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Can a Charge Configuration with Extreme Excess Charge be Stable?

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Abstract The classical electrodynamics of a macroscopic system with extreme excess charge is considered. The crucial assumption is here made that the electrodynamics of the system is completely dominated by the huge Coulomb repulsion, but counteracted by the resulting, violently oscillating acceleration field. It is shown that a special case of a system of this kind in the form of a charged spherical shell with powerful excess charge may exhibit violent such oscillations, effectively freezing it from expansion and thus in principle showing a counter-intuitive stability, albeit presumably short-lived. This effect may possibly have a bearing on, *e g*, some types of ball lightning observations, and is here also discussed in relation to the virial theorem, which traditionally is considered to exclude stable, localised electromagnetic configurations of this type. The oscillation frequency obtained for confinement of the type discussed in the article agrees well with what has been shown in laboratory experiments using microwaves to produce stationary luminous plasmoids in test tubes.

Keywords: *electromagnetic acceleration field, violent charge oscillations, luminous plasmoids, ball lightning, virial theorem*

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1. Introduction

The present paper studies the electrodynamics due to the extreme electromagnetic self-field in an ensemble of unipolar charges (specifically positive ions), each with charge *q*. In the system considered, there may also be a population of electrons that to some extent neutralizes the positive charges, but the dynamics of the system is here considered to be completely dominated by the powerful excess of positive charges. Electrons are tens of thousands times lighter than the positive ions in, say, air. Thus it is here assumed that under special circumstances an electron population may possibly become dislocated over large distances due to transient external fields with respect to the heavier, more sluggish ion population, leaving such an excess of positive charges in a localized region.

Such a situation could perhaps be created in the vicinity of a lightning stroke, or in a localized area on the ground due to extended atmospheric electric fields above – situations when sometimes observations of ball lightning [1,2] are reported: luminous balls that float around in the air for some short period of time (although the reliability of such reports may frequently be questioned, see *e g* [3]).

Secs. 2 through 4 below contain a detailed derivation of the electrodynamics of such a charge configuration with extreme excess charge, and aims at pointing out that there may be a conceivable mechanism for such a configuration to show a surprising – but possibly short-lived – stability.

Sec. 5 describes good agreement between the theory presented here and experiments with luminous plasmoids created by radiofrequency fields in test tubes.

In Sec. 6 finally, the possible temporary stability of such a charge distribution is discussed in relation to the virial theorem, which essentially states that a system of this type containing an electrostatic energy must expand, and which traditionally has been used to exclude the existence of stable such systems. However, attention is here drawn to the fact that the virial theorem may not be applicable to systems like the present case, which contain discrete charged elements that cannot themselves be expanded (a charged ionic crystal is another such example).

The present problem has previously been studied using a dielectric model [4] of the effect of acceleration in the electrodynamics of such a system with extreme excess charge. The present approach allows for a more detailed study of the electrodynamics such a system.

An important aspect of the present approach is the assumption made that the violently fluctuations of the acceleration field never give time for any appreciable net radial velocity to build up, and that the system – at least on short time-scales – thus behaves as if essentially “frozen”, like a solid.

2. Motion in a Strong Electromagnetic Field

From the Liénard-Wiechert potentials [5], the exact expressions in the general case for the electric field $\mathbf{E}(\mathbf{r}, \tau)$

and magnetic field $\mathbf{B}(\mathbf{r}, \tau)$ at position $\mathbf{r}$ and time τ from a point charge q at position $\mathbf{r}_j$ and time τ', and moving with velocity $\boldsymbol{\beta}$, can be derived to be as follows [6,7],

$$\mathbf{E}(\mathbf{r},\tau) = \frac{q}{4\pi\,\varepsilon_0}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r}-\mathbf{r}_j|^2} + \frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times \mathrm{d}\boldsymbol{\beta}/\mathrm{d}\tau)}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r}-\mathbf{r}_j|}\right]_{\tau'} \quad (2.1)$$

$$\mathbf{B}(\mathbf{r},\tau) = [\mathbf{n}]_{\tau'} \times \mathbf{E}(\mathbf{r},\tau), \quad (2.2)$$

where $\gamma^2 = 1-\beta^2$, and the unit vector $\mathbf{n}$ is defined as

$$\mathbf{n} = (\mathbf{r}-\mathbf{r}_j)/|\mathbf{r}-\mathbf{r}_j|\,. \quad (2.3)$$

The square parentheses in (2.1) and (2.2) are to be calculated at retarded time τ', *i e* taking into account the time it takes for the field to propagate from the source point $\mathbf{r}_j$ to the field point $\mathbf{r}$. The velocity is here expressed as $\boldsymbol{\beta} = \mathbf{v}/c$, where $\mathbf{v}$ is the velocity vector in normal units and c is the velocity of light, and where also $\tau = c\,t$, with t being the time in normal units.

The corresponding electromagnetic Lorentz force on a charge q is

$$\mathbf{F}(\mathbf{r},\tau) = q\,(\mathbf{E}(\mathbf{r},\tau) + \boldsymbol{\beta}\times\mathbf{B}(\mathbf{r},\tau)). \quad (2.4)$$

In the present study we will consider the case when the electrodynamics of the system is completely dominated by the Coulomb field and the acceleration term as described in in (2.1). We thus assume that we have almost chaotic fluctuations of the electric field due to successive, extremely rapid accelerations and decelerations of the particles relative to each other. These fluctuations are assumed to be so rapid that the velocities themselves of the charged particles remain very low before the next acceleration change, so that to a sufficiently good approximation we can thus set $\beta = 0$ (remember that we are here considering the motion of the more sluggish positive ions as discussed above.) In the following calculations we can then for the same reason also neglect the effects of the magnetic field on the motion of the charged particles. These assumptions make the calculations presented in this paper manageable.

The violent fluctuations of the accelerations of the ions can be expected to lead to radiation effects, as in the luminous plasmoids discussed in Sec. 5 below.

In the simplified case thus defined, expression (2.1) above reduces to

$$\mathbf{E}(\mathbf{r},\tau) = \frac{q}{4\pi\,\varepsilon_0}\left[\frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}_j|^2} + \frac{\mathbf{n}\times(\mathbf{n}\times \mathrm{d}\boldsymbol{\beta}/\mathrm{d}\tau)}{|\mathbf{r}-\mathbf{r}_j|}\right]_{\tau'}, \quad (2.5)$$

where the expression on the right-hand side is to be calculated at the source point $\mathbf{r}_j$ (and at retarded time τ', as discussed above).

Defining the acceleration $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta}/\mathrm{d}\tau, \quad (2.6)$$

and expanding the cross product, expression (2.5) can be rewritten to give

$$\mathbf{E}(\mathbf{r},\tau) = \frac{q}{4\pi\,\varepsilon_0}\left[\frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}_j|^2} + \frac{\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\alpha})-\boldsymbol{\alpha}}{|\mathbf{r}-\mathbf{r}_j|}\right]_{\tau'}. \quad (2.7)$$

From an ensemble of distributed discrete charges, the field in (2.7) then gives the following force in (2.4) on a charge q at the field point $\mathbf{r}$ and time τ,

$$\mathbf{F}(\mathbf{r},\tau) = \sum_j \frac{q^2}{4\pi\,\varepsilon_0}\left[\frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}_j|^2} + \frac{\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\alpha}_j)}{|\mathbf{r}-\mathbf{r}_j|} - \frac{\boldsymbol{\alpha}_j}{|\mathbf{r}-\mathbf{r}_j|}\right]_{\tau'}, \quad (2.8)$$

where again the expression in the parenthesis is to be calculated at the respective source points at retarded times.

We note that for pure radial accelerations, the two acceleration terms on the right-hand side cancel, and we thus have a case with a pure Coulomb force in accordance with Gauss' theorem. However – as we shall see – *when retardation effects are important*, the momentary acceleration field acting on a specific charge may *not* be a pure radial field even if (or rather especially when) the oscillations are only in the radial direction (*cf* Figure 1). This makes the situation more complicated, as will now be discussed.

Gauss' theorem is valid for the special (and normal) case of an inversely quadratic field [8], but not necessarily for other cases. Specifically, if we express $\boldsymbol{\alpha}_j$ in one component $\boldsymbol{\alpha}_{//}$ in the $\mathbf{n}$ direction as defined in (2.3) and one component $\boldsymbol{\alpha}_{\perp}$ perpendicular to $\boldsymbol{\alpha}_{//}$ in the $\mathbf{r}\mathbf{r}_j$-plane, then we see that the expression (2.8) deviates from a pure Coulomb field, and becomes

$$\mathbf{F}(\mathbf{r},\tau) = \kappa\sum_j\left[\frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}_j|^2} - \frac{\boldsymbol{\alpha}_{\perp}}{|\mathbf{r}-\mathbf{r}_j|}\right]_{\tau'} \quad (2.9)$$

where

$$\kappa = q^2/(4\pi\,\varepsilon_0). \quad (2.10)$$

This thus means that for a spherically-symmetric charge distribution that is momentarily accelerating radially outwards, there may be momentary reaction forces directed inwards. For, in the simplest case, consider a thin, radially-symmetric sheet of the charge distribution with radius r. Assume every element of this charged sheet to be initially accelerating radially outwards due to the Coulomb force. But then every element in this sheet will according to (2.9) be exposed to an instantaneous reaction force from its immediate neighbors due to this acceleration, and directed inwards and perpendicular to $\mathbf{r}-\mathbf{r}_j$, leading to violent oscillations. This effect, which thus resembles a surface tension, will in the following be studied for the special case of a thin spherical shell of charge.

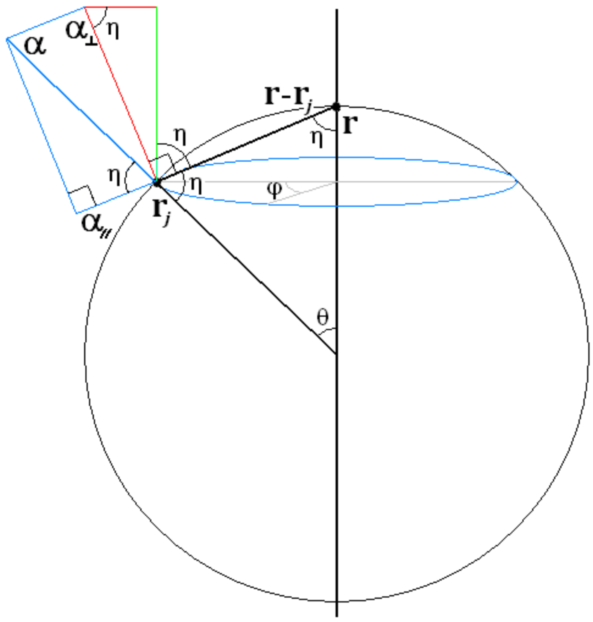

Figure 1. Accelerations $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_{\perp}$, $\boldsymbol{\alpha}_{//}$ and vectors $\mathbf{r}, \mathbf{r}_j, \mathbf{r}-\mathbf{r}_j$

3. Self-field of an Accelerating Charged Spherical Shell

Consider a charged spherical shell ("a bubble of charges"), and specifically a charge q at field point $\mathbf{r}$ and a charge q at source point $\mathbf{r}_j$ (see Figure 1). Assume from symmetry that the acceleration $\boldsymbol{\alpha}$ in Figure 1 is momentarily in the radial direction outwards with respect to the origin at the center $\mathbf{r} = 0$ of the spherical shell. In the derivation below, remember the assumption made above of very rapid fluctuations of the field, so that the momentary changes in the positions due to velocities of the charges can be ignored during the extremely rapid field fluctuations we are considering.

From Sec. 2 and Figure 1, we get for the time lapse $\tau - \tau'$ for a signal travelling from source point $\mathbf{r}_j$ to field point $\mathbf{r}$,

$$\tau - \tau' = \left|\mathbf{r} - \mathbf{r}_j\right| = 2\, r \sin(\theta/2)\,, \qquad (3.1)$$

and where also

$$\sin\eta = \cos(\theta/2), \quad \cos\eta = \sin(\theta/2)\,. \qquad (3.2)$$

The acceleration $\boldsymbol{\alpha}(\mathbf{r}_j, \tau')$ at source point $\mathbf{r}_j$ and time τ', due to the electric field there, has one component $\boldsymbol{\alpha}_\perp(\mathbf{r}_j, \tau')$ perpendicular to the direction to the field point $\mathbf{r}$, and one component $\boldsymbol{\alpha}_{//}(\mathbf{r}_j, \tau')$ in the direction to the field point, as shown in Figure 1. As derived above in (2.9), only the component perpendicular to the direction to the field point gives any contribution to the acceleration of the charge at the field point, and is given by the vector $\boldsymbol{\alpha}_\perp$ in Figure 1. In addition, from symmetry only the component of $\boldsymbol{\alpha}_\perp$ in the radial direction at the field point gives any net contribution to the acceleration at field point $\mathbf{r}$ and time τ. The resulting acceleration component due to $\boldsymbol{\alpha}$ in the radial direction at field point $\mathbf{r}$ is then given by the radial acceleration $\boldsymbol{\alpha}$ at the source point $\mathbf{r}_j$, reduced by a factor $\sin\eta = \cos(\theta/2)$ in (3.2) applied twice (from blue vector to red vector and then to green vector as shown in Figure 1).

Using (3.1) and (3.2), and integrating (2.9) over azimuth angles $0 \le \varphi \le 2\pi$ and polar angle $0 \le \theta \le \pi$, we then get the following expression for the acceleration due to the electric force at radius $\mathbf{r}$ and time τ from elements at $\mathbf{r}_j$ of a spherical shell at retarded times τ',

$$\alpha(\tau) = k \int_0^\pi \left[\frac{\sin(\theta/2)}{(2\, r \sin(\theta/2))^2} - \frac{\alpha(\tau')\cos^2(\theta/2)}{2\, r \sin(\theta/2)}\right]_{\tau'} \times$$
$$\times\, 2\pi\, r \sin\theta\; r\, d\theta\,, \qquad (3.3)$$

where $k = \kappa/m = q^2/(4\pi\ \varepsilon_0\ m)$ from (2.10) and with m denoting the particle mass.

Differentiating (3.1) gives

$$d\tau' = -\, r\cos(\theta/2)\; d\theta\,, \qquad (3.4)$$

which, after solving for $d\theta$ and inserting into (3.3), gives after simplification,

$$\alpha(\tau) = -K \int_0^{2r} \left[1/r - 2\,\alpha(\tau')\cos^2(\theta/2)\,\right] d\tau'\,, \quad (3.5)$$

or after using (3.1),

$$\alpha(\tau) = -K\int_0^{2r}\left[1/r - 2\alpha(\tau')(1-(\tau-\tau')^2/(2r)^2)\right]d\tau', \quad (3.6)$$

where $K = q^2/(4\ \varepsilon_0\, m)$.

It should be emphasized that the electric force on an element of the spherical shell of charge thus in this case results not only from the pure Coulomb force, *but also* from a reaction force due to the acceleration of its immediate neighbors. This reaction force thus closely resembles a surface tension.

4. Can a Charged Spherical Shell be Momentarily Stable?

We now ask whether a charged spherical shell as considered above can be stable, at least for some limited period of time, *i e* if the acceleration given by (3.6) above integrated over time τ might vanish, *i e* under what circumstances – if any – we might have that

$$\int_0^{2r} \alpha(\tau)d\tau$$
$$= -K\int_0^{2r}\int_0^{2r}\left[1/r - 2\alpha(\tau')(1-(\tau-\tau')^2/(2r)^2)\right]d\tau' d\tau \quad (4.1)$$
$$= 0,$$

where the time τ according to (3.1) should be taken between correspondingly 0 and $2r$.

Changing the order of the integrations in (4.1), we get

$$\int_0^{2r}\int_0^{2r}\left[1/r - 2\alpha(\tau')(1-(\tau-\tau')^2/(2r)^2)\right]d\tau' d\tau = 0. \quad (4.2)$$

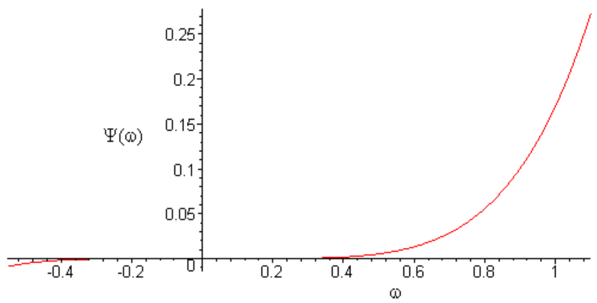

Figure 2. The function $\Psi(r, \omega)$ in (4.6) for $r = 3/5$, showing it to be essentially constant = 0 for $\omega < 0.4$

which after evaluating the inner integral becomes

$$\int_0^{2r} [\,1 - \alpha(\tau')\,(4\,r/\,3 + \tau' - \tau'^2/(2\,r))]\; d\tau' = 0\,, \quad (4.3)$$

or

$$1 - \int_0^{2r} [\alpha(\tau')\,(2/3 + \tau'/(2\,r) - \tau'^2/(2\,r)^2)]\; d\tau' = 0. \quad (4.4)$$

Assuming now that with suitable origin for τ' we can write $\alpha(\tau')$ as an oscillating function, *i e*

$$\alpha(\tau') = \cos(\omega\,\tau'), \qquad (4.5)$$

then (4.4) integrates to the following equation (after using (3.1) and simplifying),

$$\Psi(r, \omega) = 2\,r^2\omega^3 - \tfrac{4}{3}\,r^2\omega^2 \sin(2\,\omega\, r)$$
$$+\, \omega\, r\,(1 + \cos(\omega\, r)) - \sin(2\,\omega\, r) = 0\,, \qquad (4.6)$$

Expanding the function $\Psi(r, \omega)$ in (4.6) as function of ω and r (and dividing both members by 2) gives the equation

$$\left(\frac{1}{r} - \frac{5}{3}\right) r^3\omega^3 + \frac{49}{45}\, r^5\omega^5 - \frac{22}{105}\, r^7\omega^7 \ldots = 0\,, \quad (4.7)$$

It should be remarked that the first term $\frac{1}{r}$ in (4.7) is what originates from the Coulomb term in (2.9), and which can then be followed through (3.5), (4.2), and the first term $2\,r^2\omega^3$ in (4.6), to finally end up as this first term $\frac{1}{r}$ in (4.7). This is thus the point where we can explicitly see how the Coulomb acceleration $\frac{1}{r}$ is counteracted by the acceleration due to the oscillations of the spherical shell as described by the second term $-\frac{5}{3}$ in (4.7) (followed by a series expansion).

For sufficiently small values of $r\omega$, the equation (4.7) has the solution

$$r = \frac{3}{5}, \qquad (4.8)$$

corresponding to a situation where the acceleration averaged over an oscillation period vanishes. As seen in Figure 2, the function $\Psi(r, \omega)$ in (4.6) for $r = 3/5$ is further essentially constant = 0 for a considerable range $\omega < 0.4$ in the angular frequency ω.

For sufficient small angular frequencies ω (*e g* $\omega < 0.4$) of the oscillations (although they may still be violently fast), the excess charge configuration would thus seem to display a surprising, non-intuitive, and at least short-term stability. For larger values of ω, the charge configuration evaporates, as it would intuitively have been expected to do over the entire range of ω.

Under the assumption that the charge configuration has an extreme excess charge, the intuitive picture of an immediate violent explosion of such a charge configuration may thus have to be qualified. However, the system might still be susceptible to minor disturbances building up, which may cause the stability derived above to be short-lived.

5. Comparison with Experiments

As a concrete example of the assumed confinement mechanism discussed above, consider a spherical charge configuration of this type with a radius of the order of a few centimeters. This is a typical size in reported ball lightning observations [1,2] – but where many observations may be due to, or disturbed by, other types of phenomena like burning debris from a lightning stroke [9] (or possibly even just being afterimages on the retina from lightning strokes).

The time $\tau = c\,t$ (in units of meters as used above) then corresponds to the ordinary time $t = \tau / c = \tau / (3\;10^8)$ in seconds. The corresponding angular frequency ω (in units of radians/meter as used above) then corresponds to an ordinary angular frequency $w = \omega\, c$ in units of radians/sec. The condition for stability derived above, *i e* $|\omega| < 0.4$, thus corresponds to a condition for the frequency w in normal units equal to $|w| < 0.4 \times 3\;10^8 = 1.2\;10^8$ radians/sec, *i e* a frequency in the 20 MHz region.

It is interesting to note that this is exactly in the frequency region in which R. W. Wood [10] was able to create luminous plasmoids resembling ball lightning in confined laboratory plasmas [11] using high-frequency electrical fields. The present analysis thus points to the possibility that plasmoids of this type could under favorable conditions indeed exist by themselves as self-contained, freely moving objects with sufficient life-times to be observed as ball lightning.

6. Comments on the Virial Theorem

In an electromagnetic field, the virial theorem [12] is given as [13],

$$\frac{1}{2}\frac{d^2I}{dt^2} + \int_V x_k \frac{\partial G_k}{\partial t} d^3r =$$
$$= 2(T + U) + W_E + W_M - \int x_k\,(p_{ik} + T_{ik})\,dS_i\;, \qquad (5.1)$$

where I is the moment of inertia, G is the momentum density, T is the kinetic energy, U is the energy, and $W_E + W_M$ is the electric and magnetic energy content of the volume V. In the last term, p_{ik} is the pressure tensor,

$$p_{ik} = \sum m^\sigma n^\sigma \langle v_i v_k \rangle^\sigma - V_i V_k \sum m^\sigma n^\sigma\;, \qquad (5.2)$$

and T_{ik} is electromagnetic stress-energy tensor,

$$T_{ik} = \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)\delta_{ik} - \left(\varepsilon_0 E_i E_k + \frac{B_i B_k}{\mu_0}\right). \qquad (5.3)$$

According to the virial theorem, the positive electromagnetic energy of a system of electric charges will thus cause a system without a confining boundary to experience an accelerated expansion, which would thus exclude a system as discussed here to show even a temporary stability of the type discussed earlier in this paper.

Still there are in nature objects like ionic crystals, which even if they have some excess charge are still stable. This obviously contradicts [14] the conclusion just drawn from the virial theorem (*cf* also [15]).

The resolution to this apparent contradiction lies in the fact that the charge distribution inside the electric charges in the ionic crystal and similar systems [15] is not available for use in the expansion of the system. The virial theorem as described above assumes a continuum distribution of charges, and is not applicable in the present case, where the charge distributions in the individual ions are held together by forces that make the electric energies within the ions unavailable for use in a possible expansion of the crystal.

Another similar example is a soap bubble drifting in the air, where again the electric energy confined within the atoms in the water molecules is not available for use in expansion of the bubble. As discussed above, the electric forces involved in the mechanism discussed in this paper are very similar to surface tension, where thus again the energy in the atoms involved is not available for use in the expansion of the system. Thus the virial theorem in this case does not exclude a stability of the type derived in this paper.

7. Conclusions

The above discussion thus seems to show that a charge configuration with extreme excess charge may experience fast, almost chaotic oscillations between radial expansions due to the Coulomb field and radial contractions due to violent reaction forces to the Coulomb repulsion.

Detailed study of the electrodynamics of a system of this type in the form of a spherical shell of charge shows that a system of this kind may experience a type of at least short-term stability, somewhat similar to that of a soap bubble, where in this case transverse reaction forces to the electric repulsive radial acceleration play the same role as surface tension does in the case of the soap bubble. The radius for an assumed quasi-stable structure of this kind is here indicated to be of the same order of magnitude as reported from actual ball lightning observations. Also the oscillation frequencies calculated for this mechanism agree with experiments where luminous plasmoids have been created by RF oscillations.

The virial theorem is normally invoked to counter claims of self-stable electrostatic systems of this kind, but is in this case not applicable since fields from discrete charges are here fundamentally in action, rather than fields from the essentially continuous charge distributions for which an expansion due to the virial theorem is applicable.

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