

Bifurcation and Stability Analysis of Pulsating Solitons

A. Kamagaté^{1,2,*}, S. Chouli³, P. C. Bakala^{2,4}

¹Ecole Supérieure Africaine des Technologies d'Information et Communication, Abidjan, Côte d'Ivoire

²Agence Nationale de la Recherche, Paris, France

³Faculté des sciences, Université M'Hamed Bougara de Boumerdes, Alger, Algérie

⁴Université Paris Diderot, Paris, France

*Corresponding author: alkamagate@gmail.com

Received September 27, 2018; Revised November 21, 2018; Accepted December 04, 2018

Abstract Pulsating soliton solutions bifurcation analysis of the two-dimensional (2D) Complex Swift-Hohenberg equation (CSHE) is presented. The approach is based on a reduction from an infinite-dimensional dynamical dissipative system to a finite-dimensional model. Thanks to the collective variable approach, we investigated the influence of the nonlinear gain and the saturation of the Kerr nonlinearity on the pulsations of the solitons. Research has shown that the transformation between pulsating soliton and fronts can be realized through a series of period-doubling bifurcations. The complete bifurcation diagrams of the total energy have been obtained for a definite range of the nonlinear gain and the saturation of the Kerr nonlinearity values. The detailed analysis reveals that when the saturation of the Kerr nonlinearity increases one-period pulsating solution bifurcates to double-period pulsations. While the increase of the nonlinear gain leads the double-period pulsations to return into one-period pulsation before transforming into a stationary pulsating solitons.

Keywords: *pulsating solution, dissipative soliton, bifurcation diagram, spectral filtering, complex Swift-Hohenberg equation*

Cite This Article: A. Kamagaté, S. Chouli, and P. C. Bakala, "Bifurcation and Stability Analysis of Pulsating Solitons." *International Journal of Physics*, vol. 6, no. 5 (2018): 166-173. doi: 10.12691/ijp-6-5-5.

1. Introduction

Solitons in dissipative systems have attracted a great deal of attention in recent studies. Dissipative solitons are a rich dynamic and are not simply extensions of the solitons that have been studied in Hamiltonian systems. The dynamic of dissipative solitons is more complicated than Hamiltonian ones because these solitary waves result from a double balance: on the one hand between dispersive and nonlinear conservative affects and on the other hand between gain and loss [1]. Another relevant feature of dissipative solitons is that they include energy exchange with external sources. In contrast to Hamiltonian solitons, dissipative ones admit several major classes, such as, stationary, pulsating or chaotic [2,3] and exhibit many other types of dynamics. However, extensive research efforts show that each class can be listed into subclasses with more complicate dynamic [2,4]. There are all governed by the parameters of the dissipative system.

Pulsating solutions are one of outstanding features of dissipative systems. There are more complicated and have complex and variable dynamics in the sense that, in addition to dispersion and nonlinearity, they include energy exchange with external sources. One of the basic equations for modelling modulated amplitude waves [5], spatiotemporal dynamics and to describe a vast variety of nonlinear phenomena in a variety of nonlinear dissipative systems [6] is the complex Ginzburg-Landau equation (CGLE). This

equation admits stable solutions and its several parameters make the dynamical system highly complicated and appreciated [3]. Moreover, the CGLE confesses to having one-, two- or three-dimensional spatiotemporal soliton solutions. In the case of the one-dimensional, exact analytical solutions have been found, but all of them have been proved unstable. Only numerical simulations help to find stable soliton solutions. For the higher dimensions of the CGLE, stable soliton solutions have been studied extensively, either numerically [7] or with the semi-analytical methods [2,8,9]. These facts indicate that, in addition to stationary solitons, there are also pulsating ones. Such localized pulsations exist, in different forms in optics, biology and chemistry. The pulsating solitons have been intensely studied, numerically [10] semi-analytically [11] and observed experimentally [12] in a fiber laser. They have attracted a great deal of attention and find particular applications in pulse generation by passively mode-locked soliton lasers [13,14]. The pulsating soliton also can be applied to a large class of solid-state lasers [15], signal transmission in all-optical communication lines [16] and semiconductor oscillators [17].

In many experiments, the gain spectrum is usually wide with multiple peaks. But however, the CGLE can only describe a spectral responses with a single maximum [18], hence, fails to trace a gain spectrum with multiple peaks. So, to take account this assertion and to make the model more lifelike, it's sound to add others terms of higher-order spectral filtering to the CGLE, this leads to the complex Swift-Hohenberg equation (CSHE). In this

work, we present pulsating soliton solutions and bifurcation boundaries of the two-dimensional CSHE. It is tedious task as the 2D CSHE has several parameters that define both stationary and pulsating solitons. Hence, the bifurcation boundaries are surfaces in this multi-dimensional space of the parameters.

The remainder of this present study is organized as follows. We present in section 2 the model of our study and our approach of determination of the pulsating solitons. The influence of both the saturation of the Kerr nonlinearity and the nonlinear gain is analysed in section 3. The section 4 is devoted to some concluding remarks.

2. Study Model

This present study is based on an extended complex Swift-Hohenberg equation that includes cubic and quintic nonlinear terms. This CSHE can be useful to investigate soliton propagation in optical systems with nonlinear gain and spectral filtering such as communication links with lumped fast saturable absorbers or fiber lasers with additive-pulse mode-locking or nonlinear polarization rotation. Likewise, the CSHE includes the higher order of the spectral filter, which is extremely essential to analyse the generation of more complex impulse. This equation can describe the major physical effects that occur in laser cavity such as dispersion, self-phase modulation, and the spectral filter, the linear or nonlinear gains, and the linear and nonlinear losses. The normalized propagation equation reads [19,20]:

$$\begin{aligned} \psi_z - i\frac{D}{2}\psi_{tt} - i\frac{1}{2}\psi_{rr} - i\gamma|\psi|^2\psi - i\nu|\psi|^4\psi \\ = \delta\psi + \varepsilon|\psi|^2\psi + \beta\psi_{tt} + \mu|\psi|^4\psi + \gamma_2\psi_{ttt}. \end{aligned} \quad (1)$$

Without the additive term $\gamma_2\psi_{ttt}$ the equation (1) is the same as the CGLE one. With t the normalized time in a frame of reference moving with the group velocity, for passively mode-locked lasers. z represents the propagation distance along the cavity round-trip number. In a laser system $\psi = \psi(r, t, z)$ represents the normalized optical envelope and is function of three real variables. r ($r = \sqrt{x^2 + y^2}$) is the transverse coordinate, taking account of the spatial diffraction effects. The coefficients D , γ , ν , δ , ε , β , μ and γ_2 are real constants. The physical meaning of each term depends on the real problem that must be examined. Thereby, the left-hand side contains the conservative terms: namely, D denotes the cavity dispersion, being anomalous when $D > 0$ and normal $D < 0$, and ν which represents, if negative, the saturation coefficient of the Kerr nonlinearity. γ represents the Kerr nonlinearity coefficient. In the following, the dispersion is anomalous, and ν is kept relatively small. Dissipative terms are written on the right-hand-side of equation (1), and the meaning of the corresponding parameters is the following: δ represents the coefficient for linear loss (if negative), ε denotes the nonlinear gain (if positive), β is the spectral filtering (if positive) term and μ is the saturation of the nonlinear gain (if negative). Finally γ_2 represents the higher-order spectral filter term, which is very important and another distinguishing of our present study.

The CSHE admits a variety of localized solutions, from stationary [18] to pulsating, as well as period-doubling bifurcations [21]. These stable solutions were found numerically or with semi-analytical methods. The task is not simple as the CSHE has several parameters that define the existence of stable soliton solutions. So far, there has been no progress in finding analytic expressions of the 2D CSHE pulsating solutions and bifurcation boundaries. In addition, solving numerically the 2D CSHE equation (1) for a given set of parameters and an initial condition is an extremely difficult [2]. To overcome this complexity, we use a master equation approach. This way helps to reduce an infinite-dimensional to an ordinary differential equation (ODE), which can be solved numerically with relative ease. The resulting dynamical system controls the evolution of a finite number of parameters such as the pulse amplitude, width, and chirp. We used the same approach in reference [2,9,11] to find with some effectiveness the stationary solutions of the CSHE, and the pulsating and stationary solutions of the CGLE. In practice, the collective variable technique [22] is based on a trial function theory with a finite number of parameters, and this is the way to obtain a significant reduction in the number of variables used for the description of the soliton dynamics. The principle consists to associate collective variables with the pulse's parameters of interest for which equations of motion may be derived.

Accordingly, we introduce n collective variables, z dependent (X_i with $i = 1, 2, \dots, n$). Each X_i can correctly describe a fundamental parameter of the pulse (amplitude, width, chirp ...) [23], thus, the optical field can be dissociate in the following form:

$$\psi(r, t, z) = f(X_1, X_2, \dots, X_n, t) + q(z, t) \quad (2)$$

with f the trial function, dependent on the collective variables, and q the residual field that describes all other excitations in the system (radiation, dressing field, noise, etc.). To the success of the theory of collective variables, the choice of the trial function must be essential. Thereby, one must select the trial function that best represents the suitable characteristics of the field to be studied. After choosing the trial function, one can suppose, as in most practical studies that the optical pulse $\psi(r, t, z)$ can be completely characterized by the ansatz function. Under these conditions, we select, on the one hand a Gaussian function as trial function, and on the other hand neglect the residual field ($q = 0$), *the bare approximation* [22]. Thereafter we get, with relative ease, a jet of differential equations, which govern the evolution of the optical pulse parameters propagating in space and time (for more details, see [2,9,11,18,22]). Thus, the equation (1) of propagation is transformed into a system of differential equations. This approach provides the basic parameters of the fixed points, and help to find the different types of soliton solutions, thereby reducing by several orders of magnitude the volume of calculation required usually.

The benefit of the collective variable approach also lies in the fact that it makes it possible to express the total energy with respect to the parameters of the soliton. In fact, this fundamental parameter gives us the main information about the soliton dynamics, and helps to control the state of the soliton and its stability. For

dissipative systems, the total energy is not conserved but evolves in accordance with the so-called balance equation. When the pulsating solution is reached, the total energy is an oscillating function of z and when we have the stationary soliton, the total energy converges to a constant value. Here the total energy is given by the following equation:

$$Q(z) = \int_{-\infty}^{+\infty} \int_0^{+\infty} 2\pi r |\psi(r,t,z)|^2 dr dt. \quad (3)$$

3. Analysis of Pulsating Solitons

Pulsating soliton solutions in dissipative systems can be describe as a limit cycle of in infinite-dimensional dynamical system [1], and they exist on an equal basis with stationary solitons. In addition, they repeat periodically in the direction of propagation and oscillate. They can have distinctive features and we have studied in [24] the regions in the parameter space where they exist just as their main characteristic. The pulsating solitons correspond to the unstable fixed points of the system, obtained from the ordinary differential equation. To investigate the complex behaviour of the pulsating dynamic, we fix the parameters of the CSHE $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$, $\nu = -0.247$, $\epsilon = 0.50$ and $\gamma_2 = 0.05$, and we use the following initial condition:

$$\psi(r,t,0) = 2.058 \exp\left(-\frac{t^2}{18.096} - \frac{r^2}{18.326}\right). \quad (4)$$

Then, the Newton-Raphson allows looking for the fixed point and we study its stability. It is important to specify that the pulsating soliton can occupy significant regions and can be excited from a wide range of initial conditions. Usually the dissipative solutions experienced periodic changes mostly in their width while keeping almost constant their peak amplitude [24]. In this study, we decide to evaluate the effects of the nonlinear gain and the saturation of the Kerr nonlinearity on the pulsations.

3.1. Pulsating Solitons under Influence of the Saturation of the Kerr Nonlinearity

First, we started to highlight the impact of the saturation of the Kerr nonlinearity on the pulsations. This parameter is very important as it plays a key role in the dynamics of dissipative solitons. To illustrate the influence of this parameter ν , we set all the parameters of the CSHE, namely $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$, $\epsilon = 0.50$ and $\gamma_2 = 0.05$, by varying ν . The first figure below corresponds to $\nu = -0.246$ (Figure 1). It illustrates the evolution of the total energy of the dissipative pulsating soliton. Figure 1 b) represents the enlarged view of Figure 1 a). As in most cases, the solution begins with an unstable dynamics, and after the onset of oscillations, the pulsating dynamics becomes steady. The short transitional phase is followed by a permanent regime with large oscillations between the two constant limits. The duration of this transitional phase generally depends on the initial condition. The stable dynamic as illustrated by the

Figure 1 b), oscillates between the values 640 (minimum) and 710 (maximum).

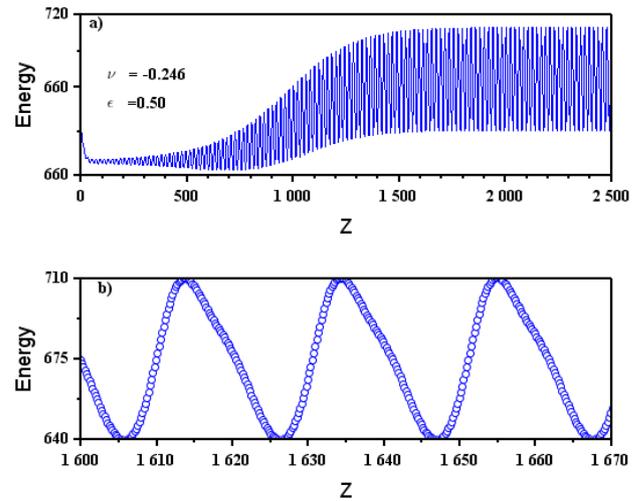


Figure 1. Evolution of the total energy of pulsating dissipative soliton, showing the onset of pulsations a), and enlarged view of the stable pulsations b). The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

We confirm numerically in Figure 2 that the evolution of the pulsating soliton corresponds to stable limit cycle. The blue cross in Figure 2 corresponds to the initial point; afterwards, all the trajectories tend to form a closed figure. When the system returns to a steady state, we observe the closed trajectory, which describes perfect pulsations behavior of the dissipative soliton. One can notice that any small perturbation from this closed trajectory causes the system to return to it, making the system stick to the limit cycle.

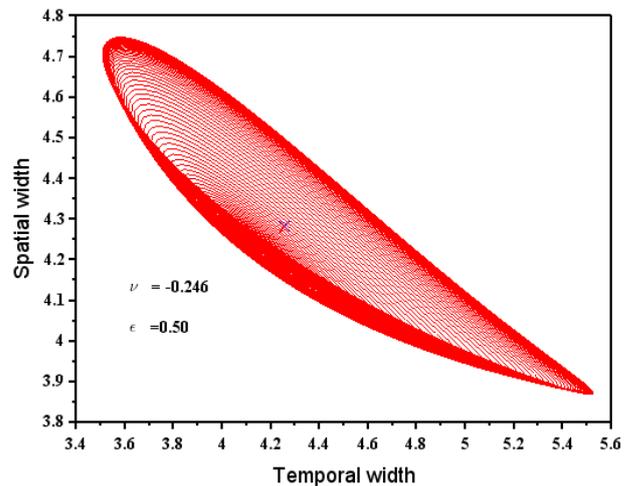


Figure 2. Example of stable limit cycle of the system in temporal and spatial widths plane. The blue cross corresponds to the initial point. The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

The parameter ν is very important in nonlinear optic for example it plays a crucial role in the filamentation and the ionization. Here, a slight modification of this parameter of the saturation of the Kerr nonlinearity has also a real impact on the oscillations of the dissipative as shown in the Figure 3. Indeed, we have changed the value of this parameter from $\nu = -0.246$ to $\nu = -0.240$ while

keeping the other parameters constant. The result of this analysis is given in Figure 3. Unlike the previous case (Figure 1 a)), we can notice that the transitional phase is short, and the dissipative soliton quickly enters to a permanent dynamic. Another difference illustrated by the Figure 2 b), shows that the solution oscillates between two maximums (840 and 817) and two minimums (680 and 787). It clear shows that the saturation of the Kerr nonlinearity real effects on the dynamic of the pulsating solitons. The example of stable limit cycle in the temporal and spatial widths plane of such solution is plotted in the Figure 4. Right here the trajectories are less compressed than that of the Figure 2. This, once again, confirms the two dynamics are dramatically very different.

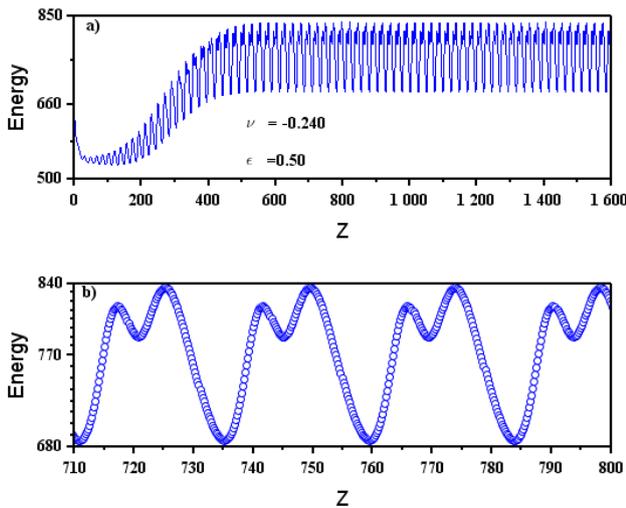


Figure 3. Evolution of the total energy of pulsating dissipative soliton, showing the onset of pulsations a), and enlarged view of the stable pulsations b). The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

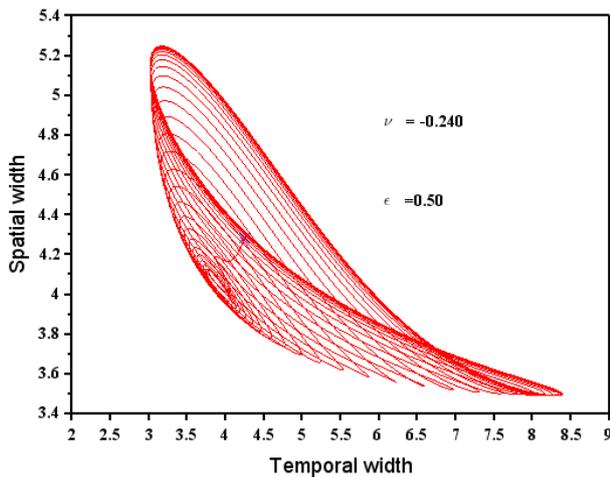


Figure 4. Example of stable limit cycle of the system in temporal and spatial widths plane. The blue cross corresponds to the initial point. The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

To complete this part of the study, we choose a third value of parameter of the saturation of the Kerr nonlinearity larger than the two ones, while keeping the other parameters constant. We fix the parameter $\nu = -0.238$ and plot the evolution of the total energy of the soliton in the Figure 5. It shows the main information about the

soliton dynamics, the total energy is an oscillating function of z . The dynamic starts with a brief transitional period characterized by small oscillations due to the adjustment of the initial condition, and after the onset of oscillations, it enters to the stable permanent regime. In Figure 5 a), we can see that the transitional phase is shorter than the two ones Figure 1 a) and Figure 3 a). The enlarged view of the dynamics illustrated by Figure 5 b), shows that the pulsations also are not similar. As in the case Figure 3, the total energy oscillates between two maximums (1100 and 886) and two minimums (849 and 750). Qualitatively the evolution resemble the one of the Figure 3 b), but the intrinsic characteristics are very different. Quantitatively we observe that the maximum (minimum) of the two energies are not the same 1100 (750) for $\nu = -0.238$ and 840 (680) for $\nu = -0.240$.

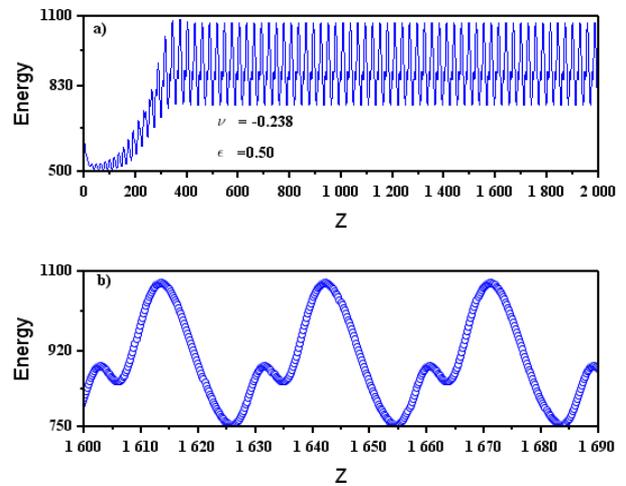


Figure 5. Evolution of the total energy of pulsating dissipative soliton, showing the onset of pulsations a), and enlarged view of the stable pulsations b). The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

The stable limit cycle in the temporal and spatial widths plane of such pulsating soliton is likewise reveal by the Figure 6, which has the same characteristic as that of the Figure 4. However, we remark that the trajectories are less compressed than that of the Figure 4.

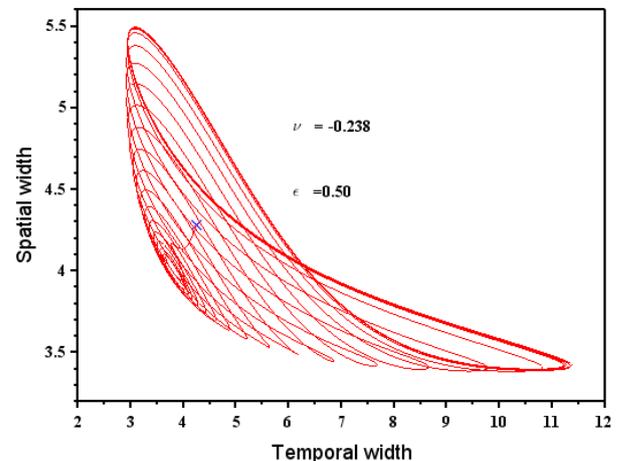


Figure 6. Example of stable limit cycle of the system in temporal and spatial widths plane. The blue cross corresponds to the initial point. The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

To resume this section, the analyses clearly demonstrate that the saturation of the Kerr nonlinearity influence drastically the pulsations of dissipative solitons and their stable limit cycles. For a given set of the parameters and varying ν to $-0.246, -0.240$ and -0.238 the soliton solution becomes pulsating with one-period, and then the pulsating soliton bifurcates to double-period pulsations. In addition, when the parameter ν increases and leads to a pulsation mode whose spectrum contains two (three) main frequencies, the trajectories of stable limit cycle are less tightly packed against each other.

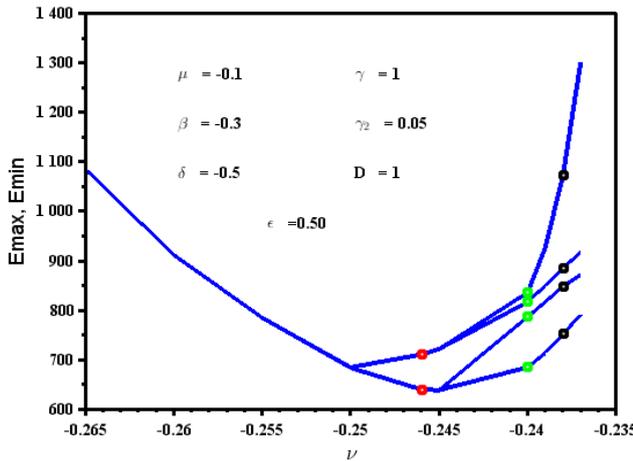


Figure 7. Bifurcation diagram representing the maximal and minimal values of the energy. The circles red corresponds to $\nu = -0.246$ (Figure 1), the green ones to $\nu = -0.238$ (Figure 3) and the black to $\nu = -0.236$ (Figure 5). Other CSHE parameters appear inside the Figure 7

To give a great overview of the dynamic and the impact of the parameter ν , we show in Figure 7 a bifurcation diagram obtained when varying the saturation of the Kerr nonlinearity from -0.265 to -0.235 while keeping the rest of parameters fixed. From this figure, we can see a local maximum or minimum of the total energy. We observe that the pulsating soliton bifurcations occur more than once with the variation of ν . When the value of the saturation of the nonlinearity is larger than -0.25 the curve is divided into several branches, which means that the maximum and minimum of the total energy differ. From -0.25 to -0.245 for the value of ν , the soliton oscillates between one maximum and one minimum, and it presents a single period pulsation solution: it's the first bifurcation. When the value of ν is larger than -0.245 a second bifurcation occurs, the soliton oscillates again between four values which presents double periods with two frequencies.

3.2. Pulsations under Influence of Nonlinear Gain

One of the most favourable term for the occurrence of stable solitons in dissipative systems is the nonlinear gain. It has been intensely studied for stationary and pulsating solitons. In [25] the authors have shown that at the stability boundary, a bifurcation occurs giving rise to stable oscillatory solitons in the CGLE for higher values of the nonlinear gain, and they can even exhibit more complex dynamics such as period-doubling. Indeed the pulsating solution in the CSHE can also be quasiperiodic, with several incommensurate periods involved in the

evolution; this is the purpose of this section. To elucidate the effect of the nonlinear gain ϵ on the pulsation, we set all the parameters of the CSHE, namely $D = \gamma = 1, \beta = -0.3, \delta = -0.5, \mu = -0.1, \nu = -0.247$ and $\gamma_2 = 0.05$, for different values of ϵ ($\epsilon = 0.50, 0.498$ and 0.496).

The Figure 8 illustrates the evolution of the total energy of the pulsating soliton solution for the value of nonlinear gain $\epsilon = 0.50$. Figure 8 b) represents the enlarged view of Figure 8 a). The evolution begins with a short transitional phase, which is followed by a permanent dynamics with large oscillations between the two constant limits. The energy remains in this stable state when the permanent dynamic is reached, as illustrated by the Figure 8 b), and oscillates between the two constant values 640 and 700.

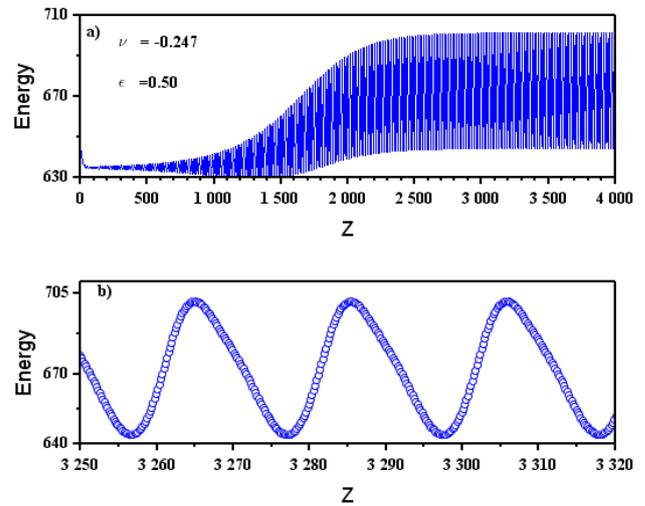


Figure 8. Evolution of the total energy of pulsating dissipative soliton, showing the onset of pulsations a), and enlarged view of the stable pulsations b). The values of other parameters are $D = \gamma = 1, \beta = -0.3, \delta = -0.5, \mu = -0.1$ and $\gamma_2 = 0.05$

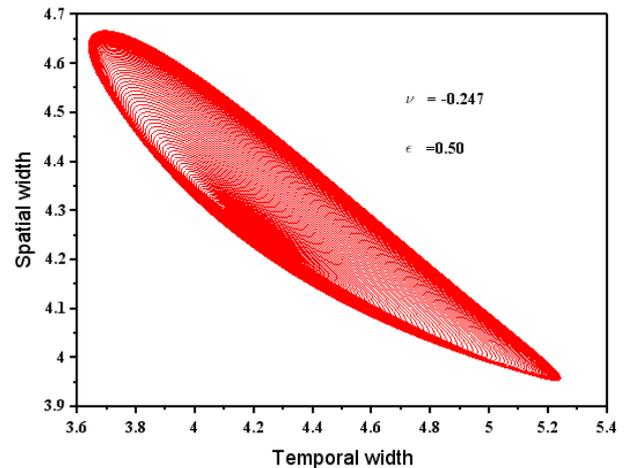


Figure 9. Example of stable limit cycle of the system in temporal and spatial widths plane. The blue cross corresponds to the initial point. The values of other parameters are $D = \gamma = 1, \beta = -0.3, \delta = -0.5, \mu = -0.1$ and $\gamma_2 = 0.05$

The trajectory of the stable limit cycle in the temporal and spatial widths plane of such pulsating soliton is resume by the Figure 9. The blue cross corresponds to the initial point; afterwards, all the trajectories tend to form a closed figure. When the steady state, is reached, we

observe the closed trajectory, which describes perfect pulsations behavior of the dissipative soliton. All the trajectories are well compressed due to the fluctuation of the initial condition.

The Figure 10 corresponds to the next value of the nonlinear gain $\epsilon = 0.498$. The total energy produces a quasiperiodic asymmetric pulsations when we decrease the nonlinear gain from $\epsilon = 0.50$ to $\epsilon = 0.498$. The pulsating soliton undergoes in stable permanent regime faster than in the previous case (Figure 8 a)), due to the decrease of the nonlinear gain. It appears that a slight decrease of the nonlinear gain parameter results in an increase of the oscillation amplitudes, which become anharmonic. The dynamic shows that the total energy is not conserved but evolves in accordance with the so-called balance equation. The solution oscillates between two maximums (685 and 720) and two minimums (600 and 680).

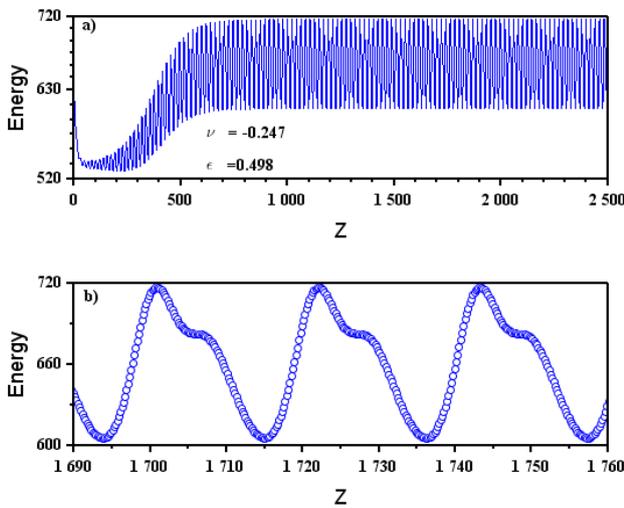


Figure 10. Evolution of the total energy of pulsating dissipative soliton, showing the onset of pulsations a), and enlarged view of the stable pulsations b). The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

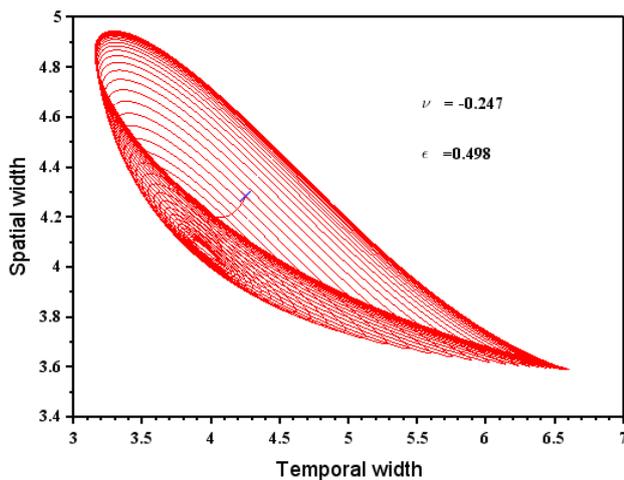


Figure 11. Example of stable limit cycle of the system in temporal and spatial widths plane. The blue cross corresponds to the initial point. The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

In the Figure 11, the example of stable limit cycle in the temporal and spatial widths plane corresponding to such solution is plotted. The blue cross represents to the initial

point and all the trajectories are less compressed than that of the Figure 9.

The last value of nonlinear gain chosen for the analysis of pulsating soliton is $\epsilon = 0.498$. For the same initial condition as the previous ones, the soliton evolves first with instable oscillations, but after a certain distance, its energy increases and takes a stable pulsating behaviour. The onset of pulsations occurs quicker than the previous ones, but regular oscillations appear after the initial amplitude overflowing effect. The soliton has the same dynamic behaviour as described above during the pulsating state Figure 5. When the permanent regime is reached the solution oscillates between two maximums (805 and 900) and two minimums (655 and 755). We clearly observe two distinct periods with different characteristics. The peak within each period of pulsations changes according to the value of the nonlinear gain. The trajectory of the stable limit cycle in the temporal and spatial widths plane of the corresponding pulsating soliton is represented in the Figure 13. The blue cross illustrates the initial point of the trajectory. One can notice that all the trajectories are less compressed than that of the Figure 11.

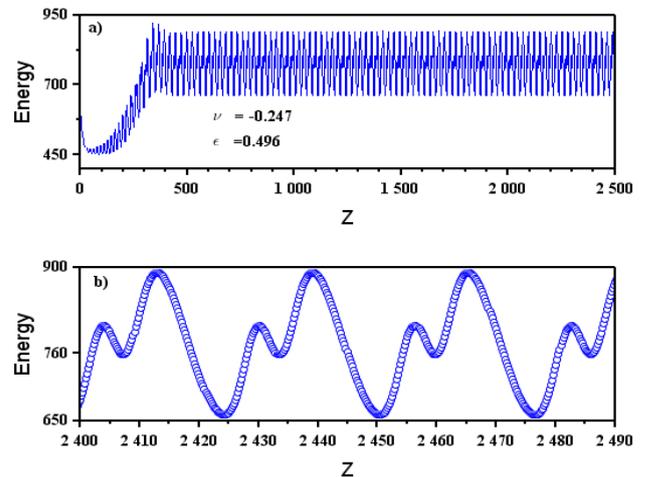


Figure 12. Evolution of the total energy of pulsating dissipative soliton, showing the onset of pulsations a), and enlarged view of the steady pulsations b). The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

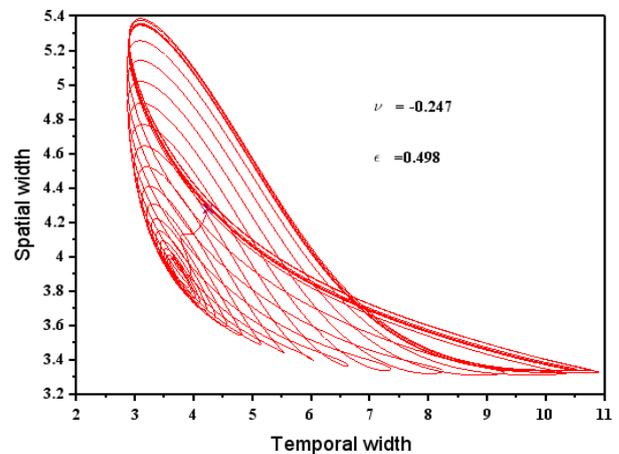


Figure 13. Example of stable limit cycle of the system in temporal and spatial widths plane. The blue cross corresponds to the initial point. The values of other parameters are $D = \gamma = 1$, $\beta = -0.3$, $\delta = -0.5$, $\mu = -0.1$ and $\gamma_2 = 0.05$

In this section, it clearly appears that the nonlinear gain perfectly controls the oscillations of dissipative solitons for a given set of the CSHE parameters. When the nonlinear gain ε increases, the double period pulsations turns into one period pulsating soliton. Furthermore, the trajectories of the stable limit cycles in the temporal and spatial widths plane of the corresponding pulsating solitons are less compressed when the nonlinear gain ε increases. In the ranges of the CSHE parameters explored in this study, ($\varepsilon = 0.496, 0.498, 0.500$), the pulsating soliton experiences a bifurcation that leads to a stationary. We observe a local maximum or minimum of the total energy evolution when varying the nonlinear gain from 0.495 to 0.504. Figure 14 shows clearly a variety of qualitatively different behaviours. When the value of nonlinear gain is smaller than 0.502, the maximum of energy is different to the minimum that means that the dissipative solitons are not in stationary state: they are pulsating.

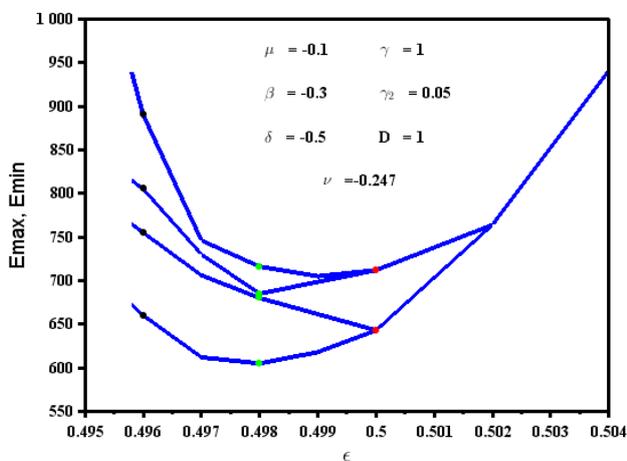


Figure 14. Bifurcation diagram representing the maximal and minimal values of the energy. The circles red corresponds to $\varepsilon = 0.500$ (Figure 8), the green ones to $\varepsilon = 0.496$ (Figure 10) and the black to $\nu = 0.498$ (Figure 12). Other CSHE parameters appear inside the Figure 14

From 0.495 to 0.500 for the value of the nonlinear gain, the solitons oscillate between two maximums and two minimums, and present double-period pulsations: it is the first bifurcation. Then from 0.500 to 0.502 we notice the second bifurcation where the soliton oscillates between one maximum and one minimum synonym of one-period pulsations. It shows that the double-period pulsations return into one-period pulsation before transforming into a stationary pulsating solitons.

3. Conclusion

We have reported a set of detailed simulations of pulsating solitons described by the Complex Swift-Hohenberg equation in two-dimensional. The investigation reveals very interesting bifurcation sequences of these pulses as the CSHE parameters are varied. Namely, the complete bifurcation diagrams of the total energy have been obtained for a definite range of the nonlinear gain and of the saturation of the Kerr nonlinearity values. The detailed analysis of the diagrams reveals that the pulsating solitons experience the period-doubling bifurcations for smaller

values of the nonlinear gain and for the larger values of the saturation of the Kerr nonlinearity. In addition, this study highlights the fact that the stable limit cycles can help to describe the dynamics of pulsating solitons. We have seen that when they are well compressed, they correspond to one-period pulsations and when the trajectories of stable limit cycle are less tightly packed, they correspond to double-period pulsations. We hope that these results can be extended to describe the pulsed operation of passively mode-locked lasers governed by the Complex Swift-Hohenberg equation or to explain specific aspects that occur in wide-aperture laser cavity.

References

- [1] Akhmediev, N. and Ankiewicz, *Dissipative Solitons*, Springer, Berlin, 2005.
- [2] Kamagaté, A., Grelu, Ph., Tchofo-Dinda, P., Soto-Crespo, J.M., Akhmediev, N., "Stationary and pulsating dissipative light bullets from a collective variable approach," *Phys. Rev. E*, 79, 026609, 2009.
- [3] Akhmediev, N., Soto-Crespo, J. M. and Town, G. "Pulsating solitons, chaotic solitons, period doubling, and pulse coexistence in mode-locked lasers: Complex Ginzburg-Landau equation approach," *Phys. Rev. E*, 63, 056602. April 2001.
- [4] Soto-Crespo, J. M., Akhmediev, N. and Ankiewicz A. "Pulsating, Creeping, and Erupting Solitons in Dissipative Systems," *Phys. Rev. Lett.* 85, 2937. October 2000.
- [5] Brusch, L., Torcini, A., Van Hecke, M., Zimmermann, M. G. and Bär, M. "Modulated amplitude waves and defect formation in the one-dimensional complex Ginzburg-Landau equation" *Physica D*, 160, 127-148, 2001.
- [6] Aranson, I. S. and Kramer, L. "The world of the complex Ginzburg-Landau equation", *Rev. Mod. Phys.* 74, 99. February 2002.
- [7] Soto-Crespo, J. M., Akhmediev, N. N., Afanasjev, V. V. and Wabnitz, S. "Pulse solutions of the cubic-quintic complex Ginzburg-Landau equation in the case of normal dispersion," *Phys. Rev. E* 55, 4783. April 1997.
- [8] Tsoy, E. N., Ankiewicz, A. and Akhmediev, N. "Dynamical models for dissipative localized waves of the complex Ginzburg-Landau equation," *Phys. Rev. E* 73, 036621. March 2006.
- [9] Konaté, A., Soro, E., Asseu, O., Kamagaté, A. and Yoboué, P. "Efficient Approach for 3D Stationary Optical Solitons in Dissipative Systems," *Journal of Applied Mathematics and Physics*, 3, 1239-1248. October 2015.
- [10] Deissler R. J. and Brand H. R. "Periodic, quasiperiodic, and chaotic localized solutions of the quintic complex Ginzburg-Landau equation," *Phys. Rev. Lett.* 72, 478. January 1994.
- [11] Asseu, O., Diby, A., Yoboué, P. and Kamagaté, A. "Spatio-Temporal Pulsating Dissipative Solitons through Collective Variable Methods," *Journal of Applied Mathematics and Physics*, 4, 1032-1041. June 2016.
- [12] Soto-Crespo, J. M., Grapinet, M., Grelu Ph. and Akhmediev, N. "Bifurcations and multiple-period soliton pulsations in a passively mode-locked fiber laser," *Phys. Rev. E*, 70, 066612. December 2004.
- [13] Ippen, E. P. "Principles of passive mode locking," *Appl. Phys. B* 58, 159-170. March 1994.
- [14] Grelu Ph. and Akhmediev, N. "Dissipative solitons for mode-locked lasers," *Nat. Photonics* 6, 84-92. February 2012.
- [15] Kalashnikov, V. L., Podivilov, E., Chernykh, A., Naumov, S., Fernandez, A., Graf, R. and A. Apolonski, "Approaching the microjoule frontier with femtosecond laser oscillators: theory and comparison with experiment," *New J. Phys.* 7, 217. 2005.
- [16] Bakonyi, Z., Michaelis, D., Peschel, U., Onishchukov, G. and Lederer, F. "Dissipative solitons and their critical slowing down near a supercritical bifurcation," *J. Opt. Soc. Am. B*, 19, 487-491. 2002.
- [17] Ultanir, E. A., Stegeman, G. I., Michaelis, D., Lange, C. H. and Lederer, F., "Stable dissipative solitons in semiconductor optical amplifiers," *Phys. Rev. Lett.* 90, 253903. 2003.

- [18] Yoboue, P., Diby, A., Asseu, O. and Kamagaté, A. "Stability of Dissipative Optical Solitons in the 2D Complex Swift-Hohenberg Equation," *International Journal of Physics*, vol. 4, no. 4. 78-84. 2016.
- [19] Akhmediev N. and Ankiewicz, A. *Solitons of the Complex Ginzburg-Landau equation*, Springer, Berlin, 2001.
- [20] Akhmediev N. and Ankiewicz A. *Dissipative solitons*. Ed. Springer, Heidelberg. 2005.
- [21] Wu, L., Guo Z. J., Song, L.-J. "Properties of pulsating solitons in dissipative systems," *Chin. Phys. B*, 19, No. 8 080512. 2010.
- [22] Tchofo-Dinda, P., Moubissi, A.B. and Nakkeeran, K. "Collective Variable Theory for Optical Solitons in Fibers," *Physical Review E*, 64. 2001.
- [23] Mirzazadeh, M., Arnous, A. H., Mahmood, M. F., Zerrad E. and Biswas, A. "Solitons solutions to resonant nonlinear Schrodinger's equation with time dependent coefficients by trial solution approach," *Nonlinear Dynamics*, 81, Issue 1-2. 277-282. 2015.
- [24] Kamagate, A. and Moubissi, A-B. "Pulsating solitons in the two-dimensional Complex Swift-Hohenberg Equation," *Journal of Applied Mathematics and Physics*, 6, 2127-2141.2018.
- [25] Facão, M. and Carvalho, M. I. "Plain and oscillatory solitons of the cubic complex Ginzburg-Landau equation with nonlinear gradient terms," *Physical Review E*, 96, 042220 –October. 2017.