

In the situation of equilibrium of a system consisting of equal mass particles with spherical distribution, the virial theorem states that

$$2T + W_{g/N} = 0, \qquad (4.114)$$

where $W_{g/N}$ is the total potential energy of the Newton's gravitational field. Then, for the situation of presence of magnetic field, it has been extended to include the energy of heat and the total energy of the magnetic field [39,40], as,

$$2T + 3(\gamma - 1)W_{heat} + W_B + W_{g/N} = 0, \qquad (4.115)$$

where γ is the ration of the specific heat, W_{heat} and W_B are the heat energy and the total energy of the magnetic field, respectively.

Note based on Gravito-EM, the total potential energy of the static gravitational field can be represented in terms of field strength **g** as

$$W_{g/N} = \frac{1}{2}\int g^2 d\tau.$$

Now, with the presence of Gravito-magnetic field, we propose to extend the virial theorem, Eq. (4.114), as,

$$2T + W_{g/N} + W_{B_g} = 0, \qquad (4.116)$$

Where

$$W_{B_g} = \frac{1}{2}\int B_g^2 d\tau,$$

is the total energy of the gravito-magnetic field.

Moreover extending Eq. (4.115) to include the total energy of gravito-magnetic field, we obtain the extended-virial theorem

$$2T + 3(\gamma - 1)W_{heat} + W_B + W_{B_g} + W_{g/N} = 0, \qquad (4.117)$$

To get a taste of how to apply the virial theorem of Gravito-EM to build a modified rotation curve, let's consider a simplest situation, for which, Eq. (4.117) only have gravitation related terms,

$$2T + W_{B_g} + W_{g/N} = 0. \qquad (4.118)$$

4.6. Poynting-type Theorem in Gravito-EM

Let's consider g-charges moving with velocity $\sum_i \mathbf{v}_i$ and interacting with gravito-electric and gravito-magnetic fields. The gravitational field changes the kinematic energy of g-charges, i.e., does the work, so decrease its field energy. The change of the kinetic energy K_g of g-charges is negative of the work done by the gravitational field, which implies that the increase of kinematic energy of g-charges is equal to the decrease of gravitational field energy,

$$dK_g = -\mathbf{F}\cdot d\mathbf{l}. \qquad (4.119)$$

Substituting Gravito-Lorentz-type force, then

$$\mathbf{F}\cdot d\mathbf{l} = q_g\left(\mathbf{g} + \mathbf{v}\times\mathbf{B_g}\right)\cdot\mathbf{v}dt = q_g\mathbf{g}\cdot\mathbf{v}dt = -\int(\mathbf{g}\cdot\mathbf{j_g})d\tau dt,$$

$$\frac{dK_g}{dt} = \int(\mathbf{g}\cdot\mathbf{j_g})d\tau, \qquad (4.120)$$

where $d\mathbf{l} = \mathbf{v}dt$, $q_g = \int \rho_g d\tau$ and $\mathbf{j_g} = -\rho_g\mathbf{v}$.

Substituting, $\nabla\times\mathbf{B_g} = \mathbf{j_g} + \frac{\partial\mathbf{g}}{\partial\mathbf{t}} - \mathbf{g}(\nabla\cdot\mathbf{v}) + (\mathbf{g}\cdot\nabla)\mathbf{v}$, we obtain

$$\begin{aligned}\mathbf{g}\cdot\mathbf{j_g} &= \mathbf{g}\cdot\left(\nabla\times\mathbf{B}_g\right) - \mathbf{g}\cdot\frac{\partial\mathbf{g}}{\partial\mathbf{t}} \\ &+ g^2(\nabla\cdot\mathbf{v}) - \mathbf{g}\cdot\left[(\mathbf{g}\cdot\nabla)\mathbf{v}\right].\end{aligned} \qquad (4.121)$$

The formula of vector analysis gives

$$\mathbf{g}\cdot\left(\nabla\times\mathbf{B}_g\right) \equiv \mathbf{B}_g\cdot(\nabla\times\mathbf{g}) - \nabla\cdot\left(\mathbf{g}\times\mathbf{B}_g\right).$$

Substituting, $\nabla\times\mathbf{g} = -\frac{\partial\mathbf{B_g}}{\partial t} + \mathbf{B}_g(\nabla\cdot\mathbf{v}) - \left(\mathbf{B}_g\cdot\nabla\right)\mathbf{v}$, into it, we obtain,

$$\begin{aligned}\mathbf{g}\cdot\left(\nabla\times\mathbf{B}_g\right) &= -\frac{1}{2}\frac{\partial B_g^2}{\partial t} + B_g^2(\nabla\cdot\mathbf{v}) \\ &-\mathbf{B}_g\cdot\left[\left(\mathbf{B}_g\cdot\nabla\right)\mathbf{v}\right] - \nabla\cdot\left(\mathbf{g}\times\mathbf{B}_g\right).\end{aligned} \qquad (4.122)$$

Substituting Eq. (4.122) into Eq. (4.121), we obtain

$$\begin{aligned}\mathbf{g}\cdot\mathbf{j_g} &= -\frac{1}{2}\frac{\partial}{\partial t}\left(B_g^2 + g^2\right) - \nabla\cdot\left(\mathbf{g}\times\mathbf{B}_g\right) \\ &+\left(B_g^2 + g^2\right)(\nabla\cdot\mathbf{v}) - \mathbf{B}_g\cdot\left[\left(\mathbf{B}_g\cdot\nabla\right)\mathbf{v}\right] \\ &-\mathbf{g}\cdot\left[(\mathbf{g}\cdot\nabla)\mathbf{v}\right].\end{aligned} \qquad (4.123)$$

Substituting Eq. (4.123) into Eq. (4.120), we derived the *extended-Poynting-type theorem of gravity*,

$$\begin{aligned}\frac{dK_g}{dt} &= -\frac{dW_g}{dt} - \oiint \mathbf{S}_g\cdot d\mathbf{a} + \\ &+\int\left\{\begin{aligned}&\left(B_g^2 + g^2\right)(\nabla\cdot\mathbf{v}) \\ &-\mathbf{B}_g\cdot\left[\left(\mathbf{B}_g\cdot\nabla\right)\mathbf{v}\right] \\ &-\mathbf{g}\cdot\left[(\mathbf{g}\cdot\nabla)\mathbf{v}\right]\end{aligned}\right\}d\tau,\end{aligned} \qquad (4.124)$$

Where the Poynting's vector S is defined as,

$$\mathbf{S}_g \equiv \mathbf{g}\times\mathbf{B}_g. \qquad (4.125)$$

For the special situation of non-spatially-varying velocity, $\nabla\cdot\mathbf{v} = \left(\mathbf{B}_g\cdot\nabla\right)\mathbf{v} = (\mathbf{g}\cdot\nabla)\mathbf{v} = 0$, the extended-Poynting-type theorem is simplified to the Poynting-type theorem of gravity,

$$\frac{dK_g}{dt} = -\frac{dW_g}{dt} - \oiint \mathbf{S}_g\cdot d\mathbf{a}. \qquad (4.126)$$

K_g is the kinetic energy of all of the g-charges in the volume V. The $\mathbf{S}_g$ is the gravitational counterpart of the Poynting vector in EM.

We interpret Eq. (4.126) as: the increase of the kinetic energy of all of the g-charges in certain volume is equal to the decrease of the energy the gravitational field lost, less the energy that flowed out through the surface.

When the kinetic energy of the gravitational charges is negligible, Eq. (4.126) indicates that the change of energy of the gravitational field is equivalent to the energy flowed out through the surface,

$$\frac{d}{dt}\int\frac{\mathbf{B}_g^2 + \mathbf{g}^2}{2}d\tau = -\oint \mathbf{S}_g\cdot d\mathbf{a}. \qquad (4.127)$$

The density of momentum of the gravitational field store in the field is,

$$\mathbf{p}_g = \mathbf{S}_g = \frac{1}{4\pi}\mathbf{g}\times\mathbf{B}_g. \qquad (4.128)$$

The gravitational field also carries angular momentum,

$$\mathbf{l}_g = \mathbf{r}\times\mathbf{p}_g. \qquad (4.129)$$

4.7. Energy-Momentum Tensor of Gravito-EM Fields

For this aim, let's introduce the Lagrangian density Λ of vector Gravito-EM fields,

$$\Lambda = -\frac{1}{16\pi} F_{g\alpha\beta} F_g^{\alpha\beta}, \tag{4.130}$$

and define the energy-momentum tensor of Gravito-EM fields as

$$T_{g\alpha}^{\beta} = \frac{\partial A_{g\gamma}}{\partial x^{\alpha}} \frac{\partial \Lambda}{\partial \left[\dfrac{\partial A_{g\gamma}}{\partial x^{\beta}}\right]} - \delta_{\alpha}^{\beta}\Lambda. \tag{4.131}$$

Substituting Eq. (4.130) into Eq. (4.131), we obtain the symmetric energy-momentum tensor of Gravito-EM field,

$$T_g^{\alpha\beta} = \frac{1}{4\pi}\left[-F_g^{\alpha\gamma}F_{g\gamma}^{\beta} + \frac{1}{4}\eta^{\alpha\beta}F_{g\gamma\delta}F_g^{\gamma\delta}\right], \tag{4.132}$$

$$T_{g\alpha}^{\alpha} = 0. \tag{4.133}$$

Note here: (1)The energy-momentum of vector Gravito-EM fields, $T_g^{\alpha\beta}$, is a tensor; on the contrary, $T^{\alpha\beta}$ in GR is a pseudo-tensor, which is considered as one of issue of GR; (2) $T_g^{\alpha\beta}$ represents either $T_{g+}^{\alpha\beta}$ related with positive g-charge/g-current, or $T_{g-}^{\alpha\beta}$ related with negative g-charge/g-current, or $T_{gN}^{\alpha\beta}$ related with net g-charge/g-current; (3) Zero trace, $T_{g\mu}^{\mu} = 0$; (4) $T_g^{\mu\nu}$ is symmetric, $T_g^{\mu\nu} = T_g^{\nu\mu}$; (5) Positive energy density, $T_g^{00} > 0$; (6) Conserved, $\frac{\partial T_{g\nu}^{\mu}}{\partial x_{\mu}} = 0$; (7) Localizable.

4.8. Retarded Gravitational Potentials

The gravito-retarded potentials of a point g-charge moving arbitrarily are,

$$V_g = -\frac{Q_g}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}, \tag{4.134}$$

$$\mathbf{A}_g = -\frac{Q_g\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})} = \boldsymbol{\beta}V_g. \tag{4.135}$$

We denote Eq. (4.134) and Eq. (4.135) as the *Lienard-Wiechert-type potentials.*

4.9. Gravitational Radiation

Lienard-type formula: The total radiation power of a non-relativistic single g-charge is proportional to the square of the acceleration of the charge,

$$I = \frac{2Gm^2}{3c^3}\mathbf{a}^2 \tag{4.136}$$

Where the non-radiation term, $\sim 1/R^2$, have been ignored. G is Newton constant. We denote Eq. (4.136) as the *Larmor-type formula of Gravitational wave (G-Wave).*

Lienard-type formula: Now let's consider a relativistic g-charge moving with velocity, $\mathbf{V} \sim C$, and acceleration, "**a**", relative to an observer. According to SR, transferring to the observer's system, we obtain the energy of radiation,

$$W = \frac{2Gm^2}{3C^3}\int_{-\infty}^{\infty}\frac{1}{\left(1-\beta^2\right)^3}\left[a^2 - (\boldsymbol{\beta}\times\mathbf{a})^2\right]dt. \tag{4.137}$$

We denote Eq. (4.137) as the *Lienard-type formula*. In terms of the gravitational field strength, we obtain,

$$W = \frac{2G^2m^2}{3C^3}\int_{-\infty}^{\infty}\frac{1}{\left(1-\beta^2\right)^2}\begin{bmatrix}(\mathbf{g}+\boldsymbol{\beta}\times\mathbf{B_g})^2 \\ -(\boldsymbol{\beta}\cdot\mathbf{g})^2\end{bmatrix}dt. \tag{4.138}$$

Bremsstrahlung-type radiation: For the angular distribution of the radiation energy at a long-distance R, by keeping only the lowest term in 1/R, we obtain the intensity radiated into the solid angle $d\Omega$,

$$dI = \frac{Gm^2}{4\pi C^3}\left\{\begin{array}{l}\dfrac{\mathbf{a}^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^4} + \dfrac{2(\mathbf{n}\cdot\mathbf{a})(\boldsymbol{\beta}\cdot\mathbf{a})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5} \\ -\dfrac{\left(1-\beta^2\right)(\mathbf{n}\cdot\mathbf{a})^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^6}\end{array}\right\}d\Omega. \tag{4.139}$$

When the velocity and acceleration of a relativistic g-charge are collinear, i.e., $\mathbf{v}\times\mathbf{a} = 0$, we obtain the *bremsstrahlung-type radiation*,

$$I = \frac{2Gm^2a^2}{3C^3\left(1-\beta^2\right)^3}. \tag{4.140}$$

Synchrotron-type radiation: Let's consider G-Wave radiation of a star orbiting around a rotating massive neutron star or a rotating black hole that generates a gravito-magnetic field $\mathbf{B}_g$ around itself. The radius r of the orbit, the cyclic frequency ω_s of the motion, the gravito-magnetic field $\mathbf{B}_g$, and the velocity **v** of the star are related by,

$$r = \frac{CV}{\sqrt{G}B_g\sqrt{1-\beta^2}}, \tag{4.141}$$

$$\omega_{gs} = \frac{\sqrt{G}B_g}{C}\sqrt{1-\beta^2}. \tag{4.142}$$

When ignoring the time integral for simplicity as well as setting $\mathbf{g} = 0$ and $\mathbf{B_g} \perp \mathbf{v}$, we have the total intensity of radiation,

$$I = \frac{2G^2B_g^2p^2}{3C^5}, \tag{4.143}$$

where $\mathbf{p} = m\mathbf{v}/\sqrt{1-\beta^2}$ is the momentum of the star. The total intensity of radiation is proportional to the square of momentum **p**. This radiation is the *synchrotron-type radiation or gravito-magnetic bremsstrahlung.*

4.10. Transformation Law of Energy Density of G-Wave

From SR, the transformation law of the energy density, $W_{G-Wave} = \frac{1}{8\pi}\left(g^2 + B_g^2\right)$, of a plane G-Wave from one inertial reference system K to another system K′ is

$$W_{G-Wave} = \frac{\left(1+\beta\cos\alpha'\right)^2}{1-\beta^2} W'_{G-Wave}. \quad (4.144)$$

The α' is the angle in the K' system between the direction of velocity **v** and the direction of propagation of G-Wave; $\beta = v/c$.

4.11. Doppler-type Effect of G-Wave

Now let's derive the *Doppler-type effect* for G-Wave. As mentioned above, Gravito-EM is compatible with SR. An important conclusion is that the redshifts of G-Wave follows the rule same as that of EM-Wave. According to the rule of transformation of the four-vector from a source-rest system K_s to an observer system K_o moving with velocity "$-v$" relative to the system K_s, we obtain the *Doppler-type effect,*

$$\omega_{go} = \omega_{gs} \frac{\sqrt{1-\beta^2}}{1-\beta\cos\alpha}, \quad (4.145)$$

$$\lambda_{go} = \lambda_{gs} \frac{1-\beta\cos\alpha}{\sqrt{1-\beta^2}}, \quad (4.146)$$

where ω_{gs} (λ_{gs}) and ω_{go} (λ_{go}) are, respectively, the frequencies (wavelength) of G-Wave in the source-rest system K_s and in the observer system K_o, respectively; α is the angle, in the K_o system, between the direction of emission of G-Wave and the direction of motion of the source. Eq. (4.146) is the gravitational counterpart of the Doppler effect of EM-Wave. Now we can find the frequency/wavelength of emitted G-Wave from observed G-Wave.

The formulas describing both EM-Wave Doppler effect and the gravito-Doppler effect have the same form. The redshifts of EM-Wave and G-Wave radiated by the same source carrying both e-charge and g-charge are the same. Therefore, by measuring the redshift, $\mathbb{Z}_{EM-W}$, of EM-Wave, we obtain the "redshift velocity" of the source of both EM-Wave and G-Wave from the following formula,

$$\mathbb{Z}_{G-Wave} = \mathbb{Z}_{EM-Wave} \equiv \frac{1-\beta\cos\alpha}{\sqrt{1-\beta^2}} - 1. \quad (4.147)$$

Returning back to G-Wave, substituting Eq. (4.147) into Eq. (4.146), we obtain

$$\omega_{gs} = \omega_{g0}\left(1 + \mathbb{Z}_{EM-Wave}\right), \quad (4.148)$$

$$\lambda_{gs} = \frac{\lambda_{go}}{1+\mathbb{Z}_{EM-Wave}}. \quad (4.149)$$

By Eq. (4.149), we can obtain the emitted frequency and wavelength of G-Wave from the redshift of EM-Wave.

For $\alpha = 90^0$, Eq. (4.149) gives the *transverse Doppler-type effect* of G-Wave,

$$\lambda_{gs} = \lambda_{go}\sqrt{1-\beta^2}, \quad (4.150)$$

$$\omega_{gs} = \omega_{go}\frac{1}{\sqrt{1-\beta^2}}. \quad (4.151)$$

For the case of $v \ll c$ and α not close to 90^0, Eqs. (4.148) and (4.149) becomes respectively,

$$\omega_{gs} \cong \omega_{go}(1-\beta\cos\alpha), \quad (4.152)$$

$$\lambda_{gs} \cong \frac{\lambda_{go}}{\left(1-\beta\cos\alpha\right)}. \quad (4.153)$$

Eqs. (4.148) to (4.153) give the relations between wavelength/frequency and the velocity of the source of G-Wave.

For the redshift of g-wave emitted by an accelerating source, please see Appendix E.

4.12. Hubble Constant and Redshift of G-Wave

For a source radiating only G-Wave, how can we find redshift of G-Wave? For this aim, we need to find the relations between frequency/wavelength and the distance of the source of G-Wave. For the time scale measuring G-Wave, we only consider the constant Hubble parameter. An expression for Hubble's Law is,

$$v = H_0 R_0, \quad (4.154)$$

where H_0 is Hubble's constant, R_0 is the proper distance.

Substituting Eq. (4.154) into Eq. (4.147), we obtain the redshift,

$$\mathbb{Z}_{G-Wave} \equiv \frac{1-\frac{H_0R_0}{c}\cos\alpha}{\sqrt{1-\left(\frac{H_0R_0}{c}\right)^2}} - 1. \quad (4.155)$$

Substituting Eq. (4.155) into Eq. (4.148) and Eq. (4.149), we obtain the emitted frequency ω_{gs} and wavelength λ_{gs}, respectively,

$$\omega_{gs} = \omega_{g0}\frac{1-\frac{H_0R_0}{c}\cos\alpha}{\sqrt{1-\left(\frac{H_0R_0}{c}\right)^2}}, \quad (4.156)$$

$$\lambda_{gs} = \frac{\lambda_{go}\sqrt{1-\left(\frac{H_0R_0}{c}\right)^2}}{1-\frac{H_0R_0}{c}\cos\alpha}. \quad (4.157)$$

Eq. (4.157) is a useful tool to determine the emitted G-Wave's frequency and wavelength once we know the distance of a G-Wave source in the universe.

We need to distinguish emitted frequency/wavelength of G-Wave from measured frequency/wavelength on Earth.

The calculations of detected G-Wave's parameters related with frequency, e.g., those in the pocket formulae, such as Scaling amplitude, Chirp/Chirp waveform, Chirp mass, G-Wave phase, G-Wave form, and Luminosity distance, etc., need to include the effects of redshift of G-Wave.

Hubble's law has been extended to contain the term of acceleration of expansion of the universe. For detail, please see Appendix F.

4.13. Tully-Fisher-type Law of G-Wave

Substituting $v = \omega r$ and $a = \omega^2 r$ into the Larmor-type formula, Eq. (4.136), we obtain the *Tully-Fisher-type law* for G-Wave,

$$I = \frac{2Gm^2}{3C^3 r^2} v^4. \quad (4.158)$$

Eq. (4.158) shows that the intensity of G-Wave is proportional to the v^4. The Tully-Fisher-type relation is a correlation for spiral galaxies between their total energy of G-Wave emitted by the spiral galaxies per unit time and how fast they are rotating.

4.14. Dipole Radiation of Non-Relativistic Binary System

As an example, let's consider a non-relativistic binary system. For a system of two-point positive g-charges, Q_{g1+} and Q_{g2+}, e.g., a system of regular binary stars, moving with low velocities, $v_i \ll c$, we choose the origin of the reference system at the center of masses. Then the gravitational dipole moment has the form,

$$\mathbf{d_T} = Q_{g1+}\mathbf{r}_1 + Q_{g2+}\mathbf{r}_2 = \mu\left(\frac{Q_{g1+}}{m_{01}} - \frac{Q_{g2+}}{m_{02}}\right)\mathbf{r}. \quad (4.159)$$

Where $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ and $\mu = \frac{m_{01}m_{02}}{m_{01}+m_{02}}$. Substituting the definition of positive g-charge, $Q_{g+} = \sqrt{G}m_0$, into Eq. (4.159), we have $\mathbf{d_T} = 0$, i.e., the gravitational dipole moment vanishes, no G-wave.

4.15. Dipole Radiation of Relativistic Binary System

For a system of two relativistic g-charges, $v_i \sim c$, we obtain the *generalized Lienard-type equation*,

$$I_T = \sum_k I_k = \sum_k \frac{2Gm_k^2}{3C^3\left(1-\beta_k^2\right)^3}\left[a_k^2 - \left(\boldsymbol{\beta}_\mathbf{k} \times \mathbf{a}_\mathbf{k}\right)^2\right]. (4.159)$$

For a binary system, we have,

$$I_T = \frac{2G}{3C^3}\left[\begin{array}{c}\frac{m_1^2 a_1^2}{\left(1-\beta_1^2\right)^3} + \frac{m_2^2 a_2^2}{\left(1-\beta_2^2\right)^3} \\ -\frac{m_1^2\left|\boldsymbol{\beta}_1 \times \mathbf{a}_1\right|^2}{\left(1-\beta_1^2\right)^3} - \frac{m_2^2\left|\boldsymbol{\beta}_2 \times \mathbf{a}_2\right|^2}{\left(1-\beta_2^2\right)^3}\end{array}\right]. \quad (4.160)$$

For a special situation where two g-charges are equal, then $m_1 = m_2$, $a_1 = a_2$, $\beta_1 = \beta_2$, we have the intensity of G-Wave radiated by the dipole momentum of the binary system,

$$I_T = \frac{4Gm^2}{3C^3\left(1-\beta^2\right)^3}\left[a^2 - \left|\boldsymbol{\beta} \times \mathbf{a}\right|^2\right]. \quad (4.161)$$

4.16. Quadrupole Radiation of Relativistic Binary System

Let's consider a binary system of gravitational charges having equal charges, $Q_{g1} = Q_{g2}$, and moving circularly. At a starting time, the velocities of two gravitational charges are slow. We have

$$\begin{aligned} I &= \frac{16G^4m^5}{5C^5R^5} = \frac{2Gm^2}{5C^5}\frac{a^4}{\omega^2} \\ &= \frac{2G}{5C^5m^2\omega^2}\left(\frac{d\mathbf{P}}{dt}\cdot\frac{d\mathbf{P}}{dt}\right)^2, \end{aligned} \quad (4.162)$$

where $a\ (= \omega^2 r)$ is the acceleration of the g-charge.

Since two charges lose energy by G-Wave radiation, their orbits shrink and speeds become faster and closer to that of light. To deriving the intensity of the G-Wave quadruple radiation of this relativistic binary system, we write the Lorentz invariant generalization of Eq. (4.162),

$$I = \frac{G}{20C^5}\dddot{D}_{m\alpha\beta}^2 = \frac{2G}{5C^5m^2\omega^2}\left(\frac{dP^\mu}{d\tau}\frac{dP_\mu}{d\tau}\right)^2, (4.163)$$

where $d\tau = dt/\gamma$.

We obtain the intensity of G-Wave radiated by the gravitational quadrupole momentum of the relativistic binary system,

$$\begin{aligned} I &= \frac{G}{20C^5}\dddot{D}_{m\alpha\beta}^2 \\ &= \frac{2Gm^2}{5C^3\omega^2}\frac{1}{\left(1-\beta^2\right)^6}\left[a^2 - \left(\boldsymbol{\beta}\times\mathbf{a}\right)^2\right]^2. \end{aligned} \quad (4.164)$$

The change rate of quadrupole radiation intensity with respect to velocity is,

$$\begin{aligned} &\frac{dI_{quadrupole,relativistic}}{dV} \\ &= \frac{8Gm^2}{5R^2}\left(\frac{\beta}{1-\beta^2}\right)^3. \end{aligned} \quad (4.165)$$

4.17. Gravito-Bremsstrahlung-type Radiation of Two Relativistic Stars Collision

For a special case of head-on collision of two relativistic g-charges, e.g., two relativistic stars, carrying similar g-charges, the *generalized gravitational bremsstrahlung-type radiation* is,

$$I_T = \sum_K I_K = \frac{2G}{3C^3}\left[\frac{m_1^2 a_1^2}{\left(1-\beta_1^2\right)^3} + \frac{m_2^2 a_2^2}{\left(1-\beta_2^2\right)^3}\right]. \quad (4.166)$$

In the case of two relativistic star merge, we need to take into account the generalized gravitational bremsstrahlung.

4.18. Energy Loss by Radiation: Non-Relativistic Binary System

Let's consider a simple example of the radiation damping, a system of two equal g-charges attracting each other and slowly moving circularly. Here we are not use the conventional dipole method, since it will give zero dipole radiation incorrectly. Instead, we use the particle approach developed in this article, i.e., treat the dipole radiation of each g-charge independently, and the quadrupole radiation of the system as a whole. The total energy loss due to the dipole and quadrupole radiation is the summation,

$$W_T = \frac{2}{3C^3}\sum_K Q_{gK}^2 \overline{\mathbf{a}_{gK}^2} + \frac{G}{20C^5}\overline{\dddot{D}_{m\alpha\beta}^2}. \quad (4.167)$$

For a binary non-relativistic system of equal g-charges, then the total lost energy is obtained,

$$\begin{aligned} W_T &= W_{dipole} + W_{quadrupole} \\ &= \frac{4Gm^2}{3C^3}\mathbf{a}^2 + \frac{16G^4m^5}{5C^5R^5}. \end{aligned} \quad (4.168)$$

4.19. Energy Loss by Radiation: Relativistic Binary System

For a binary relativistic system of equal g-charges, the total lost energy is,

$$\begin{aligned} W_T &= W_{dipole} + W_{quadrupole} \\ &= \frac{4Gm^2}{3C^3\left(1-\beta^2\right)^3}\left[a^2 - (\boldsymbol{\beta}\times\mathbf{a})^2\right] \\ &+ \frac{26Gm^2}{15C^3}\frac{1}{\left(1-\beta^2\right)^3}\left[a^2 - (\boldsymbol{\beta}\times\mathbf{a})^2\right] \\ &= \frac{46Gm^2}{15C^3}\frac{1}{\left(1-\beta^2\right)^3}\left[a^2 - (\boldsymbol{\beta}\times\mathbf{a})^2\right]. \end{aligned} \quad (4.169)$$

4.20. Discussions and Remark

In this Section, we have systematically study G-Wave in the framework of Gravito-EM theoretically and shown the following:

1) The physical characteristics of G-Wave have been studied by following Electrodynamics and, especially, in terms of the field strengths, $\mathbf{g}$ and $\mathbf{B}_g$; thus, the directions of looking for new effects of G-Wave are conceptually clear, and the related calculations are simple.
2) The concept of gravitational field strength has been introduced into GR. For the components of T^{00} and T^{0i}, the linearized Einstein equation has been written as [28],

$$\nabla\cdot\mathbf{g} = -4\pi\rho_g,$$
$$\nabla\cdot\mathbf{B_g} = 0,$$
$$\nabla\times\mathbf{g} = -\frac{1}{C}\frac{\partial\mathbf{B_g}}{\partial\mathbf{t}},$$
$$\nabla\times\mathbf{B_g} = -\frac{4\pi}{C}\mathbf{J}_g + \frac{1}{C}\frac{\partial\mathbf{g}}{\partial\mathbf{t}}.$$
$$\mathbf{F} = Q_g\mathbf{g} + 4Q_g\mathbf{V}\times\mathbf{B}_g.$$

This set of linearized Einstein equations has the form same to that of Gravito-EM. All of conclusions in this section are derived from Gravito-EM. The linearized GR, therefore, will give the same results, except a factor of 4 in the expression of gravitational force.
3) There is gravitational synchrotron radiation from a relativistic star moving in the gravitomagnetic field of a rotating massive object, such as black hole.
4) There is gravitational Bremsstrahlung-type radiation from a head-on collision of two relativistic stars.
5) The gravitational Doppler-type effect is derived, which needs to be considered in the detection of G-Wave, especially for a faraway source.
6) The correlation between the redshifts of both G-Wave and EM-Wave is derived.

Appendix

A. Duality

In Mathematics, duality is one of the most fruitful ideas, is a 'principle' and is very powerful and useful [41]. In physics, the concept of duality has played an important role in the development of physical theories. The underlying idea is that the analogy between different phenomena in Nature is not a mere coincidence.

Moreover, duality gives one point of view of looking at the different objects, which, in principle, are all duals. When equations/theories are mathematically equivalent, then they are dual to each other. We argue that: "Mathematical identities lead to mathematical duality that lead to physical duality. Duality discloses the similarity of intrinsic nature of apparently different physical interactions and leads to unification". We show that this point of view of duality is powerful, fruitful and guidance.

'Duality' may be advanced to 'symmetry', e.g., the duality between EM and Gravito-EM leads to the e-charge/g-charge symmetry, ultra-symmetry.

Let's study mathematical duality between different quantities.

A1. Type-1 duality and Type-2 duality

For studying dualities clearly and conveniently, let's introduce different level axial vector fields:

First level axial vector field (abbreviated FAF): "**c**" is defined as the cross product of **a** and b,

$$\mathbf{c} \equiv \mathbf{a} \times \mathbf{b},$$

where both the "**a**" and "**b**" are *vector field*, denote "**c**" as FAF.

Second level axial vector field (abbreviated SAF): "**d**" is defined as the cross product of **e** and f,

$$\mathbf{d} \equiv \mathbf{e} \times \mathbf{f},$$

where the "**e**" is a *vector field* and "**f**" is a *first level axial vector field*, denote "**d**" as SAF.

Third level axial vector field (abbreviated TAF): "**n**" is defined as the cross product of **q** and p,

$$\mathbf{n} \equiv \mathbf{q} \times \mathbf{p},$$

where both the "**q**" and "**p**" are *first level axial vector fields*, denote "**n**" as TAF.

Now let's introduce two categories of dualities as following:

Type-1 duality: for an axial field $\mathbf{W}_{\text{before}} \equiv \mathbf{S} \times \mathbf{T}$, under transformation(s) of either **S** or **T** or both **S** and **T**, the field $\mathbf{W}_{\text{before}}$ transfers to $\mathbf{W}_{\text{after}}$. Under two conditions: (1) $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ are same level axial vector field; (2) the field equations describing respectively $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ have either the same form or are mathematically equivalent; then there is a Type-1 duality between $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$.

Type-2 duality: for an axial field $\mathbf{W}_{\text{before}} \equiv \mathbf{S} \times \mathbf{T}$, under transformation(s) of either **S** or **T** or both **S** and **T**, the field $\mathbf{W}_{\text{before}}$ transfers to $\mathbf{W}_{\text{after}}$. Under two conditions: (1) $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ are different level axial vector fields; (2) the field equations describing respectively $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$ have either the same form or are mathematically equivalent; then there is a Type-2 duality between $\mathbf{W}_{\text{after}}$ and $\mathbf{W}_{\text{before}}$.

Where the "**T**" is either a vector field $\mathbf{T}_{\text{vector}}$ (abbreviated $\mathbf{T_v}$) or an axial vector field $\mathbf{T}_{\text{axial}-\text{vector}}$ (abbreviated $\mathbf{T_{av}}$) or a field $\mathbf{T}_{\text{combination}}$ (abbreviated $\mathbf{T_c}$) that is the combination of a vector and an axial vector $\mathbf{T_c} = \mathbf{T_v} + \mathbf{T_{av}}$. The "S" may be either a vector field $\mathbf{S}_{\text{vector}}$ (abbreviated $\mathbf{S_v}$) or an axial vector field $\mathbf{S}_{\text{axial}-\text{vector}}$ (abbreviated $\mathbf{S_{av}}$).

Note: A special situation is that, during the transformation, one term in equation becomes zero. For keeping the same form of equations, we still keep the zero-term for the purpose of discussing duality. Then, in later calculation, ignore those zero-terms.

Specific examples of type-1 duality:

In the following examples, "n" is an integer and n = 1, 2, 3….

Example 1: Corresponding to different axial vector fields $\mathbf{T}_{\mathbf{av}/\mathrm{n}}$, the fields $\mathbf{v} \times \mathbf{T}_{\mathbf{av}/\mathrm{n}}$ are SAF, where **v** is a vector. Dualities between those SAFs are type-1 duality, i.e., under transformation,

$$\mathbf{T}_{\mathbf{av}/1} \leftrightarrow \cdots \leftrightarrow \mathbf{T}_{\mathbf{av}/\mathrm{n}}$$

there are conversions between the SAFs,

$$\mathbf{v} \times \mathbf{T}_{\mathbf{av}/1} \leftrightarrow \cdots \leftrightarrow \mathbf{v} \times \mathbf{T}_{\mathbf{av}/\mathrm{n}}, (\mathrm{n} \neq 1)$$

and between UMFT describing them.

Example 2: Corresponding to different axial vector fields $\mathbf{T}_{\mathbf{av}/\mathrm{n}}$, the fields $\mathbf{S_v} \times \mathbf{T}_{\mathbf{av}/\mathrm{n}}$ are TAF. Dualities between those TAFs are type-1 dualities, i.e., under transformation,

$$\mathbf{T}_{\mathbf{av}/1} \leftrightarrow \cdots \leftrightarrow \mathbf{T}_{\mathbf{av}/\mathrm{n}}$$

we have conversions between the TAFs,

$$\mathbf{S_v} \times \mathbf{T}_{\mathbf{av}/1} \leftrightarrow \cdots \leftrightarrow \mathbf{S_v} \times \mathbf{T}_{\mathbf{av}/\mathrm{n}}, (\mathrm{n} \neq 1),$$

and between UMFT describing them.

Specific examples of type-2 duality:

Example 1: Corresponding to a vector field $\mathbf{T}_{\mathbf{v}/\mathrm{n}}$ and an axial vector field $\mathbf{T}_{\mathbf{av}/\mathrm{n}}$, the fields $\mathbf{v} \times \mathbf{T}_{\mathbf{v}/\mathrm{n}}$ and fields $\mathbf{v} \times \mathbf{T}_{\mathbf{av}/\mathrm{n}}$ are FAF and SAF respectively. Duality between $\mathbf{v} \times \mathbf{T}_{\mathbf{v}/\mathrm{n}}$ and $\mathbf{v} \times \mathbf{T}_{\mathbf{av}/\mathrm{n}}$ is type-2 duality.

Example 2: same as Example 1, but replace vector **v** with vector $\mathbf{S_v}$.

Example 3: Corresponding to a vector field $\mathbf{T}_{\mathbf{v}/\mathrm{n}}$, the field $\mathbf{v} \times \mathbf{T}_{\mathbf{v}/\mathrm{n}}$ and field $\mathbf{S_{av}} \times \mathbf{T}_{\mathbf{v}/\mathbf{n}}$ are FAF and SAF respectively. Duality between $\mathbf{v} \times \mathbf{T}_{\mathbf{v}/\mathrm{n}}$ and $\mathbf{S_{av}} \times \mathbf{T}_{\mathbf{v}/\mathrm{n}}$ is type-2 duality.

Example 4: same as Example 3, but replace $\mathbf{T}_{\mathbf{v}/\mathrm{n}}$ with $\mathbf{T}_{\mathbf{av}/\mathrm{n}}$.

A2. Transferability between Dualities of Same Type

The mathematical type-1 duality and type-2 duality can be transferred. We propose Transfer Rules:

1). There are type-1 dualities between A and B and between C and D. If the duality between A and C is type-1, then the duality between B and D is type-1, and vice versa;

2). There are type-2 dualities between A and B and between C and D. If the duality between A and C is type-1, and if B and D are the same lever axial fields, then the duality between B and D is type-1, and vice versa;

3) There are type-1 (or type 2) dualities between A and B; and C is mathematical equivalent to A. There is D that is mathematical equivalent to B, and thus is type-1 (or type 2) dual to C.

4) There is type-1 (or type 2) duality between A and B; and C is mathematical equivalent to A. There is a type-1 (or type 2) dual of C, which is mathematical equivalent to B.

A3. Duality and Quasi-Duality between Physical Theories

The conventional duality is a one-to-one mapping and equivalence between two physical theories, which have similar formulations but have different values of corresponding physical quantities, so that we should describe different interactions with one theory, rather than with two theories.

To study the above topics further, we restudy the concept of duality:

Duality: between two theories should satisfy the following requirements:

Requirement 1: both theories have corresponding symmetries.

Requirement 2: both theories have the formulas of same forms and corresponding quantities of different values, e.g., linear-to-linear and non-linear-to-non-linear theories.

Requirement 3: each theory has charge conjugation.

In physics, duality with same kind of symmetries will disclose deeper relation between two theories. Based on Noether's theorem, Requirement 1 leads to Requirement 2. But Requirement 2 does not necessarily lead to Requirement 1 before proving.

In physics of gravity, the conventional Gauge/Gravity Duality doesn't satisfy Requirement 1. The gauge theories are dictated by internal and CPT symmetries. On the contrary, there are no such symmetries in conventional Einstein theory of General Relativity (GR).

Thus, we introduce a new less-restrict concept of Quasi-Duality:

Quasi-Duality: two theories satisfy part of Requirements of Duality, e.g., both theories with either non-corresponding symmetries or one of theories with unknown symmetry of the kind that other theory has.

A4. Theory-independent and Transferability of Duality and Quasi-Duality

With theory-to-theory Duality (Quasi-Duality), we postulate following conjectures:

***Conjecture* 1**: the intrinsic nature of quantities disclosed by a Duality (Quasi-Duality) must be *theory-independent*.

***Conjecture* 2**: Dualities (Quasi-Dualities) are transferable.

***Conjecture* 3**: Quasi-Duality should advance to Duality eventually; otherwise, at least, one of theories needs to be modified.

Duality reflects intrinsic nature of the real world, while Quasi-Duality reflects only partial nature of the real world. A theory attempting to describe Nature is limited by present knowledge. Thus, a Duality, as well as a Quasi-Duality, should not depend on theories, since theory will be modified eventually. Conjecture 1 reflects this fact. An example of Conjecture 2 is that if there are theory-A/theory-B Duality (Quasi-Duality) and theory-B/theory-C Duality (Quasi-Duality), we must have theory-A/theory-C Duality (Quasi-Duality). Conjecture 3 implies that our limited understanding of Nature will gradually progress and reach, hopefully, full understanding ultimately.

The general goals of understanding gravity are: (1) Find out the internal symmetry associates with gravity; (2) Suggest a gravitational charge conjugate; (3) Reformulate gravity in the same terms as that of other forces. We have completed the above tasks [3], so gravity and other three force are on the same footing. We have accomplished the tasks: (1) Quantize gravity; (2) unify gravity with other forces. Gravity is a physical field as other three interactions in real world 4-dimention spacetime.

B. Resolving Negative Energy Issue of Vector Gravity

We define the work done by an external force to dismantle a collection of same signs of g-charge, which converts to the energy of the gravitational field, as the potential energy of gravitational field, which is positive. As a summary, the energy, W_g, stored in the Gravito-EM field is positive, denoted as "energy" of gravito-EM fields,

$$W_g = \frac{1}{2}\int(\mathbf{g}^2 + \mathbf{B}_g^2)d\tau \tag{B1}$$

Where $\mathbf{g}$ and $\mathbf{B}_g$ are the gravito-electric and gravito-magnetic field strengths respectively, which are created by either positive or negative or net g-charges/g-current respectively. The negative energy issue of the vector gravito-magnetic field is resolved.

The Lagrangian for Gravito-EM is

$$L_g = \frac{1}{8\pi}\left(g^2 - B_g^2\right) \tag{B2}$$

C. Lorentz Transformation of Gravito-EM Field

Writing Eq. (4.61), Eq. (4.72), Eq. (4.73) and Eq. (4.64) in tensor form,

$$\frac{\partial F_g^{\alpha\beta}}{\partial x^\beta} = J_g^\alpha, \tag{C1}$$

$$\frac{\partial F_g^{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_g^{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_g^{\gamma\alpha}}{\partial x^\beta} = 0, \tag{C2}$$

$$J_g^\alpha = J_{g+}^\alpha + J_{g-}^\alpha = (\rho_g, \mathbf{J}_g),$$

Where $F_g^{\alpha\beta} = \partial^\alpha A_g^\beta - \partial^\beta A_g^\alpha$ is the field strength tensor, A_g^α is four-vector gravitational potential, J_g^α is gravitational four-current. Eq. (C1) and Eq. (C2) satisfy Lorentz transformation and comply with Special Relativity.

For transformation of gravito-electric and gravito-magnetic field strengths between two frames moving relative to each other with constant velocity along the x-axis, the transformation law is,

$$B'_{gx} = B_{gx}, B'_{gy} = \gamma(B_{gy} - vg_z), B'_{gz} = \gamma(B_{gz} - vg_y), \tag{C3}$$

$$g'_x = g_x, g'_y = \gamma(g_y + vB_{gz}), g'_z = \gamma(g_z + vB_{gy}), \tag{C4}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. They have the same forms as that of EM in SR. In analogy to the transformation of EM fields in SR, we obtain the same rule for the transformation of Gravito-EM fields, including extended-Newtonian gravitational field and gravito-magnetic fields. In the tensor form, the gravitational fields in one inertial frame S' can be expressed in terms of gravitational fields in another frame S,

$$F'^{\mu\nu}_{g\pm} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}F^{\alpha\beta}_{g\pm}. \tag{C5}$$

In the vector form, the gravitational field strengths transfer as,

$$\left.\begin{aligned} \mathbf{g}'_\pm &= \gamma\left(\mathbf{g}_\pm + \boldsymbol{\beta}\times\mathbf{B}_{g\pm}\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{g}_\pm\right) \\ \mathbf{B}'_{g\pm} &= \gamma\left(\mathbf{B}_{g\pm} - \boldsymbol{\beta}\times\mathbf{g}_\pm\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}_{g\pm}\right) \end{aligned}\right\} \tag{C6}$$

When the velocity is slow, $\gamma \approx 1$, in the laboratory frame,

$$\mathbf{B}_g \approx \mathbf{B}'_g + \mathbf{v}\times\mathbf{g}. \tag{C7}$$

Where $\mathbf{B}'_g$ is the Gravito-magnetic field at the circuit d$\mathbf{l}$ in a reference frame in which d$\mathbf{l}$ is at rest. The $\mathbf{g}$ is a gravitational field at the neighborhood of the circuit d**l**. The **v** is the velocity of the circuit d**l** relative to the laboratory frame. While a gravitational field **g** in the laboratory frame transform as,

$$\mathbf{g} \approx \mathbf{g}' - \mathbf{v} \times \mathbf{B}_g \quad \text{(C8)}$$

where $\mathbf{g}'$ is the gravitational field at the circuit d**l** in a reference frame, in which d**l** is at rest. The $\mathbf{B}_g$ is a gravito-magnetic field at the neighborhood of the circuit d**l**. The v is the velocity of the circuit d**l** relative to the laboratory frame.

For a special situation of the gravito-electric field, $\mathbf{g}'$, and gravito-magnetic, $\mathbf{B}_g'$, do not exist in the circuit d**l** rest frame, we have transformation law,

$$\mathbf{B}_g = \mathbf{v} \times \mathbf{g}. \quad \text{(C9)}$$

$$\mathbf{g} = -\mathbf{v} \times \mathbf{B}_g. \quad \text{(C10)}$$

Eq. (C9) implies that a gravito-magnetic field must accompany with a moving g-charge.

Following EM, the invariant quantities of Gravito-EM are,

$$F_{g+\alpha\beta} F_{g+}^{\alpha\beta} = invariant, \quad \text{(C11)}$$

$$\epsilon_{\alpha\beta\gamma\delta}\ \epsilon_{\alpha\beta\gamma\delta} F_{g+}^{\alpha\beta} F_{g+}^{\gamma\delta} = invariant, \quad \text{(C12)}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely anti-symmetry unit tensor of fourth-rank.

In three vector forms of gravito-electric and gravito-magnetic field strength, Eq. (C11) and Eq. (C12) gives

$$B_{g+}^2 - g_+^2 = invariant, \quad \text{(C13)}$$

$$\mathbf{g}_+ \cdot \mathbf{B}_{g+} = invariant. \quad \text{(C14)}$$

For the gravitational fields created by negative g-charge/g-current, we have the same invariants,

$$F_{g-\alpha\beta} F_{g-}^{\alpha\beta} = invariant, \quad \text{(C15)}$$

$$\epsilon_{\alpha\beta\gamma\delta}\ \epsilon_{\alpha\beta\gamma\delta} F_{g-}^{\alpha\beta} F_{g-}^{\gamma\delta} = invariant, \quad \text{(C16)}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely anti-symmetry unit tensor of fourth-rank.

D. Perihelion Precession of Mercury

We can rewrite the equation of motion for a g-charge in external fields. At the "zero approximation", the Newtonian Lagrangian is,

$$L_{gi}^{(0)} = \frac{1}{2} m_i \mathbf{v}_i^2 - Q_{gi} V_g, \quad \text{(D1)}$$

where $\mathbf{v}_i$ is the velocity of the i^{th} g-charge of a system of g-charges; V_g is the potential of the external field. Since the existence of gravito-magnetic field $\mathbf{A}_g$ in Gravito-EM, the relativistic Lagrangian is

$$L_{gi} = -m_i c^2 \sqrt{1-\beta_i^2} - Q_{gi} V_g + \frac{1}{C} Q_{gi} \mathbf{A}_g \cdot \mathbf{v_i}. \quad \text{(D2)}$$

Expanding the term $m_i c^2 \sqrt{1-\beta_i^2}$ in power of v_i/c and keeping the terms of second order, and substituting the retarded potentials, we obtain the *Darwin-type gravitational Lagrangian* for the whole system, from Eq. (D2),

$$L_g = \sum_i \left(\frac{1}{2} m_i v_i^2 + \frac{1}{8c^2} m_i v_i^4 \right) - \sum_{i>j} \left(\frac{Q_{gi} Q_{gj}}{R_{ij}} - \frac{Q_{gi} Q_{gj}}{2c^2 R_{ij}} \left[\mathbf{v_i} \cdot \mathbf{v_j} + \left(\mathbf{v_i} \cdot \mathbf{n_{ij}} \right)\left(\mathbf{v_j} \cdot \mathbf{n_{ij}} \right) \right] \right), \quad \text{(D3)}$$

where $\mathbf{A}_g = \dfrac{Q_{gi}\left[\mathbf{v} + (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\right]}{2cR}$ has been used.

From a vector theory of gravity and taking into account vector potential, Borodikhin [42] proposed a Lagrange function,

$$L_g = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 + \frac{G m_1 m_2}{R_{12}} + \frac{G m_1 m_2 V^2}{c^2 R_{12}}, \quad \text{(D4)}$$

and show that the Lagrange function can explain the perihelion precession of Mercury.

For a binary system, the gravito-Darwin-type Lagrangian, Eq. (D3), gives Borodikhin's Lagrange function, Eq. (D4), which implies that Gravito-EM explains the perihelion precession of Mercury.

E. Extended-Doppler Effect of G-Wave of Accelerating Source

The stars are acceleratingly receding from each other. We suggest that the acceleration causes the redshift of g-wave, denoted it as "Acceleration-Redshift". Both receding velocity and acceleration of source cause the wavelength of g-wave shifted to red. The total redshift is a summation of Doppler-type redshift and Acceleration-Redshift. Moreover, we derived General Redshift and General Blueshift formulas that show that not only velocity, but also acceleration, jerk, and higher order derivative shift wavelength. The General Redshift formula is identical with the cosmological redshift, which implies that the cosmological redshift intrinsically includes Acceleration-Redshift. We believe that to describe the accelerating universe, physics laws should include acceleration for accuracy and completeness. The acceleration-redshift of electromagnetic wave has been derived [43]. Due to the duality between EM and Gravito-EM, we directly copy the results of EM-wave here for several situations.

Situation (A): Source of g-wave is receding with acceleration.

A source emitting g-wave of wavelength λ_s is receding with velocity v_s and constant acceleration a_s relative to an observer on the earth. The observed wavelength λ_o and acceleration-velocity-redshift z of g-wave are, respectively,

$$\lambda_o = \lambda_s \left[1 + \frac{v_s}{c} + \frac{a_s \lambda_s}{2c^2} \right], \quad \text{(E1)}$$

$$cz \equiv \frac{\lambda_o}{\lambda_s} - 1 = v_s + a_s\left(\frac{\lambda_s}{2c}\right), \qquad \text{(E2)}$$

where "c" is the speed of g-wave, which is equal to that of light.

Eq. (E2) implies that, the acceleration of a receding source does shift the wavelength. The first terms on the right-hand sides of Eq. (E2) is the regular Doppler-type redshift for g-wave; the second term is redshift due to acceleration, "Acceleration-redshift".

A significant difference between redshifts of constantly moving source and accelerating source is that the observed wavelength and redshift are time dependent. Taking derivative with respect to time, we obtain,

$$\frac{c}{f_s}\frac{df_o}{dt} = -\frac{a_s}{\left[1 + \frac{v_s}{c} + \frac{a_s\lambda_s}{2c^2}\right]^2}, \qquad \text{(E3)}$$

$$c\frac{dz}{dt} = a_s \qquad \text{(E4)}$$

Eq. (E4) give the relation between the acceleration of source and the redshift.

Situation (B): Source of g-wave is receding with deceleration d_s.

Wavelength, deceleration-velocity-redshift and time change of redshift are respectively,

$$\lambda_o = \lambda_s\left[1 + \frac{v_s}{c} - \frac{1}{2}d_s\frac{\lambda_s}{c^2}\right], \qquad \text{(E5)}$$

$$cz = v_s - d_s\left(\frac{\lambda_s}{2c}\right), \qquad \text{(E6)}$$

$$c\frac{dz}{dt} = -d_s. \qquad \text{(E7)}$$

The redshift is decreasing with time, which is due to the deceleration of source. The deceleration, d_s, of a source may be measured by detecting the time change of shifts of wavelength/frequency.

F. Extended-Hubble Law for Accelerating Universe

In the 1998, Scientists reported the observations that the expansion of the universe is accelerating, $\ddot{r} > 0$, and the acceleration is assumed to be uniform, $\dddot{r} = 0$. In the 2016, Scientists report that even that acceleration is faster than expected. If the 2016 observation is confirmed, which could mean that the acceleration is accelerating, $\dddot{r} > 0$, there is a discrepancy in Hubble constant obtained from the 2016 observation and from the 2013 data of European Planck spacecraft. Following classical mechanics, call $\dddot{r}(t)$ the "jerk". There is no evidence showing whether the jerk is uniform or not.

To describe an accelerating system, the physics laws should contain the term of the acceleration of the system for accuracy and completeness. Hubble's law was derived based on the uniformly expanding universe. We argue that to describe the accelerating universe, Hubble law should be extended to fit the accelerating and jerking universe. For this aim, we generalize the linear distance-velocity and distance-redshift relations of Hubble law to a non-linear distance-movement and a distance-redshift-movement relation of the extended Hubble law [44]. We suggest that the Extended-Hubble law is worth to pursue and may be tested by introducing it into cosmological study.

The extended-Hubble law is expressed in different forms as the following [44],

$$\frac{\dot{r}(t_1)}{r(t_1)} = H\left(1 + \frac{1}{2}\frac{\ddot{r}}{r}\frac{1}{H^2}\right), \qquad \text{(F1)}$$

$$H = \frac{1}{2}\frac{\dot{r}}{r}\left\{1 + \sqrt{1 - 2\frac{\ddot{r}(t_1)r(t_1)}{\dot{r}^2}}\right\}, \qquad \text{(F2)}$$

$$\frac{\dot{r}(t_1)}{r(t_1)} = H\left\{1 - \frac{1}{2}q_2 + \frac{1}{3!}q_3\right\}, \qquad \text{(F3)}$$

where

$$q_n \equiv -\frac{r^{(n)}(t_1)}{r(t_1)H^n}, \qquad \text{(F4)}$$

and q_2 and q_3 are the "deceleration-parameter" and "jerk-parameter", respectively.

Note with acceleration, $H \neq \frac{\dot{r}}{r}$, i.e., the original definition of Hubble parameter is no longer valid. Eq. (F2) may be considered as a new definition of Hubble parameter in terms of velocity and acceleration.

Table F: Summary

	Distance-Movement Relation	Distance-Redshift-Movement Relation: $1 + Z = \frac{r(t_0)}{r(t)}$
Uniformly Expanding Universe	$r = \frac{\dot{r}}{H}$	$Z = Z_{H0} = \frac{H_0 r(t_0)}{c}$
Accelerating Universe	$r = \frac{\dot{r}}{H} - \frac{\ddot{r}}{2H^2}$	$Z = Z_{H0} - \frac{\ddot{r}(t_0)r(t_0)}{2\dot{r}^2(t_0)}Z_{H0}^2$ $Z = Z_{H0} + \frac{1}{2}q_{R0}^2 Z_{H0}^2$
Jerking Universe	$r = \frac{\dot{r}}{H} - \frac{\ddot{r}}{2H^2} + \frac{\dddot{r}}{3!\,H^3}$	$Z = Z_{H0} - \frac{\ddot{r}(t_0)r(t_0)}{2\dot{r}^2(t_0)}Z_{H0}^2 + \frac{\dddot{r}(t_0)r^2(t_0)}{3!\,\dot{r}^3(t_0)}Z_{H0}^3$ $Z = Z_{H0} + \frac{1}{2}q_{R0}^2 Z_{H0}^2 - \frac{1}{3!}q_{R0}^3 Z_{H0}^3$

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