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On Mathematical Foundations of Quantum Collisions and Nuclear Reactions and Outcome of Certain Physical Phenomena from Them

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Abstract I. Analytical structure of the non-relativistic unitary and non-unitary *S*-matrix is reviewed for the cases of any interactions with any motion equations inside a sphere of radius *a*, enclosed by centrifugal and rapidly decreasing (exponentially or by the Yukawian law or by the more rapidly decreasing) potentials. Some kinds of the symmetry conditions are imposed. The Schroedinger equation for the particle motion in the external region where $r > a$ and the completeness of the correspondent wave functions are assumed. The connection of the obtained results with the causality is examined. Partially some analytical properties for the multi-channel *S*-matrix are reviewed and the sum rules for mean compound-nucleus time delays and the density of compound-nucleus levels. Sometimes (as physical manifestations of the profound general methodic and in very good consistent accordance with the experiment) observable physical effects, such as parity violation enhancement and time resonances or explosions, are appeared. Finally a scientific program of future search is presented as a clear continuation and extension of the obtained results. II. It is already known the appearance of time advance (due to distortion by the non-resonant background) instead of the expected time delay in the region of a compound-nucleus resonance in the center-of-mass (C-) system. Here at the same conditions we study cross sections and durations of the neutron-nucleus scattering in the laboratory (L-) system. Here it is shown that such time advance is a virtual paradox but in the L-system the time-advance phenomenon does not occur and only the trivial time delay is observed. At the same time the transformations from C-system into the L-system appeared to be different from the standard kinematical transformations because in the C-system the motion of a compound nucleus is absent but it is present in the L-system. We analyze the initial wave-packet motion (after the collision origin) and the cross section in the laboratory (L-) system. Also here (as physical revelations of profound general methodic and in very good consistent accordance with the experiment) several results of the calculated cross sections for the neutron-nucleus in comparison with the experimental data in the L-system at the range of one or two overlapped compound resonances are presented. It is shown in the space-time approach that the standard kinematical transformations of cross sections from the C-system to the L-system are not valid because it is necessary to consider the center-of-mass motion in the L-system. Finally on a correct self-consistent base of the space-time description of the nuclear processes in the laboratory system with 3 particles in the final channel, it is shown the validity of the former approach, obtained for the space-time description of the nuclear processes with 2-particle channels earlier.

Keywords: *(I) S-matrix, condition of completeness of external wave functions, external centrifugal and rapidly decreasing potentials, causality, time resonances or explosions, parity violation enhancement; (II) space-time approach to nuclear collision, time delay, time advance, transformations of cross sections from the C-system to the L-system, interference phenomena*

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1. General Introduction (Pre-face)

The known Russian mathematician and physicist-theoretician N.N. Bogolyubov claimed [1] that mathematics now becomes partly the range of the theoretical physics (namely the quantum collision theory, analytical theory of the S-matrix, dispersion ratio and quite recently - maximal hermitian time operator for quantum systems with continuous energy spectra [3]). Also the known Russian physicist-theoretician Landau has said [2] a new good method in physics is better than any effect because it can bring us to some or even many new effects which can in a new way explain the experimental data. And namely there is a reincarnation of these ideas in my paper. The both remarks are clearly manifested in both parts of this paper. There are new physical effects which had followed almost directly from my methods (they are developed from mathematical methods, reviewed in the part I, and from my new theoretical method, generalized in the part II).

In part I there are the new effect of the time resonances (or explosions), which had followed from the Simonius multichannel *S*-matrix, and the parity-violation-enhancement effect, which had get out the analytical structure of the *S*-matrix for the interactions with the violated parity. In the part II there are new transformations between the C- and L-system for cross sections with two and more interaction mechanisms, which generalize the well-known standard cinematic transformations between the C- and L-systems for cross sections with one direct (or prompt) mechanism, and also the virtual delay-advance effect in the C-system and absence of it in L-system (both revealed firstly by me). Both results or effects had followed from the space-time analysis (or method) of neutron-nucleus scattering, which has also firstly been elaborated by me.

So, the presented paper is evident manifestation of such remarks of two known physicists, which appeared to be unexpected for me.

2. Part I. Analytic Properties of the S-Matrix for any Interactions, Enclosed by Centrifugal and Exponentially or More Rapidly Decreasing Potentials

2.1. Introduction of Part I

Many papers and books on non-relativistic quantum collision theory are dedicated to the analysis of the solutions of the Schroedinger equation and of the analytical properties of the correspondent *S*-matrix for various potentials of different forms, extended in all the three-dimensional space with radial coordinate along the axis (0, ∞). And only a rather small number of papers are concentrated on the study of the analytical properties of the *S*-matrix with the minimal number of assumptions on the interactions on small distances (practically nothing, with the exception of very general physical and mathematical principles, such as certain symmetry properties, causality or the condition of the completeness of the wave functions at the external interaction range, and also the possibility of the *S*-matrix analytic continuation at the complex plane of the kinetic energies or of the wave numbers). This approach ascends to the old idea of Heisenberg [1] (see also [2,3,4,5] and precedent references therein) on the unique fundamental quantity (the *S*-matrix) which is sufficient for the predictions of many observable quantities basing only on the general physical and mathematical principles.

Now we shall outline the main results of [4, part II] for the unitary *S*-matrix, since they will be an initial base of the further reviewed results of papers [6-12]. Namely in [4, part II] it had been obtained the analytical expression of the function $S_l(k)$, which defines the relation between the amplitudes of ingoing and outgoing *l*-waves for the elastic scattering of non-relativistic particles without spin (with *l*=0) for arbitrary interaction, localized inside the sphere of radius *a*, starting from the unitary condition

$$S_l(k) S_l^*(k^*) = 1, \qquad (1)$$

the symmetry condition

$$S_l(k) S_l(-k) = 1 \qquad (2)$$

or

$$S_l^*(k) S_l\left(-k^*\right) = 1 \qquad (3)$$

and the particular "causality" condition (*if the ingoing wave packet is normalized so, that at t = –∞ it represents one particle, then the total probability to find the particle in any successive time moment (for instance, t=0) outside the interaction sphere cannot be more than 1*). Strictly speaking, this condition is not the causality but the conservation of the total probability (more correctly, its analytical continuation in the complex plane of k). In [13] it was shown that it does directly follow from the orthogonality of the eigen functions of a self-adjoint operator, describing the motion and interaction of the colliding particles.

Then, it had been also assumed the existence of the analytic continuation of $S_l(k)$ into the complex plane of *k* and the condition of the quadratic integrability of the weight functions of the wave packets which in turn ensured the uniform convergence (at the range *r*>*a*) of the integrals over momentum in the Fourier-expansions of the wave packets. Finally it was obtained the following expression for $S_0(k)$:

$$S_0(k) = \exp(-2ik\alpha) \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{(k_s - k)(k_s^* + k)}{(k_s^* - k)(k_s + k)}, \qquad (4)$$

where $\alpha \le a$, k_λ are zeros on the imaginary axis (which are simple on the lower semi-axis), k_s are the zeros in the upper half-plane D^+, the products $\prod_\lambda$ and $\prod_s$ converge on the real axis *k*. In [14] it was shown that zeros k_λ on the lower and upper imaginary semi-axes and zeros k_s correspond to bound, virtual (anti-bound) and resonance states, respectively.

If the interaction is described by a local central potential *V*(*r*), independent from *k*, and the conditions

$$\int_0^\infty dr r^n \left|V(r)\right| < \infty, \quad n = 1, 2, \qquad (5)$$

and

$$V(r) \equiv 0 \;\; for \;\; r > a, \qquad (6)$$

are fulfilled, the expression (4) is valid also for arbitrary values of *l*, with α = *a* and the product over λ contains a finite number of poles on the upper imaginary semi-axis. But if only the condition (5) is fulfilled, then the expression (4) is, generally speaking, invalid and one does often use the following expression

$$S_l(k) = \frac{f_{l-}(k)}{f_{l+}(k)}, \qquad (7)$$

where $f_{0\pm}(k) = f_{0\pm}(k, 0)$ for $l = 0$ and

$$f_{l\pm}(k) = \frac{k^l \exp(\pm il\pi/2)}{(2l-1)!!} \lim_{r\to 0} r^l f_{l\pm}(k, r)$$

for $l > 0$, $f_{l\pm}(k,r)$ is the solution of the radial Schroedinger equation or of the equivalent to it integral equation

$$f_{l\pm}(k,r) = \pm i\exp(\pm ilp/2)kr\, h_l^{(1,2)}(kr) - \frac{2\mu}{\hbar^2 k}\int_0^\infty dr' g_l(k;r,r')V(r') f_{l\pm}(k,r'), \quad (8)$$

with the boundary condition

$$\lim_{r\to\infty} f_{l\pm}(k,r)\exp(\text{m}ikr) = 1 \quad (9)$$

where

$$g_l(k;r,) = \frac{ikrr'}{2}\begin{bmatrix} h_l^{(1)}(\text{k})h_l^{(2)}(kr) \\ -h_l^{(1)}(kr)h_l^{(2)}(\text{k}) \end{bmatrix},$$

and $h_l^{(1,2)}(kr) = j_l(kr) \pm in_l(kr)$ are the Hankel spherical functions of the first and the second kind, respectively ($j_l(kr)$, $n_l(kr)$ are the Bessel and the Neiman spherical function, respectively). At such conditions the function $S_l(k)$ can have, besides the singularities described by (4), additional singularities, corresponding to the singularities of $f_{l\pm}(k,r)$.

The author's (partly with his collaborators) papers [6-12] are presented the review of the results of that approach, published gradually during 1961- 2006 (mainly in the Russia and Ukraine), and can be evidently continued in the future. The second part of this paper contains another review, dedicated to the space-time description of cross sections and durations of neutron-nucleus scattering near 1-2 resonances in the C- and L-systems. In the final sections of both parts of the present review, the scientific program is presented which is connected with the remained tasks, problems and also the gradually revealed perspective, unexpected previously, – how the rigorous mathematical method or approach can help to reveal quite concrete and sometimes paradox physical phenomena.

2.2. The Properties of the Non-Unitary One-Channel *S*-matrix for the Arbitrary Interactions Enclosed by the centrifugal Barrier and a Potential, which is Decreasing More Rapidly then any Exponential Function

Now, following [7], we consider a generalized case when the interaction and motion equation inside the sphere of radius a are unknown as before, but at $r>a$ contains the centrifugal barrier $\hbar^2\, l(l+1)/r^2$ and a potential $V(r)$, and there is not only a scattering but also a partial particle absorption or generation. For the convenience let us introduce new interaction characteristics – a complex "interaction constant" γ. We agree conventionally that its real part Reγ will characterize that interaction part which cause by itself the scattering only without the particle absorption or generating. And we agree to set up the negative (positive) value of Imγ in correspondence with that interaction part, the absence of which causes the absence of the particle absorption (or generating). If we further connect the particle absorption and generating with the simple decreasing or increasing of the flux of the scattered particles in comparison with the flux of bombarding particles, assuming the conservation of their impulse and other characteristics, then it will be natural to impose the following conditions:

$$0 < \left|S_l(\gamma,k)\right|^2 \le 1, \quad (10a)$$

$$1 \le \left|S_l(\gamma^*,k)\right|^2 < \infty, \quad (10b)$$

with Im$\gamma < 0$, for real positive k. Since the conditions (10a) and (10b) are evidently insufficient for the study of the analytic properties of $S_l(\gamma, k)$, let introduce, generalizing (1)-(3), the new symmetry properties (typical for central interactions)

$$S_l(\gamma,k)S_l(\gamma,-k) = 1, \quad (2a)$$

$$S_l(\gamma^*,k)S_l(\gamma^*,-k) = 1, \quad (3a)$$

and the generalized "unitarity" condition

$$S_l(\gamma,k)S_l^*(\gamma^*,-k^*) = 1, \quad (1a)$$

thus selecting for any interaction with the constant γ(Im γ, 0) the "conjugate " interaction with the complex conjugate constant γ^*.

One can easily check that the conditions (1a),(2a),(3a) and (10a), (10b) are automatically fulfilled in the case when the interaction can be described by the complex potential which satisfies the condition (5) [14,15]. In that case the values γ and γ^* are not only conventional but also factual parameters of the potential $V(\gamma, r)$ = Re $\gamma V_1(r) + i$ Im $\gamma V_2(r)$.

Instead of the "causality" condition from [4, part II], we shall use the condition of the completeness for the wave functions outside the sphere of unknown interaction, factually assuming in this region (i.e. for $r \ge a$) the possibility of describing the colliding particles by the Schroedinger equation with a self-adjoint Hamiltonian:

$$\frac{2}{\pi}\int_0^\infty k^2 dk R_l^{(+)}(\gamma,k,r)R_l^{(+)*}(\gamma,k,r') + \sum_n R_{nl}(\gamma,k_{nl},r)R_{nl}(\gamma,k_{nl},r') = \frac{\delta(r-r')}{r^2} \quad (11)$$

where

$$R_l^{(+)}(\gamma,k,r) = \frac{i}{2kr}\begin{bmatrix} f_{l-}(k,r)\exp(il\pi/2) \\ -S_l(\gamma,k) f_{l+}(k,r)\exp(-il\pi/2)\end{bmatrix},$$

$$R_{nl} = \frac{1}{\sqrt{2\pi}}B_{nl}(\gamma,k_{nl}) f_{l+}(k_{nl},k)/r,$$

functions $f_{l\pm}(k, r)$ are defined by equation (8); Im $k_{nl} > 0$ and consequently the functions R_{nl} are integrable together with their squares (at least, at the range $a \le r < \infty$) ; all the information on the interaction inside the sphere with the radius $r < a$ contains in the functions $S_l(\gamma, k)$ and the constants $B_{nl}(\gamma,k_{nl})$. Let note that we (tacitly) assume that $R_{nl}(\gamma,k_{nl},r) = R_{nl}^{\bullet}(\gamma^{\bullet},k_{nl}^{\bullet},r)$ in (11).

Eq.(11) represents a generalization of the completeness relation for the eigen functions of the most simple classes of the non-Hermitian Hamiltonians [14] for the cases when all the eigen values k_{nl} are simple (non-multiple) and are situated outside the real axis k. When γ = Re γ, the functions R_{nl} describe simply bounds states of the system. For the complex values of γ they have the same boundary conditions as the bound states and their properties for the non-singular potentials with the negative imaginary part are partially described in [16].

In order to be sure that $S_l(\gamma, k)$ can have the analytic continuation into the complex plane of k, one has to impose some limitations on the potential tails of at the range of $r > a$. In the correspondence with the study of the potential scattering in [14,15,16,17], we can try, at least, to limit ourselves by the cases when at the range $r > a$, besides the centrifugal barrier, there is present a potential which satisfies the following condition

$$\int_0^\infty drr\left|V(r)\right|\exp(br) < \infty \tag{12}$$

at least, with any arbitrarily small b.

Using the properties

$$f_{l+}^*(k^\bullet, r) = f_{l+}(-k, r) = f_{l-}(k, r) \tag{13}$$

for real k and relations (1a),(2a) and (3a) for $S_l(\gamma, k)$, one can transform (11) into the form

$$\begin{aligned}&\frac{1}{rr'}\int_C dkf_{l+}(k,r)f_{l-}(k,r')\\ &-\frac{(-1)^l}{rr'}\int_C dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')\\ &+\frac{1}{rr'}\sum_n (B_{nl})^2 f_{l+}(k_{nl},r)f_{l+}(k_{nl},r') = \frac{2\pi\delta(r-r')}{r^2}\end{aligned} \tag{14}$$

where the integration trajectory C goes along the real axis k from $-\infty$ to ∞, bypassing the point $k = 0$ where $f_{l\pm}$ have the pole of the l-th order by a semi-circle of the infinitesimal small radius, located in the upper semi-space.

We shall limit ourselves by the case when $f_{l\pm}(k, r)$ behavior as $\exp(\pm ikr)$ in all the complex plane at $|k| \to \infty$. Then, shifting the integration contour into D^+, enclosing all the singularities and utilizing equalities

$$\begin{aligned}&\int_{\Gamma+} dkf_{l+}(k,r)f_{l-}(k,r) = \int_{\Gamma+} dke^{ik(r-r')}\\ &= \int_{-\infty}^{\infty} dke^{ik(r-r')} = 2\pi\delta(r-r'),\end{aligned} \tag{15}$$

$$\int_{\Gamma+} dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r') = \int_{\Gamma+} dkS_l(\gamma,k)e^{ik(r+r')}, \tag{16}$$

we obtain

$$\begin{aligned}&\sum_n \oint_{k_m} dkf_{l+}(k,r)f_{l-}(k,r') + \sum_p \oint_{\gamma_p} dkf_{l+}(k,r)f_{l-}(k,r')\\ &-(-1)^l\sum_j \oint_{k_j} dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')\\ &-(-1)^l\sum_q \oint_{\gamma_q} dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')\\ &-(-1)^l\sum_n (B_{nl})^2 f_{l+}(k_{nl},r)f_{nl}(k_{nl},r') = 0,\end{aligned} \tag{17}$$

where $\int_{\Gamma^+}$ is the integral over the infinitely large semi-circle above the real axis, $\oint_{k_n}$ is the integral over an infinitesimal circle around an isolate singular point, $\oint_{\gamma_p}$ is the integral over a contour which envelops a non-isolate singularity (for instance, over the edges of the cut conducted for a branch point). Since all these contours are independent, equality (17) is equivalent to the following system of equalities:

$$\begin{aligned}&(-1)^l \oint_{k_n} dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')\\ &= (B_{nl})^2 f_{l+}(k_{nl},r)f_{l+}(k_{nl},r'),\end{aligned} \tag{18}$$

$$\begin{aligned}&(-1)^l \oint_{k_n} dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')\\ &= \oint_{k_n} dkf_{l+}(k,r)f_{l-}(k,r'),\end{aligned} \tag{19}$$

$$\begin{aligned}&(-1)^l \oint_{\gamma_p} dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')\\ &= \oint_{\gamma_p} dkf_{l+}(k,r)f_{l-}(k,r'),\end{aligned} \tag{20}$$

$$\int_\Gamma dkS_l(\gamma,k)e^{ik(r+r')} = 0. \tag{21}$$

A simple analysis of equation (18) shows that $S_l(\gamma, k)$ has the poles of the first order on the positive imaginary semi-axis (which at $\gamma = \mathrm{Re}\gamma$ correspond to bound states) with residues $(-1)^{l+1} i\frac{(B_{nl})^2}{2\pi}$. Re-writing eq.(19) in the form

$$\oint_{k_n} dkf_{l+}(k,r)f_{l+}(k,r')[\frac{f_{l-}(k,r')}{f_{l+}(k,r')} - (-1)^l S_l(\gamma,k)] = 0, \tag{19a}$$

after simple reasoning one can easily to conclude that $S_l(\gamma, k)$ has to have additional isolate singularities D^+, coincident with those isolate singularities of $f_{l-}(k,r)$ in the upper semi-space near which

$$\lim_{k\to k_{mn}} f_{l-}(k,r) = \lim_{k\to k_{mn}} D_m(k)f_{l+}(k,r), \tag{22}$$

where the function $D_m(k)$ does not depend on r and has an isolate singular point k_m. Similarly one can study non-isolate singularities of $S_l(\gamma, k)$ coming from analysis of eq.(20).

In the simplest case when outside the sphere of radius a there is only a centrifugal potential, functions $f_{l\pm}(k,r)$ have the form

$$f_{l\pm}(k,r) = (\pm i)\exp(\pm ilp/2)krh_l^{(1,2)}(kr). \tag{8a}$$

Since functions $h_l^{(1,2)}(kr)$ are analytical in the whole complex plane k, with the exception of points $k = 0$ and ∞, then we can choose at $|k| \to \infty$

$$(\pm i)\exp(\pm ilp/2)krh_l^{(1,2)}(kr) \underset{|k|\to\infty}{\to} \exp(\pm ikr) \qquad (9a)$$

in the whole complex plane k, and so, in correspondence with eq.(18)-(21), the function $\exp(2ik\alpha)\, S_l(\gamma,k)$, $\alpha \le a$, is regular everywhere in the whole D^+, except isolate singularities k_{nl} which for γ =Re γ are localized on the positive imaginary semi-axis. In this last case, we can find the product expansion of the type (4), where points k_λ are the zeros k_{nl} on the lower imaginary semi-axis, corresponding to bound states, and also the zeros on the upper imaginary semi-axis, which define virtual (anti-bound) states and correspond to the poles situated, at least, by one between the poles k_{nl} and $k_{n+1,}$ following an approach that was outlined in [9,13].

For the complex values of γ the final result for $S_l(\gamma,k)$ can be also represented in the analytical form. Considering that the zeros (poles) of $S_l(\gamma,k)$ in the first and the second quadrants do in the consequence of the symmetry conditions (2a) correspond to the poles (zeros) in the third and the forth quadrants, mirror-like symmetrical to them relative to the direct lines Im k = Re k and Im k = – Re k, respectively, we can find the product expansion of $S_l(\gamma,k)$, following an approach that was outlined in [10,12]. Its derivation is performed in Appendix I, and the obtained there final forms are:

$$S_l(\gamma,k) = \exp(-2i\alpha k)\prod_n \frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda \frac{k_\lambda-k}{k_\lambda+k}\prod_s \frac{k_s-k}{k_s+k}\prod_{s'} \frac{k_{s'}-k}{k_{s'}+k}, \qquad (23a)$$

$$S_l(\gamma^*,k) = \exp(-2i\alpha k)\prod_n \frac{k_{nl}^\bullet+k}{k_{nl}^\bullet-k}\prod_\lambda \frac{k_\lambda^\bullet-k}{k_\lambda^\bullet+k}\prod_s \frac{k_s^\bullet-k}{k_s^\bullet+k}\prod_{s'} \frac{k_{s'}^\bullet-k}{k_{s'}^\bullet+k}, \qquad (23b)$$

which generalizes (4), taking conditions (1a)-(3a) into account. Here k_{nl} are the poles in the lower half-space D^-, k_λ are the zeros in D^+, k_s and $k_{s'}$ are the zeros in the first and the second quadrants, respectively. The results (23a, b) had been explicitly obtained in [7] firstly and had not been analyzed before even for the simple interactions described by the complex potentials.

The written above simplified assumptions on the eigen values k_{nl} in the completeness condition (11) factually brings to an insignificant limitation of the interaction class. The absence of values k_{nl} on the real axis k, i.e. the absence of poles and zeros (spectral points) of $S_l(\gamma,k)$ and $S_l(\gamma^*,k)$ corresponding to them (as well as the absence of values of k_s and $k_{s'}$), does simply signify the rejection the cases of the total absorption of bombarding particles and also the rejection of the infinite increasing of the new-particle birth for the physical values of $k \ge 0$. The condition of the absence of the eigen values k_{nl} with the multiplicity of more than 1 apparently does not also bring to the essential limitation of the interaction class. Really, if one naturally assumes that a smooth change of the interaction parameter γ brings to the smooth shift of the values k_{nl}, then the arbitrarily small change of the parameter γ will bring to a certain small divergence of the various trajectories $k_{nl}(\gamma)$ from the point of their (k_{nl}) coincidence. In [9] it was shown (with the help of another method) that expressions (23a,b) are valid for local potentials inside $r \le a$ with a hard (infinite) core of radius $r_0 < a$, for non-local separable potentials of the type $v(r)\, v(r')$ with $0 < r,\ r' < a$, for non-local separable potentials with a hard(infinite) core of radius $r_0 < a$. And expressions (23a,b) were generalized for local complex potentials with multiple zeros – k_{nl}, k_λ and k_s. In the last case in (23a,b) there will be present the factors of the type $\left(\frac{k_{nl}+k}{k_{nl}-k}\right)^{\alpha_{nl}}$ $\left(\frac{k_\lambda-k}{k_\lambda+k}\right)^{\alpha_\lambda}$ $\left(\frac{k_s-k}{k_s+k}\right)^{\alpha_s}$, where α_{nl}, α_λ and α_s are the multiplicities of reros – k_{nl}, k_λ and k_s, respectively.

If at the external region, when $r \ge a$, there are the centrifugal barrier and a potential, which is decreasing more rapidly then any exponential function, then the results (23a, b) remain valid since in this case the functions $f_{l\pm}(k, r)$ are analytical everywhere (see Appendix III), besides points k=0 and ∞, and at the limit $|k| \to \infty$ they tend to exp ($\pm ikr$).

2.3. The Properties of the One-Channel S-matrix for the Arbitrary Interactions Enclosed by the Centrifugal Barrier and a Potential, which is Decreasing More Rapidly then any Exponential Function

If at the external region, where $r \ge a$, there are the centrifugal barrier and an exponential potential of the type $V = V_0 \exp(-br)$, $V_0, b > 0$, then the functions $f_{l\pm}(k, r)$ have the simple poles in points $k = \mp i\,\frac{b}{2}m$ (m =1,2,…) and at the limit $|k| \to \infty$ they tend to exp ($\pm ikr$). Similar results can be obtained for the Eckart, Hulthén and Woods-Saxon potentials [13]. And if at the external region (with $r \ge a$) there are the centrifugal barrier and a potential of the type $V = V_0 P_n(r) \exp(-br)$, where $P_n(r)$ is an n-th-order polynomial and b>0, then the function $f_{l\pm}(-k,r)$ has poles of an order not higher than n+1 at the points $\mp ib/2$, $\mp ib$, $\mp 3ib/2$,..., and it is analytic at all other points of the complex plane.

Moreover, in the Appendix II the following theorem is proved:

For $f_0(-k,r)$ to have poles of the order not higher than (n_1+1) at the points $ib_1/2$, ib_1, $3ib_1/2$,..., not higher than (n_2+1) at the points $ib_2/2$, ib_2, $3ib_2/2$,..., not higher than (n_m+1) at the points $ib_m/2$, ib_m, $3ib_m/2$,..., it is necessary and sufficient that the corresponding potential would have a term $\sum_{n\ m} P_{n_m}(r) \exp(-b_m r)$. (This theorem had been firstly proved in [6]). Then, in the Appendix III, with adopting from [12] and [14] the appropriate integral equation, which allows computing $f_l(-k,r)$ from $f_0(-k,r)$, it is shown that in this case $f_l(-k,r)$ has the same isolate singular points as $f_0(-k,r)$.

Thus, coming from (18)-(21), one will also in this case obtain the results (23a,b), where in $\prod_m$ one must include the factors, corresponding to “redundant” poles $\frac{i}{2}bm'$ $(m' = 1, 2, ...)$ of the first order at the presence of an exponential potential tail $V = V_0 \exp(-br)$, and the factors of the type $\prod_{m''}(\frac{k_{m''}-k}{k_{m''}+k})^n$ corresponding to multiple "redundant" poles $\frac{i}{2}bm''$ $(m''=1,2,...)$ at the presence of the potential tail of the type $V_0 \sum_n P_n(r) \exp(-br)$.

2.4. The Properties of the One-Channel S-matrix for the Arbitrary Interactions Enclosed by the Centrifugal Barrier and a Potential, which is Decreasing as an Yukawa Potential

If at the external region, where $r \geq a$, there are the centrifugal barrier and a central Yukawa potential of the type $V = V_0 [(br)^{-1}\exp(-br)]$, V_0, $b^{-1} \sim a$, we can consider the analytic properties of $f_{l\pm}(k,r)$, following [11]. Firstly, according to ref.[14], for the case $l = 0$, one has

$$f_{0\pm}(k,r) = \left[1+\int_b^\infty db' S_\pm(b',k)\exp(-b'r)\right]\exp(\pm ikr), \quad (24)$$

where $S_\pm(b',k)$ is the solution of the equation

$$b(b\mp 2ik)S_\pm(b,k) = \rho + \int_b^\infty db' \rho S_\pm(b',k), \quad (25)$$

where $\rho = 2\mu V_0/\hbar^2 b^3 < 0$ and μ is the reduced mass. The solution of eq. (25) can be written as

$$S_\pm(b,k) = \rho\left\{b(b\mp 2ik)\left[1\mp\frac{i\rho}{2k}\ln\left(1\mp\frac{2ik}{b}\right)\right]^{-1}\right\} \quad (26)$$

and from eq.(24) one obtains

$$f_{0\pm}(k,r) = \left\{1+\rho\left[1\mp\frac{i\rho}{2k}\ln\left(1\mp\frac{2ik}{b}\right)\right]^{-1}\int_b^\infty db'\frac{\exp(-b'r)}{b'(b'\mp 2ik)}\right\}\exp(\pm ikr) \quad (24a)$$

In equation (26) there is a logarithmic singularity of $f_{0\pm}(k,r)$ at the point $k = k_\gamma = ib/2$. We consider the analytic properties of the factor

$$A_- = \left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}. \quad (27)$$

It is easy to verify that for complex values of k both Re k and Im k nonzero, the factor A_-^{-1} has no zeros, and A_- therefore has no poles. The same result is obtained for real k. Further, we set $k = ix$, where x is real, and rewrite eq.(27) in the form

$$A_- = \left[1+\frac{\rho}{2x}\ln\left(1-\frac{2x}{b}\right)\right]^{-1}.$$

In the case $2x/b > 1$, the factor A_-^{-1} has no zeros because the logarithm is complex and A_- therefore has no poles. In the case $0 \leq 2x/b < 1$ for $\rho < 0$, as in the case of the long-range part of the nuclear forces, A_- again has no poles. Finally, in the case $2x/b \leq 0$, poles can exist, but they must be located in the lower half of the complex k plane D^-.

In conclusion, we note that the factor A_- contains no additional singularities in D^+ except a branch point at $k_\gamma = ib/2$.

The treatment can be extended to higher angular momenta $l > 0$. The same logarithmic singularity at the point $k = k_\gamma = ib/2$ also appears in $f_{l-}(k,r)$. To show this, the following integral equation, which allows computing $f_{l-}(k,r)$ from $f_{0-}(k,r)$, can be used:

$$f_{l-}(k,r) = f_{0-}(k,r) + l(l+1)\int_r^\infty dr' G(k;r,r')(r')^{-2} f_{l-}(k,r'), \quad (28)$$

where $r > a$ and the Green's function $G(k; r,r')$ has the form [16]

$$G(k;r,r') = (2ik)^{-1}\begin{bmatrix} f_{0-}(k,r)f_{0+}(k,r') \\ -f_{0-}(k,r')f_{0+}(k,r)\end{bmatrix} = \left[f_{0+}(k,0)\right]^{-1}\begin{bmatrix} F(k,r)f(k,r') \\ -F(k,r')f(k,r)\end{bmatrix}. \quad (29)$$

Because the function $\Phi(k,r)$ is regular everywhere, the Green's function in eq.(29) has no singularity at the point k_γ. The solution $f_{l-}(k,r)$ of equation (28) for any value of l contains the same logarithmic singularity as the function $f_{0-}(k,r)$ at the point $k_\gamma = ib/2$.

We now consider equation (20) near the logarithmic singularities of $f_{l-}(k,r)$ in D^+. The contour γ_p can be chosen in the form shown in Figure 1. It consists of the almost closed circle γ_{acc} around $k_\gamma = ib/2$, with the small radius $\varepsilon \equiv (\varepsilon/b)b$ and the two infinite lines γ_{edge} along the edges of the cut with a much smaller distance between them given by $(\varepsilon/b)^\delta b$, $\delta > 2$, i.e. $\gamma_p = \gamma_{acc} + \gamma_{edge}$. We let γ_{12} denote the segment joining the lowest points 1 and 2 of the two lines, according to Figure 1. Letting γ_c denote the closed contour formed by the almost closed circle and the segment, we can write the identity γ_{acc}

$$\int_{\gamma_{acc}} = \oint_{\gamma_c} - \int_{\gamma_{12}} \underset{\varepsilon\to 0}{\longrightarrow} \oint_{\gamma_c}. \quad (30)$$

for the integrals in equation (20). Because the length of γ_{12} is $(\varepsilon/b)^\delta b$, the integrals over γ_{12} and over γ_{edge} vanish as $O(\varepsilon^{\delta-2})$ for $\varepsilon \to 0$. Therefore, only the contour integral over the closed circle γ_c centered at the point k_γ remains.

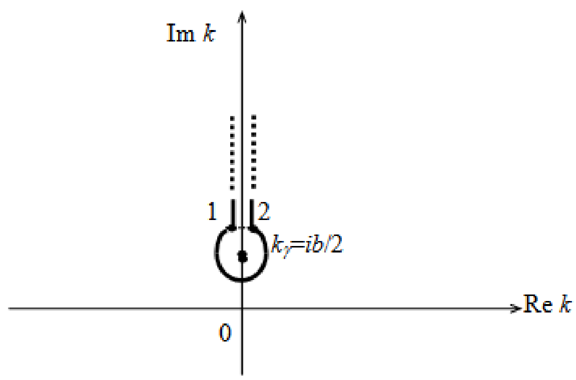

Figure 1. A form of contour γ_q

We consider the integral over γ_c in detail. The value of the integral is determined by the behavior of the integrand as the radius of the circle tends to zero. Therefore, we consider the limit

$$\lim_{k\to k_\gamma}\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_b^\infty \frac{\exp(ik-b')r}{b'(b'+2ik)}db'. \quad (31)$$

It is easy to show that the integral over the variable b' in equation (31) has a logarithmic divergence at k_γ that cancels when the corresponding factor vanishes, and the function $f_{0-}(k,r)$ therefore has no pole at k_γ. Explicitly evaluating limit of (30) shows that in the vicinity of k_γ, the function $f_{0-}(k,r)$ can be written as (see Appendix IV, with citation of reference [21])

$$f_{0-}(k,r)\to W(k,r)+[1+\frac{i\rho}{2k}\ln(1+\frac{2ik}{b})]^{-1}U(k,r), \quad (32)$$

where the functions W and U are analytic functions of k at the point k_γ and inside the small closed circle γ_c. Eq.(20) can therefore be rewritten as

$$\begin{aligned}&\oint_{\gamma_c} dkS_0(k)\,f_{0+}(k,r)\,f_{0+}(k,r')\\ &=\oint_{\gamma_c} dk\,f_{0+}(k,r)\,f_{0-}(k,r')\\ &=\oint_{\gamma_c} dkW(k,r)\,f_{0+}(k,r')\\ &+\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\oint_{\gamma_c} dkU(k,r)\,f_{0+}(k,r').\end{aligned} \quad (33)$$

Because each integral on the right-hand side vanishes, we can conclude that

$$\oint_{\gamma_c} dkS_0(k)\,f_{0+}(k,r)\,f_{0+}(k,r')=0. \quad (34)$$

It hence follows that $S_0(k)$ can contain at most a singular factor of the type

$$F=\frac{[1-\frac{i\rho}{2k}\ \ln(1-\frac{2ik}{b})]}{[1+\frac{i\rho}{2k}\ \ln(1+\frac{2ik}{b})]}, \quad (35)$$

connected with the analogous logarithmic branch points at $k=k_\gamma$. Of course, there may be no such factor, or $S_0(k)$ may contain other factors which have a logarithmic branch point but vanish at k_γ.

We consider few special cases where this factor actually occurs. If the interaction inside the sphere $r\le a$ is such that the scattering wave function in the external part ($r>a$) can be written in the form

$$\Psi_{\text{ext}}=\ f_{0-}(k,r)-S_0(k)\,f_{0+}(k,r) \quad (36)$$

and vanishes at some point $r=r_0>a$, then

$$S_0(k)=\frac{f_{0-}(k,r_0)}{f_{0+}(k,r_0)}. \quad (37)$$

It follows that S_0 *must contain* at most a factor F given by equation (35).

Another possibility occurs for a wide class of potentials [1], namely when the interaction inside the sphere $r\le a$ is such that the following continuity relations hold true

$$\Psi_{\text{int}}\equiv \text{const}\Phi(k,a)=\ f_{0-}(k,a)=S_0(k)\,f_{0+}(k,a), \quad (38)$$

$$\begin{aligned}&\frac{d\Psi_{\text{int}}}{dr}|_{r=a}=\text{const}\frac{d\Phi(k,r)}{dr}|_{r=a}\\ &=\frac{df_{0-}(k,r)}{dr}|_{r=a}=S_0\frac{df_{0+}(k,r)}{dr}|_{r=a}.\end{aligned}$$

In equation (38) the function $\Phi(k,r)$ is the regular solution of the radial Schroedinger equation inside the sphere $r\le a$ with the boundary condition $\Phi(k,0)=0$. This function is determined only by the interaction inside the sphere $r\le a$. Eqs (38) determine the const (constant) and the corresponding S-matrix

$$S_0(k)=\frac{\varphi(k,a)\dfrac{df_{0-}(k,a)}{da}-f_{0-}(k,a)\dfrac{d\varphi(k,a)}{da}}{f_{0+}(k,a)\dfrac{d\varphi(k,a)}{da}-\varphi(k,a)\dfrac{df_{0+}(k,a)}{da}}, \quad (39)$$

i.e. then $S_0(k)$ also must then contain factor (35).

According to (III,1)-(III,2), the same result is also obtained for $S_l(k)$ with $l>0$.

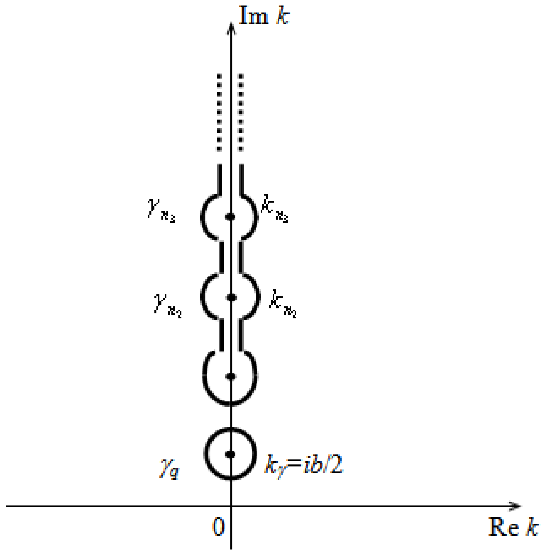

Figure 2. A disposition of the poles and the cut.for the Yukawian tail

Using the same approach, we can study a more general case where there is a covering of the cut (see Figure 2) by the poles (corresponding to bound states and/or “redundant” poles that appear when the potentials decrease exponentially). In this case it sufficiently to use equalities like (30):

$$\int_{\gamma_{accc}(k_n)} =\oint_{k_n} -\int_{\gamma 12}\ \lim_{\varepsilon\to 0}\oint_{k_n}$$

and simply repeat the reasoning proceeding (36). It is then easy to prove that for all these singularities, equation (22) continues to hold, and the results obtained previously concerning the singularities of $S_l(k)$ in D^+ also continue to hold.

Finally, the analytic continuation of the functions $S_l(k)$ to the lower half-plane D^- can be found as usually based on symmetry condition (2a),(3a) and the known general theorem about the analytic continuation. Thus, considering (23a,b) and (35), we finally obtain

$$S_l(\gamma,k)=\exp(-2i\alpha k)\ F(k)\prod_n\frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda\frac{k_\lambda-k}{k_\lambda+k}\prod_s\frac{k_s-k}{k_s+k}\prod_{s'}\frac{k_{s'}-k}{k_{s'}+k}, \quad (40a)$$

$$S_l(\gamma^*, k) = \exp(-2i\alpha k)$$
$$F(k)\prod_n \frac{k_{nl}^\bullet + k}{k_{nl}^\bullet - k}\prod_\lambda \frac{k_\lambda^\bullet - k}{k_\lambda + k}\prod_s \frac{k_s^\bullet - k}{k_s^\bullet + k}\prod_{s'} \frac{k_{s'}^\bullet - k}{k_{s'}^\bullet + k}, \quad (40b)$$

Conditions (10a,b) impose certain limitations on the distribution of zeros in D^+. We consider the following example as an illustration. Let in expression (23a) or (40a) for $S_l(\gamma,k)$ the factor $\frac{(\nu-k)(\nu'-k)}{(\nu+k)(\nu'+k)}$ does essentially dominate and, hence,

$$S_l(\gamma,k) \approx \frac{(\nu-k)(\nu'-k)}{(\nu+k)(\nu'+k)}, \quad (41)$$

according to (10a), $\nu = \alpha + i\beta$, $\nu' = -\alpha + \beta'$, $\alpha > 0$, $\beta > 0$, $\beta' > \beta$. Then the partial cross sections of scattering, absorption and both processes together are respectively equal to

$$\sigma_{scatt}^{(l)} = \frac{\pi}{k^2}(2l+1)\left|1 - S_l\right|^2 \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma_{scatt}/2)^2}{(E-E_r)^2 + \Gamma^2/4}, \quad (42)$$

$$\sigma_{absorp}^{(l)} = \frac{\pi}{k^2}(2l+1)[1 - \left|S_l\right|^2] \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma_{scatt}/2)(\Gamma_{absorp}/2)}{(E-E_r)^2 + \Gamma^2/4}, \quad (43)$$

$$\sigma_{tot}^{(l)} = \sigma_{scatt}^{(l)} + \sigma_{absorp}^{(l)} \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma_{scatt}/2)(\Gamma/2)}{(E-E_r)^2 + \Gamma^2/4}, \quad (44)$$

where $\Gamma_{scatt} = \frac{4\mu}{\hbar^2}k(\beta+\beta')$, $\Gamma_{absorp} = \frac{4\mu}{\hbar^2}\alpha(\beta'-\beta)$, $\Gamma = \Gamma_{scatt} + \Gamma_{absorp}$. The formulae (42)-(44) generalize the known results obtained in the model description of nuclear reactions (see, for instance [22]).

2.5. The Analytical Properties of the Non-Unitary *S*-matrix for any Non-Central and Parity-Violating Interactions, Enclosed by the Centrifugal Barrier and a Potential, which is Decreasing More Rapidly then any Exponential Function

We shall study this problem, following [10]. Let us suppose that the interaction between two colliding particles is such that the *S*-matrix is diagonal, as regards the total momentum *j*, does not depend on the total-momentum projection onto an arbitrary axis, and contains both diagonal and non-diagonal elements regarding the orbital momentum *l* with the mixed neighboring values $l, l' = j \pm \lambda$ of equal ($\lambda = 1$) or opposite ($\lambda = \frac{1}{2}$) parities. Particularly, there is a mixture of values $l, l' = l \pm 1$ (in the case of a tensor interaction admixture) or there is no mixture at all ($l = l' = j, \lambda = 0$), and there is a mixture $l, l' = j + \frac{1}{2}$ in the case of a parity-violating interaction like $\upsilon(r)\hat{\vec{\sigma}}\hat{\vec{p}} + \hat{\vec{\sigma}}\hat{\vec{p}}\,\upsilon(r)$, where *r* is the relative distance between two particles, $\hat{\vec{\sigma}}$ is the Pauli pseudo-vector matrix, $\hat{\vec{p}}$ is the momentum operator for the relative motion of a nucleon and a nucleus with spin 0. Of course, in the case of central interactions always $l = l' = j$ and $\lambda = 0$.

Thus, we consider the unknown non-central or parity-violating interaction inside the sphere $r < a$ surrounded by the centrifugal barrier and a central potential, which is decreasing more rapidly than any exponential function $V(r)$. Supposing that there is not only the scattering but also the absorption or the creation of particles, ii is natural as usually to put, generalizing (10a,b), the following conditions for the elements $S_{ll'}^j$ of the *S*-matrix

$$0 < \sum_{l'}\left|S_{ll'}^j(\gamma,k)\right|^2 \le 1, \quad (45a)$$

$$1 \le \sum_{l'}\left|S_{ll'}^j(\gamma,k)\right|^2 < \infty, \quad (45b)$$

and, generalizing (1a)-(3a), the extended "unitarity" condition

$$\sum_l S_{l_1 l}^j(\gamma,k)S_{ll_2}^{j\bullet}(\gamma^\bullet,k^\bullet) = \delta_{l_1 l_2} \quad (46)$$

and symmetry condition

$$S_{ll'}^{j\bullet}(\gamma^\bullet,k^\bullet) = (-1)^{l+l'}S_{ll'}^j(\gamma,-k) \quad (47)$$

(as regards the axis Im *k*), and also the condition of $S_{ll'}^j$ symmetry regarding the lower indices:

$$S_{ll'}^j(\gamma,k) = S_{l'l}^j(\gamma,k). \quad (48)$$

One easily check that the conditions (45)-(48) are automatically fulfilled in the case of central complex potential (5).

A system state for $r \ge a$ can be described by the wave functions

$$R_{ll'}^{j\bullet}(\gamma,k,r) = \frac{i}{2kr}\begin{bmatrix}\delta_{ll'}f_{l'-}(k,r)\exp(il'\pi/2) \\ -S_{ll'}^j(\gamma,k)f_{l'+}(k,r)\exp(-il'\pi/2)\end{bmatrix} \quad (49)$$

in the continuous part of the spectrum and

$$R_l^{j(n)}(\gamma,k_{nl},r) = (2\pi)^{-1/2}B_l(\gamma,k_{nl})f_{l+}(k_{nl},r)r^{-1} \quad (50)$$

in the discrete part of the spectrum.

Generalizing the completeness relation (11) for the unknown non-central or parity-violating interaction inside the sphere $r < a$, surrounded by the centrifugal barrier and a central potential, which is decreasing more rapidly then any exponential function $V(r)$, we can write

$$\frac{2}{\pi}\sum_l \int_0^\infty k^2dkR_{l'l}^{j(+)}(\gamma,k,r)R_{l''l}^{j(+)\bullet}(\gamma,k,r') + \sum_n R_{l'}^{j(n)}(\gamma,k_{nj},r)R_{l''}^{j(n)}(\gamma^\bullet,k_{nj},r') = \frac{\delta(r-r')}{r^2}\delta_{l'l''}. \quad (51)$$

Relation (51) is a generalization of the completeness condition for eigen functions of a class of non-hermitian Hamiltonians [17,18] for which all eigen values are simple (not multiple) and are situated outside the axis Re k.

As usually, in order that one be sure of the possibility of analytic continuation of $S^j_{ll'}(\gamma,k)$ in the complex plane k, one needs to put the limitation (12) in the external region ($r \geq a$).

Using the properties (49) for real k and conditions (46)-(48), one can rewrite (51) in the form

$$\begin{aligned}&\frac{1}{rr'}\int_C dk f_{l'-}(k,r) f_{l''+}(k,r')\delta_{l'l''}\\&-\frac{\exp[-i(l'+l'')\pi/2]}{rr'}\int_C dk S^j_{l'l''}(\gamma,k) f_{l'+}(k,r) f_{l''+}(k,r')\\&+\frac{1}{rr'}\sum_n B_{l'}(\gamma,k_{nj})B_{l''}(\gamma,k_{nj}) f_{l'+}(k_{nj},r) f_{l''+}(k_{nj},r')\\&=\frac{2\pi\delta(r-r')}{r^2}\delta_{l'l''},\end{aligned} \tag{52}$$

where the integration path C goes along the axis Re k from $-\infty$ to ∞, passing near the point k=0 (here $f_{l\pm}(k,r)$ have poles of l-th order) along semi-circle of the infinitely small radius in D^+.

We shall limit ourselves by the case when $f_{l\pm}(k, r)$ behavior as exp($\pm ikr$) in all the complex plane at $|k| \to \infty$. Then, shifting the integration contour into D^+, enclosing all the singularities by closed singularities (as near to them as we like) and using equalities

$$\begin{aligned}&\int_{\Gamma+} dk f_{l-}(k,r) f_{l+}(k,r') = \int_{\Gamma+} dk e^{ik(r'-r)}\\&=\int_{-\infty}^{\infty} dk e^{ik(r'-r)} = 2\pi\delta(r-r'),\end{aligned} \tag{53}$$

$$\begin{aligned}&\int_{\Gamma+} dk S^j_{l'l''}(\gamma,k) f_{l'+}(k,r) f_{l''+}(k,r')\\&=\int_{\Gamma+} dk S^j_{l'l''}(\gamma,k) e^{ik(r+r')},\end{aligned} \tag{54}$$

we obtain

$$\begin{aligned}&\delta_{ll'}\sum_m \oint_{k_m} dk f_{l-}(k,r) f_{l'+}(k,r')\\&+\delta_{ll'}\sum_p \oint_{k_p} dk f_{l-}(k,r) f_{l'+}(k,r)\\&-\exp[-i(l+l')\pi/2]\left[\begin{aligned}&\sum_\nu \oint_{k_\nu} dk S^j_{ll'}(\gamma,k) f_{l+}(k,r) f_{l'+}(k,r')\\&+\sum_q \oint_{\gamma_q} dk S^j_{ll'}(\gamma,k) f_{l+}(k,r) f_{l'+}(k,r')\\&+\int_{\Gamma^+} dk S^j_{ll'}(\gamma,k) f_{l+}(k,r) f_{l'+}(k,r')]\end{aligned}\right]\\&+\sum_n B_l(\gamma,k_{nj})B_{l'}(\gamma,k_{nj}) f_{l+}(k_{nj},r) f_{l''+}(k_{nj},r') = 0\end{aligned} \tag{55}$$

where $\int_{\Gamma^+}$ is the integral over the infinitely large semi-circle above the real axis, $\oint_{k_n}$ is the integral over an infinitesimal circle around an isolate singular point, $\oint_{\gamma_p}$ is the integral over a contour which envelops a non-isolate singularity (for instance, over the edges of the cut conducted for a branch point). Since all these contours are independent, equality (55) is equivalent to the following system of equalities:

$$\begin{aligned}&\exp\left[-i(l+l')\pi/2\right]\oint_{k_{nj}} dk S^j_{ll''}(\gamma,k) f_{l+}(k,r) f_{l'+}(k,r')\\&= B_l B_{l'} f_{l+}(k_{nj},r) f_{l'+}(k_{nj},r'),\end{aligned} \tag{56}$$

$$\begin{aligned}&\exp\left[-i(l+l')\pi/2\right]\oint_{k_\nu} dk S^j_{ll''}(\gamma,k) f_{l+}(k,r) f_{l'+}(k,r')\\&=\oint_{k_\nu} dk f_{l-}(k,r) f_{l''+}(k,r')\delta_{ll'},\end{aligned} \tag{57}$$

$$\begin{aligned}&\exp\left[-i(l+l')\pi/2\right]\oint_{\gamma_p} dk S^j_{ll'}(\gamma,k) f_{l+}(k,r) f_{l'+}(k,r')\\&=\oint_{\gamma_p} dk f_{l+}(k,r) f_{l'-}(k,r')\delta_{l'l},\end{aligned} \tag{58}$$

$$\int_{\Gamma} dk S^j_{ll'}(\gamma,k) e^{ik(r+r')} = 0. \tag{59}$$

Quite similar to the previous cases of the central unknown interactions inside the sphere $r \leq a$, it follows from equation (56) that all the elements $S^j_{ll'}(\gamma,k)$ have in D^+ poles of the first order (for γ=Reγ they are situated on the half-axis Im $k > 0$ and correspond to the bound states) with the residues

$$\exp\left[-i(l+l')\pi/2\right](2\pi i)^{-1}B_l B_{l'}.$$

Directly from equation (57) it follows that the diagonal elements $S^j_{ll}(\gamma,k)$ must have in D^+ additional isolated singularities which coincide with those isolated singularities $f_{l-}(k,r)$ in D^+, near which equation (22) is valid also here, with the function $D_m(k)$ which also does not depend on r and has an isolate singular point k_m. Similarly, it follows from eq.(58) that the diagonal elements $S^j_{ll}(\gamma,k)$ must have in D^+ branch points and non-isolated singularities which coincide with the appropriate singularities of $f_{l-}(k,r)$ in D^+.

As it was previously made for the unknown central interactions inside the small sphere of radius a, we consider several cases of potential tails in the external region with $r \geq a$.

When there is only a centrifugal barrier there, then equations (8a) and (9a) are valid and according to equations (47)-(50) all the functions $\tilde{S}^j_{ll'}(\gamma,k) = S^j_{ll''}(\gamma,k)\exp(2ika)$ are regular and limited everywhere in D^+ except the isolated points. No poles can appear on the half-axis Re r because of the conditions (36a,b) and also because of the finite values of $R^{j(+)}_{l'l}(\gamma,k,r)$ at the point $k = 0$.

It is easy to conclude from the finite value of $R^{j(+)}_{l'l}(\gamma,k,r)$ for $k \to 0$ by recalling the known behavior of $f_{l\pm}(k,r)$ and of $h^{(1,2)}_l(kr) = j_l(kr) \pm in_l(kr)$ at the point $k \to 0$ that

$$S^j_{ll'}(\gamma,k) \underset{k\to 0}{\to} \delta_{ll'}[1+O(k^{l_>+1})]+[1-\delta_{ll'}]O(k^{l_>+1}), \quad (60)$$

where $l_>$ is the larger of the two numbers l and l'.

One can determine the analytic continuation of the functions $S^j_{ll'}(\gamma,k)$ in D^- as usual on the basis of the symmetry condition (47) and the general theorem on the analytic continuation.

Solving system (46) with the use of (47) and (48) relatively to $S^j_{ll'}(\gamma,-k)$, we obtain

$$\begin{aligned} S^j_{ll}(\gamma,-k) &= S^j_{l'l'}(\gamma,k)/d_j(\gamma,k), \\ S^j_{ll'}(\gamma,-k) &= -S^j_{ll'}(\gamma,k)d_j(\gamma,k) \end{aligned} \quad (61)$$

(with $l \neq l'$ and $d_j(\gamma,k) = S^j_{ll}(\gamma,k)\, S^j_{l'l'}(\gamma,k) - [S^j_{ll'}(\gamma,k)]^2$), from which we can see that in D^- all the elements $S^j_{ll'}(\gamma,k)$ have the same poles k_{nj} (on the half-axis Im $k < 0$), k_s (in the 4-th quadrant), $k_{s'}$ (in the 3-rd quadrant), which correspond to the zeros of the function d_j in D^+, and also the zeros $-k_{nj}$, which correspond to the poles k_{nj} in D^+. Besides that, every diagonal element $S^j_{ll'}(\gamma,k)$ can have additional poles on the half-axis Re $k < 0$ (k_μ), in the 4-th quadrant (k_σ) and in the 3-rd quadrant ($k_{\sigma'}$), which correspond to the zeros $-k_\mu$, $-k_\sigma$ and $-k_{\sigma'}$ of two functions $S^j_{ll}(\gamma,k)$ and $S^j_{ll'}(\gamma,k)$ in D^+. Moreover, one can conclude from the formulae (48) that the zeros k_p (on the axis Im k), k_r (on the axis Re k), k_t (in the 1-st and 4-th quadrants) and $k_{t'}$ (in the 2-nd and 3-rd quadrant) of the diagonal element $S^j_{ll}(\gamma,k)$ correspond to the zeros $-k_p$, $-k_r$, $-k_t$ and $-k_{t'}$ of the second diagonal element $S^j_{l'l'}(\gamma,k)$, $l' \neq l$, and also that the zeros of the non-diagonal element $S^j_{ll'}(\gamma,k)$, $l' \neq l$, can appear only in pairs $\pm k_\pi$ (on the half-axis Im k), $\pm k_\rho$ (on the half-axis Re k), $\pm k_t$ (in the rest of the complex plane). Evidently, the last assertion is true for those zeros which are not general zeros of all the elements $S^j_{ll'}(\gamma,k)$.

In the considered case, $S^j_{ll'}(\gamma,k)$ cannot have any singular points in D^- besides poles, since there will be a singular point $-k_x$ of $S^j_{l'l}(\gamma,k)$ in D^+ for every singular point k_x of $S^j_{l'l'}(\gamma,k)$ in D^- because of (61), but this is in contradiction with our previous result on the analyticity of $S^j_{l'l'}(\gamma,k)$ in D^+. Thus all the elements $S^j_{l'l'}(\gamma,k)$ are meromorphic functions and consequently they can be represented in the form of a ratio of two integer analytic functions:

$$\tilde{S}^j_{ll'}(\gamma,k) = A_{ll'}(\gamma,k)\exp[g_{ll'}(k)]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1-k/k_p)(1-k/k_r)(1-k/k_t)(1-k/k_{t'})}{(1-k/k_m)(1-k/k_s)(1-k/k_{s'})}, \quad (62)$$

where $A_{ll'} = \delta_{ll'} + (1-\delta_{ll'})Ck^{l_>+1}$, $C = i$ Im C is a constant, the topology of the poles k_{nj}, k_m, k_s, $k_{s'}$ and of the zeros $-k_{nj}$, k_p, k_r, k_t, $k_{t'}$ was specified before, $g_{ll'}(k) = u_{ll'}(k) + i\theta_{ll'}(k)$.

The real function $u_{ll'}(k)$ must be non-positive in D^+ because of the analyticity of $\tilde{S}^j_{ll'}(\gamma,k)$ (and, consequently, the convergence of the infinite products of (62) in D^+) and must be non-negative in D^- owing to (61). Then the Cauchy-Riemann conditions

$$0 \geq \partial u_{ll'}/\partial\,\mathrm{Im}\,k = -\partial\theta_{ll'}/\partial k\left(\mathrm{Im}k=0\right)$$

must be satisfied on the real axis. From these conditions one can conclude that the function $\theta_{ll'}(k)$ is monotonically increasing and reaches a real value not more than once. Then $g_{ll'}(k) = = u_{ll'}(k) + i\theta_{ll'}(k)$ reaches any imaginary value not more than once, and hence must be a linear function: $g_{ll'}(k) = 2i\beta_{ll'}(k) + \gamma_{ll'}(k)$. Evidently $\beta_{ll'}(k) \geq 0$ and, since $S^j_{l'l'}(\gamma,0) = \delta_{ll'}$, $\gamma_{ll'} = 0$. Thus, considering that $\beta_{ll'} = (\beta_{ll} + \beta_{l'l'})/2$ owing to (52)61, we obtain the following final expression

$$S^j_{ll'}(\gamma,k) = A_{ll'}(\gamma,k)\exp[-i(\alpha_l+\alpha_{l'}]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1-k/k_p)(1-k/k_r)(1-k/k_t)(1-k/k_{t'})}{(1-k/k_m)(1-k/k_s)(1-k/k_{s'})}, \quad (63)$$

where $\alpha_l = a - \beta_l \leq a$. Considering (63) and on the basis of (46),(47), we can write

$$S^j_{ll'}(\gamma^\bullet,k) = A_{ll'}(\gamma^\bullet,k)\exp[-i(\alpha_l+\alpha_{l'}]\prod_n \frac{1+k/k^\bullet_{nl}}{1-k/k^\bullet_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1-k/k^\bullet_p)(1-k/k^\bullet_r)(1-k/k^\bullet_t)(1-k/k^\bullet_{t'})}{(1-k/k^\bullet_m)(1-k/k^\bullet_s)(1-k/k^\bullet_{s'})} \quad (63a)$$

In the case of $\gamma = \mathrm{Re}\gamma$, the zeros appear in the pairs $\pm k_r$ and $k_{s'} = -k_s^*$, $k_{t'} = -k_t^*$ because of the symmetry condition (47) and then

$$S^j_{ll'}(\mathrm{Re}\,\gamma,k) = A_{ll'}\exp[-i(\alpha_l+\alpha_{l'}]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1-k/k_p)(1-k/k_r)^2(1-k/k_t)(1-k/k^\bullet_t)}{(1-k/k_m)(1-k/k_s)(1-k/k^\bullet_s)} \quad (64)$$

It may appear a possible physical phenomenon of sharp enhancement of $S^j_{ll'}(\gamma,k)$ ($l' \neq l$) in comparison with $S^j_{ll}(\gamma,k)$ near an isolated resonance, noted in [10,11] and described in Appendix V.

When $l = l'$, $\lambda = 0$ (particularly, $\gamma = \gamma_c$),

$$k_p = -k_m, k_t = -k_s, k_{t'} = -k_{s'},$$

the zeros k_r are absent and then

$$S_l(\gamma_c, k) \equiv S_{ll}^l(\gamma_c, k) = \exp[-2i\alpha_l k]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s,s'} \frac{(1-k/k_p)(1-k/k_r)(1-k/k_t)(1-k/k_{t'})}{(1-k/k_m)(1-k/k_s)(1-k/k_{s'})}, \quad (65)$$

that corresponds to results [7]. In the particular case in which $l = l' = j$ and $\gamma = \gamma_c$ we have also $k_{s'} = = -k_s^*$ and hence

$$S_l(\mathrm{Re}\,\gamma_c, k) = \exp[-i2\alpha_l k]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s} \frac{(1+k/k_m)(1+k/k_s)(1-k/k_s^\bullet)}{(1-k/k_m)(1-k/k_s)(1+k/k_s^\bullet)}, \quad (66)$$

that corresponds to the results [14].

If for $r > a$ there is a centrifugal barrier and a potential decreasing more rapidly than any exponential function, results (63) and (63a) are valid because in that case $f_{l\pm}(k, r)$ are also analytic in all the plane k except the points $k = 0$ and $k = \infty$ and for $|k| \to \infty$ have the limit $\exp[\pm ikr]$ in all directions.

If for $r > a$ there is also an exponential potential of the type $V = V_0 \exp(-br)$, $V_0, b > 0$, then the functions $f_{l\pm}(k, r)$ have the simple poles in points $k = \mp i\frac{b}{2}m$ (m =1,2,…) and at the limit $|k| \to \infty$ they tend to $\exp(\pm ikr)$. Similar results can be obtained for the Eckart, Hulthén and Woods-Saxon potentials [14]. On the basis of (47)-(50) we obtain also in this case results (63), (63a), where in $\prod_n$ the factor $\prod_\nu \frac{(1+k/\kappa_\nu)}{(1-k/\kappa_\nu)}$ ($\kappa_\nu = ib_\nu/2$ (ν = 1,2,…) being the “redundant” poles which do not correspond to bound states) must be included for the diagonal elements $S_{ll}^j(\gamma, k)$.

And if at the external region (with $r \ge a$) there are the centrifugal barrier and a potential of the type $V = V_0 P_n(r) \exp(-br)$, where $P_n(r)$ is an nth-order polynomial and $b>0$, then the function $f_{l\pm}(-k,r)$ has poles of an order not higher than $n+1$ at the points $\mp ib/2, \mp ib, \mp 3ib/2, \ldots$, and it is analytic at all other points of the complex plane. Thus, coming from (18)-(21), one will also in this case obtain the results (23a,b), where in $\prod_m$ one must include the factors, corresponding to “redundant” poles $\frac{i}{2}bm'$ $(m' = 1,2,\ldots)$ of the first order at the presence of an exponential potential tail $V = V_0 \exp(-br)$, and the factors of the type $\prod_{m''}(\frac{k_{m''}-k}{k_{m''}+k})^n$, corresponding to multiple “redundant” poles $\frac{i}{2}bm''$ $(m'' = 1,2,\ldots)$ at the presence of the potential tail of the type $V_0\sum_n P_n(r)\exp(-br)$. In this case the factor with the “redundant” poles in the diagonal elements $S_{ll}^j(\gamma, k)$, the number and the degree of which depend on the type of the potential, also appears [7,14].

If at the external region, where $r \ge a$, there are the centrifugal barrier and a central Yukawa potential of the type $V = V_0[(br)^{-1}\exp(-br)]$, $V_0, b^{-1} \sim a$, then the functions $f_{l-}(k,r)$ must have the factor $[1+\frac{i\rho}{2k}\ln(1+\frac{2ik}{b})]^{-1}$ and the diagonal elements $S_{ll}^j(\gamma, k)$ must have the factor $F(k)$ with the logarithmic branch point at $k_\gamma = ib/2$ [11].

2.6. The Simonius Representation of the Multi-Channel *S*-matrix any Interactions Inside the Sphere *r > a*

Sometimes there is rather often used the Simonius parametrization of the *S*-matrix in the energy representation are used [23]:

$$\hat{S}^{(\alpha)} = \hat{U}^{(\alpha)}\prod_\nu (1 - \frac{i\Gamma_\nu^{(\alpha)}\hat{P}_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\varepsilon)}/2})\hat{U}^{(\alpha)T},$$
$$\hat{U}^{(\alpha)}\hat{U}^{(\alpha)*} = 1, \quad (67)$$
$$\hat{P}_\nu^{(\alpha)} = \hat{P}_\nu^{(\alpha)*} = \hat{P}_\nu^{(\alpha)2}, \mathrm{Trace}\hat{P}_\nu^{(\alpha)} = 1.$$

Index α in (67) signifies the set of quantum numbers of conserved quantities (usually α={J,Π}, where J and Π are the quantum numbers of the total momentum (spin) and parity of the system). In this parametrization, resonances are described by the general poles of the all elements of the *S*-matrix. According to the causality these poles must be located in the lower half-plane of the complex plane *E* (in order to describe the decays of the resonance states).

The Simonius parametrization (67) was obtained in [23], coming from the general principles of unitarity, meromorphy and T-invariance of the *S*-matrix. With this, in [23] it was noted that there is a practical difficulty of the explicit considering of T-invariance in the general case of non-symmetric and non-commuting with each other projectors $\hat{P}_\nu^{(\alpha)}$. This parametrization is the mostly convenient for overlapping and strongly overlapping resonances and was below utilized for revealing the time resonances (explosions) of compounds clots and nuclei in high-energy nuclear reactions at the range of strongly overlapping energy resonances.

It was shown in [24] that when the projectors $\hat{P}_\nu^{(\alpha)}$ do not depend on the values of any other resonance parameters ($E_\lambda^{(\alpha)}$ and $\Gamma_\nu^{(\varepsilon)}$), then $\hat{S}^{(\alpha)} = \hat{S}^{(\alpha)T}$. Really, in that case one can rewrite the resonance part $\hat{S}_{res}^{(\alpha)} \equiv \prod_{\nu=1}^{\Lambda^{(\alpha)}}(1 - \frac{i\Gamma_\nu^{(\alpha)}P_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2})$ in the form of a sum

$$\hat{S}_{res}^{(\alpha)} = 1 - i\sum_\nu \frac{\Gamma_\nu^{(\alpha)}P_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2} - \sum_{\nu'>\nu}\frac{\Gamma_\nu^{(\alpha)}\Gamma_{\nu'}^{(\alpha)}\hat{P}_\nu^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}}{(E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2)(E - E_{\nu'}^{(\alpha)} + i\Gamma_{\nu'}^{(\alpha)}/2)} + \ldots \quad (68)$$

which can be transformed to the expansion:

$$\hat{S}_{res}{}^{(\alpha)} = 1 - i\sum_{\nu} \frac{iG_{\nu}^{(\alpha)}}{E - E_{\nu}^{(\alpha)} + i\Gamma_{\nu}^{(\alpha)}/2}, \qquad (68a)$$

$$G_{\nu}^{(\alpha)} = \Gamma_{\nu}^{(\alpha)} P_{\nu}^{(\alpha)}$$

$$-i\sum_{\nu'>\nu} \frac{\Gamma_{\nu}^{(\alpha)}\Gamma_{\nu'}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}}{E_{\nu}^{(\alpha)} - E_{\nu'}^{(\alpha)} + i(\Gamma_{\nu}^{(\alpha)} - \Gamma_{\nu'}^{(\alpha)})/2}$$

$$-i\sum_{\nu''<\nu} \frac{\Gamma_{\nu}^{(\alpha)}\Gamma_{\nu''}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu''}^{(\alpha)}}{E_{\nu}^{(\alpha)} - E_{\nu''}^{(\alpha)} + i(\Gamma_{\nu}^{(\alpha)} - \Gamma_{\nu''}^{(\alpha)})/2} + \ldots$$

Taking into account (68a) and the T-invariance of the expression (68) for $\hat{S}^{(\alpha)}$, one can write

$$\hat{S}_{res}{}^{(\alpha)} = \hat{S}_{res}{}^{(\alpha)T} \qquad (69)$$

and then one can further rewrite (69) in the following form:

$$G_{\nu}^{(\alpha)} = G_{\nu}^{(\alpha)T}, \ \nu = 1, 2, \ldots, \Lambda^{(a)}. \qquad (70)$$

Relations (70) are in general too bulky as correlations between the matrices $\hat{P}_{\nu}^{(\alpha)}$ with different ν. But if the $\hat{P}_{\nu}^{(\alpha)}$ do not depend on the values of $E_{\lambda}^{(\alpha)}$ and $\Gamma_{\nu}^{(\varepsilon)}$, then the relations

$$\hat{P}_{\nu}^{(\alpha)} = \hat{P}_{\nu}^{(\alpha)T} \qquad (71)$$

$$\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)} = \hat{P}_{\nu'}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)}, \nu, \nu' = 1, 2, \ldots, \Lambda^{(a)}. \qquad (72)$$

(i.e. the matrices $\hat{P}_{\nu}^{(\alpha)}$ will be symmetric and commute with each other) are the direct consequences of (67). By the way, such simplification (the independence of $\hat{P}_{\nu}^{(\alpha)}$ from any other resonance parameters) is justified at least when $\Lambda^{(\alpha)}$ and the number N of open channels are very large. And then it follows from the properties (69) and $\hat{P}_{\nu}^{(\alpha)} = \hat{P}_{\nu}^{(\alpha)*} = \hat{P}_{\nu}^{(\alpha)2}$, Trace $\hat{P}_{\nu}^{(\alpha)}$ =1 (from (67)) that the $\hat{P}_{\nu}^{(\alpha)}$ are real, i.e.

$$\hat{P}_{\nu}^{(\alpha)} = \hat{P}_{\nu}^{(\alpha)\bullet}. \qquad (73)$$

2.7. Duration of Resonance Processes of Many-Channel Scattering

If to exclude the small threshold regions with their characteristic, then it is possible to utilize the Simonius representation [23] for the many-channel S-matrix:

$$\hat{S}^{(J)}(E) = \hat{U}^{(J)}\prod_{\nu=1}^{N}\left(1 - \frac{i\Gamma_{\nu}^{(J)}}{E - E_{\nu}^{(J)} + i\Gamma_{\nu}^{(J)}/2}\right)\hat{U}^{(J)T}, \quad (67)$$

where the unitary matrix $\hat{U}^{(J)}$ and the projection matrixes $\hat{P}_{\nu}^{(J)}$ ($\hat{P}_{\nu}^{(J)} = \hat{P}_{\nu}^{(J)\bullet} = P_{\nu}^{(J)2}$, Tr $\hat{P}_{\nu}^{(J)}$ =1) are practically do not depend on energy, $(\hat{U}^{(J)T})_{ij} = U_{ij}^{(J)}$ (i,j=1,2,…,n), n is the number of the open channels, $\tilde{S}^{(J)} = \hat{U}^{(J)}\hat{U}^{(J)T}$ is a symmetric background (non-resonance) S-matrix, $\tilde{S}_{ij}^{(J)} = \tilde{S}_{ji}^{(J)T}$, $\sum_{k=1}^{n}\tilde{S}_{ik}^{(J)}\tilde{S}_{jk}^{(J)\bullet} = \delta_{ij}$. Representation (67) is suitable, precisely speaking, only for the two-channel (binary) reactions. It has such preference that in it, in difference from the representations with the additive sets of the resonance terms, is evidently considered the propriety of the unitarity

$$\sum_{k=1}^{n} S_{ik}^{(J)} S_{jk}^{(J)\bullet} = \delta_{ij}. \qquad (74)$$

Utilizing hermitian matrix

$$\hat{Q}^{(J)}(E) = i\hbar\hat{S}^{(J)}(E)\frac{d\hat{S}^{(J)\bullet}(E)}{dE},$$

introduced in [25], it is not difficult to show, considering (67), the validity of the following relation:

$$\mathrm{Tr}Q^{(J)})E) = \sum_{\nu=1}^{N}\frac{\Gamma_{\nu}^{(J)}}{(E - E_{\nu}^{(J)})^2 + (\Gamma_{\nu}^{(J)})^2/4}, \qquad (75)$$

which does not depend on matrixes $\hat{U}^{(J)}$ and $\hat{P}_{\nu}^{(J)}$, i.e. from the smooth (practically constant) background, connected with the potential scattering and the direct reactions. We can average (75) inside the region, in which the resonances are situated, by the procedure < >. Then, supposing that the interval ΔE contains the set of resonances with $\Delta E >> \Gamma_J >> D_J$, where Γ_J and D_J are the mean width and the mean distance between the J-resonances, then we obtain

$$\mathrm{Tr} < \hat{Q}^{(J)}(E) >= \mathcal{T}_J, \qquad (76)$$

where $\mathcal{T}_J = 2\pi\hbar/D_J$ is the time of the Poincare cycle (more strictly, the time of the Poincare cycle for such system with equidistant distribution of purely discrete levels, for which $\left|E_{\nu+1}^{(J)} - E_{\nu}^{(J)}\right| = D_J$ and Γ_J =0). Utilized the transformations

$$i\hbar < S_{ij}^{(J)} dS_{ij}^{(J)\bullet}/dE >$$
$$= \hbar < \left|S_{ij}^{(J)}\right|^2 >< \Delta\tau_{ij}^{(J)} >$$
$$+ i\hbar < d\left|S_{ij}^{(J)}\right|^2/dE > /2, \ i \neq j,$$

$$i\hbar < S_{ii}^{(J)}(dS_{ii}^{(J)\bullet}/dE) >$$
$$= \hbar < \left|1 - S_{ii}^{(J)}\right|^2 >< \Delta\tau_{ii}^{(J)} >$$
$$+ \hbar\,\mathrm{Im} < dS_{ii}^{(J)}/dE > + i\hbar < d\left|S_{ii}^{(J)}\right|^2/dE > /2,$$

where it is possible to neglect by the quantity Im $< dS_{ii}^{(J)}/dE >$ for sufficiently large ΔE in the approximation of the random phases, and the equality $< d\left|S_{ij}^{(J)}\right|^2/dE >= 0$ is directly follows from the unitarity (74), we obtain from (76) the following two sum rules for $< \Delta\tau_{ij}^{(J)} >$ and $<\Delta\tau_{i}^{J}>$:

$$\sum_{i,j} <\Delta\tau_{ij}^{(J)}><\left|S_{ij}^{(J)}-\delta_{ij}\right|^2>=\mathcal{T}_J, \qquad (77a)$$

$$\sum_{i} <\Delta\tau_{i}^{(J)}>[1-\mathrm{Re}<S_{ii}^{(J)}>]=\mathcal{T}_J/2. \qquad (77b)$$

If we assume the equal durations in all the channels, i.e. $<\Delta\tau_i^{(J)}>=<\Delta\tau_{ji}^{(J)}>=<\Delta\tau^{(J)}>$ (i,j=1,2,…,n), then in the approximation $<S_{ii}^{(J)}>\approx 0$ (that, as will be seen later, can be take place for $\Gamma_J >> nD_J/2\pi$ in so called approximation of the equivalent entrance channels [26], we obtain:

$$<\Delta\tau^{(J)}>\cong \mathcal{T}_J/2n. \qquad (78)$$

If $\sum_i \mathrm{Re}<S_{ii}^{(J)}>=n-1$, that is possible in so called Newton case of the total correlation between the decay amplitudes of all resonances in the approximation of equal projection matrixes $\hat{P}_\nu^{(J)}=\hat{P}_J$ (ν=1,2,…,N) and $\hat{U}^{(J)}=1$, when the unitary S-matrix has the form $\hat{S}^{(J)}=\hat{1}+[\exp(2i\delta_J-1]\hat{P}_J$, then

$$<\Delta\tau^{(J)}>=\mathcal{T}_J/2. \qquad (79)$$

The result (79) was obtained in [27].

If $\mathrm{Re}<S_{ii}^{(J)}>=1-\pi\Gamma_J/nD_J$, which, as will be seen later, can take place for $\Gamma_J << nD_J/2\pi$, then $<\Delta\tau^{(J)}>=\hbar/\Gamma_J$, as in the case of one isolated resonance with the total width $\Gamma_J << \Delta E$.

If we start from the direct definition for $<\Delta\tau_{ij}^{(J)}>$ for usual simplifications of central and spin-less particles, one can use partial time delays $<\Delta\tau^{(J)}{}_{fi}(E)>$, defined as [28].

$$<\Delta\tau^{(J)}{}_{fi}(E)> = \hbar<\left|T^{(J)}{}_{fi}(E)\right|^2>\partial\arg T^{(J)}{}_{fi}(E)/\partial e_i>/<\left|T^{(J)}{}_{fi}(E)\right|^2>$$

(formula (4) in [37]).

Then in the approximation $\hat{P}_\nu^{(J)}=\hat{P}_J$ (ν = 1,2,…,N), it is not difficult to show that under condition $\Delta E >> \Gamma_J >> D_J$ (see also [28])

$$\hat{S}^{(J)}=\tilde{S}^{(J)}-\hat{\alpha}^{(J)}+\hat{\alpha}^{(J)}\prod_\nu \frac{E-E_\nu^{(J)}-i\Gamma_\nu^{(J)}}{E-E_\nu^{(J)}+i\Gamma_\nu^{(J)}}$$

and

$$<\Delta\tau_{ij}^{(J)}>=\left|\alpha_{ij}^{(J)}\right|^2\mathcal{T}_J\Big/[\left|\tilde{S}_{ij}^{(J)}-\alpha_{ij}^{(J)}\right|^2], i\neq j; \quad (80a)$$

$$<\Delta\tau_{ii}^{(J)}>=\left|\alpha_{ii}^{(J)}\right|^2\mathcal{T}_J/[\left|1-\tilde{S}_{ii}^{(J)}+\alpha_{ii}^{(J)}\right|^2], \quad (80b)$$

where $\alpha_{ij}^{(J)}=(\hat{U}^{(J)}\hat{P}_J\hat{U}^{(J)T})_{ij}$ with $\sum_{i,j}\left|\alpha_{ij}^{(J)}\right|^2=1$. It is directly follows from the unitarity $\hat{U}^{(J)}$.

In [24] (see also [28]) it had been derived the third sum rule, connecting mean time delay for the compound nucleus, dispersion of time-delays distributions for compound nuclei with the mean resonance density ρ_J and Γ_J. It extends the possibilities of the study main properties of the compound nuclei (the level density $\rho^{(JS\Pi)}$ and the mean total resonance width $\Gamma^{(JS\Pi)}$) and the decay laws of the compound nucleus in the range of un-resolved resonances.

2.8. The Manifestation of the Time Resonances (Explosions) of Compounds Clots and Nuclei in High-Energy Nuclear Reactions at the range of Strongly Overlapping Energy Resonances

Introduction. In the wide energy region of the bombarding particles more 1-10 GeV/nucleon (see, for instance, [29-35]) and for the great their number (from p till ^{20}Ne), number of targets and of the registered final fragments there are observed the exponentially decreasing inclusive (and some times non inclusive) energy spectra without structure. For more heavy bombarding particles such phenomena are observed also smaller energies (see, for instance, [36]). For the analysis of such reactions with heavy ions with energies till 1GeV/nucleon one can use in a certain degree the fireball model and also the model of intra-nuclear cascade [37] and the model of nuclear fluid [38] works for more high energies in the supposition of the high-dense collision-complexes formation. Between the difficulties of the fireball models there is a problem, why even for high excitations (more than 100 MeV/nucleon) there is formed the statistical equilibrium. In [39] there was proposed other model of “time compound nucleus” for the alternative explanation of high-energy nuclear reactions. This model utilized the preliminary results of eigen states of time operator in the Hamiltonian approach [40]. It was based on the introduction of the formal similarity between the meta-stable states with the eigen complex energies as the eigen states of the Schroedinger equation and the correspondent Fourier transformations with complex eigen values for the equation with time operator, canonically conjugate to the Hamiltonian. This model was only the initial step to the time-dependent approach and was not sufficiently justified.

We proposed a new version of the time-evolution approach, starting not only from the principal ideas [41,42] but also from the known correspondence between the exponential decreasing of behavior of any quantity in any (time or energy) representation and the Lorentzian behavior of its Fourier transformation in the canonically conjugate (i.e. energy or time) representation and then utilizing the results, obtained in [24,28] for the properties of compound nuclei in the range of the non-resolved strongly overlapped energy resonances. Here we introduce concretely the phenomenon of time resonance and it is explained the similarity between energy and time resonances. And also there are analyzed the energy and time properties of compound nuclei which are connected with the explosions of time resonances in the evolution decay of final particles.

The *theoretical origin of time resonances (explosions).* Our theoretical approach is based on [28,42,43]. So far let us choose the reaction amplitude $f_{\alpha\beta}(E)$ and T-matrix (E) in such forms

$$f_{\alpha\beta}\left(E\right)=C_{\alpha\beta}{}^{n}\exp(-E\tau_n/2\hbar+\ iEt_n/\hbar) \qquad (81)$$

and

$$=\exp(-E\tau_n/2+\ \mathrm{iEt_n}/\hbar), \qquad (81a)$$

Here in the certain energy region $E_{min}< E < \infty$, where τ_n and t_n are constants (with the dimension of time), τ_n and t_n define the exponential dependence on energy for the corresponding cross section and the linear dependence from energy for the amplitude phase, respectively. $\tilde{T}^n_{\alpha\beta}$ is the constant or the very smooth function (inside ΔE) on energy E of the final particle. A resonant structure of $\tilde{T}^n_{\alpha\beta}$ we so far do not taken evidently into account, supposing it the totally averaged in the limits of the energy spread (or resolution) ΔE, supposing that $\Delta E<< 2\hbar/\tau_n$.

In this case it is possible to write the following equation (see also [28])

$$\Psi_\beta(R_\beta,t)\cong\int_{E_{\min}}^{\infty}dE'A'\exp[-E'\tau_n/2\hbar+iE'(t_n-t)/\hbar], \qquad (82)$$

where $A' = \tilde{T}_{\alpha\beta}g(E')$. Utilizing the simplest rectangular form of $g(E')$,

$$g(E')=\begin{cases}(\Delta E)^{-1/2}\exp(i\arg g),\\ \text{for } E_{\min}\le E-\Delta E/2<E'<E+\Delta E/2\\ 0,\\ \text{for } E'<E-\Delta E/2 \text{ and } E'>E+\Delta E/2\end{cases} \qquad (83)$$

where arg g is the smooth function of E inside ΔE, we obtain

$$\Psi_\beta(R_\beta,t)=\frac{\text{const}}{t-t_n+i\tau_n/2}\exp[E\begin{pmatrix}-\tau_n/2\\+i(t_n-t)\end{pmatrix}/\hbar] \cdot\left[\exp\begin{bmatrix}\Delta E(-\tau_n/2+i(t_n-t)/2\hbar)\\-\exp\left[-\Delta E(-\tau_n/2+i(t_n-t)/2\hbar)\right]\end{bmatrix}\right]. \qquad (84)$$

If all energies in the large interval, beginning from E_{min}, are totally filled, i.e.

$$\begin{cases}(E+\Delta E/2)\tau_n/2 & \to\infty \quad\text{and}\\ E-\Delta E/2 & \to E_{\min},\end{cases} \qquad (85)$$

then we arrive to

$$\Psi_\beta(R_\beta,t)=\frac{\text{const}}{t-t_n+i\tau_n/2}\exp[E_{\min}\begin{pmatrix}-\tau_n/2\\+i(t_n-t)\end{pmatrix}/\hbar]. \qquad (86)$$

It is natural to call such behavior $\Psi_\beta(R_\beta,t)$ be *time resonance* due to the Lorentzian form of factor $\frac{1}{t-t_n+i\tau_n/2}$ in (86), or *explosion* (for small values of τ_n). And inversely, if $\Psi_\beta(R_\beta,t)$ has the form (86), the Fourier transformation $\Psi_\beta(R_\beta,t)$ will be equal

$$\int_{-\infty}^{\infty}dt\Psi_\beta(R_\beta,t)\exp(iEt/\hbar) = const\cdot\exp[-E\tau_n/2\hbar+iEt_n/\hbar+E_{\min}\tau_n/2\hbar]. \qquad (87)$$

It is proportional to the amplitude (81).

For $z_\beta > R_\beta$ it is possible to re-write (82) in a following way:

$$\Psi_\beta(z_\beta,t) = \int_{E_{\min}}^{\infty}dE'f^n_{\alpha\beta}N_\beta\exp(ikz_\beta)g(E')\exp\begin{bmatrix}-E'\tau_n/2\hbar\\+iE'(t_n-t)/\hbar\end{bmatrix}. \qquad (88)$$

For the small energy spread ($\Delta E << E$), utilizing the function (83) for $g(E')$ and introducing a new variable

$$y'=\sqrt{\frac{i\hbar(t-t_n-i\tau_n/2)}{2m_\beta}}\left[k-\frac{m_\beta z_\beta}{\hbar(t-t_n-i\tau_n/2)}\right], \qquad (89)$$

we finally obtain:

$$\Psi_\beta(R_\beta,t)=\begin{cases}0,\\ \text{for } z_\beta>v(t-t_n-t^0_{in});\\ \text{const}\cdot\exp\left[ikr-\frac{iE\begin{pmatrix}t-t_n-t^0_n\\-i\tau_n/2\end{pmatrix}}{\hbar}-\Delta EA(t)\right],\\ \text{for } z_\beta\le v(t-t_n-t^0_{in})\end{cases} \qquad (90)$$

where

$$A(t)=[t-t_n-t^0{}_n-z_\beta/v-it_n/2]/2\hbar.$$

The cross section, defined for the macroscopic distances, has the following exponential form:

$$\sigma_{\alpha\beta}=\left|f_{\alpha\beta}\right|^2=\text{const}\cdot\exp(-E\tau_n/\hbar). \qquad (91)$$

When $\tilde{T}_{\alpha\beta}$, or $f_{\alpha\beta}$, has the general form like

$$f_{\alpha\beta}=\sum_{n=1}^{\nu}f^n{}_{\alpha\beta}\exp\left[-E\tau_n/2\hbar+iEt_n/\hbar\right] \qquad (92)$$

with several terms (ν=2,3,…), the cross section $\sigma_{\alpha\beta} = |f_{\alpha\beta}|^2$ contains not only exponentially decreasing terms, but also oscillating terms with factors cos $[E(t_n-t_{n'})/\hbar]$ or sin $[E(t_n-t_{n'})/\hbar]$. In the case of 2 terms (ν =2) in (92), formula (92) transforms in the following expression

$$\begin{aligned}\sigma_{\alpha\beta}&=\left|f^1{}_{\alpha\beta}\right|^2\exp\left(-E\tau_1/\hbar\right)\\&+\left|f^2{}_{\alpha\beta}\right|^2\exp\left(-E\tau_2/\hbar\right)\\&+2\mathrm{Re}\left\{f^1{}_{\alpha\beta}f^2{}_{\alpha\beta}*\exp\begin{bmatrix}iE(t_1-t_2)/\hbar\\-E(\tau_1+\tau_2)/\hbar\end{bmatrix}\right\}\end{aligned} \qquad (93)$$

(where the terms with c ΔE can be neglected, if we suppose that $\Delta Et_n<<E\tau_n$ и $\Delta E\tau_n << Et_n$).

The evolution of the survival of the compound nucleus (in the time moment t after its formation) is described by the following function:

$$L^c(t)=1-\int_{t_0}^{t}dtI(t) \qquad (94)$$

where $I(t)$ is defined, relative to [28], by *the probability of the emission (for time unit)* in the proximity of the compound nucleus (near $z_\beta = R_\beta$)

$$I(t) = \frac{j_\beta(R_\beta, t)}{\int_{-\infty}^{\infty} \mathrm{d}t j_\beta(R_\beta, t)}$$

The initial moment t_0 current time it is natural to choose in the moment $t_{in}^{\ 0}$ and to suppose that $t_{in}^{\ 0} = 0$. However it is necessary to consider indeterminacy $\delta t = \hbar/\Delta E$ of the duration of the initial wave packet before the collision. Therefore

$$t_0 \cong t_n^{\ 0} - \delta t = -\delta t = -\hbar/\Delta E.$$

In the region of the time resonance (86) the function $L^c(t)$ is essentially non-exponential even in the approximation $t_0 = 0$. The qualitative form of $L^c(t)$ can be illustrated, as in [28], with the help of the strongly simplified examples for the very narrow interval near $t = t_n$, and also for all the values of t, when

$$j_\beta(R_\beta, t) = \mathrm{Re}\begin{bmatrix} \Psi_\beta(R_\beta, t) \times (i\hbar/m_\beta) \\ \lim_{z_\beta \to R_\beta} \partial\Psi^*{}_\beta / \partial(z_\beta, t)/\partial(z_\beta) \\ \cong \bar{v}\left|\Psi_\beta(R_\beta, t)\right|^2 \end{bmatrix} \quad (95)$$

with $\bar{v}$ is defined by the integral theorem on the mean value, namely by the expression

$$\begin{aligned} &\int_{E_{\min}}^{\infty} dEvA\exp(-E\tau_n/2\hbar) \\ &= \bar{v}\int_{E_{\min}}^{\infty} dEA\exp(-E\tau_n/2\hbar) \end{aligned} \quad (96)$$

(v appears here after the using (75)). Then

$$\begin{aligned} I(t) &= \frac{j_\beta(R_\beta, t)}{\int_{-\infty}^{+\infty} dtj_\beta(R_\beta, t)} \cong \frac{\left[(t - t_n)^2 + \tau_n^{\ 2}/4\right]^{-1}}{\int_{-\infty}^{+\infty} dt\left[(t - t_n)^2 + \tau_n^{\ 2}/4\right]^{-1}} \quad (97) \\ &= (\tau_n/2\pi)\frac{1}{(t - t_n)^2 + \tau_n^{\ 2}/4} \end{aligned}$$

and

$$L^c(t) = 1 - \mathrm{dt\ I\ (t)} = 1 - (1/\pi)\left[\arctan(y)\right]_{y = 2t_0/\tau_n}^{y = 2(t - t_n - t_0)/\tau_n}. \quad (98)$$

Since the curve arctan (y) has the form, depicted in Figure 3 in the case $2t_0/\tau_n \to -\infty$ (the quantity τ_n is small) the function $L^c(t)$ has the form, depicted in Figure 4(the curve 1).

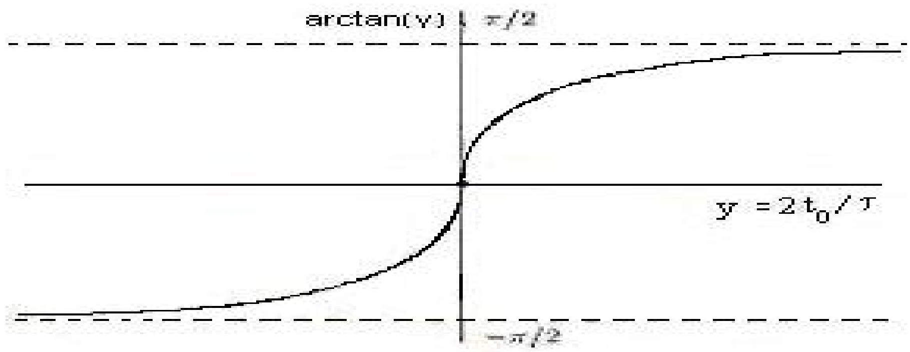

Figure 3. The function arctan(y) for $2t_0/\tau_n \to -\infty$

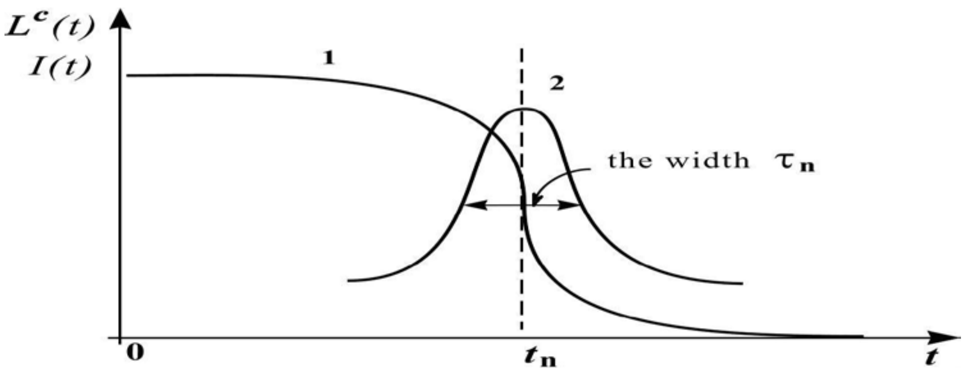

Figure 4. $L^c(t)$ (the curve 1) and $I(t)$ (the curve 2)

In this case

$$L^c(t) = 1 - \pi^{-1}\left[\arctan(2(t - t_n - t_0)/\tau_n) + \pi/2\right] \quad (99a)$$

and

$$L^c(t) = \begin{cases} 1, & \text{when } 0 \le t < t_n = 0 \ \ (\mathrm{c} - 2t_0/\tau_n \to \infty) \text{ and} \\ 0, & \text{when } t \to \infty \end{cases} \quad (99b)$$

From the simple form of Figure 4 it is easy to see that t_n can be interpreted as the Poincare period of internal motion of the compound nucleus (after its formation and before its decay), when $t_n >> \tau_n$. Such behavior of $L^c(t)$ was studied in [28,41].

If precisely consider the compound-resonance structure of $T_{\alpha\beta}$, then the strongly non exponential form of $L^c(t)$ and $I(t)$ will take place, as it is depicted in Figure 4, for the strong overlapping of the energy resonances, when

$$\Gamma_{JS\Pi} << N_{JS\Pi} \,/\, 2\pi\rho_{JS\Pi} \qquad (100)$$

($\Gamma_{JS\Pi}$ and $\rho_{JS\Pi}$ are the mean resonance width and level density, $N_{JS\Pi}$ is the number of open channels, JSΠ are the values of the total momentum, spin and parity, respectively). The small probability of the compound-nucleus decay for $t < t_n$ (inside the Poincare cycle) can be explained by the consequence of the multiply meta-stable states in the region of the overlapped energy resonances. In the case of several time resonances it can signify the superposition of several strongly overlapped groups of energy resonances with different values of JSΠ in the same compound nucleus or the formation of several compound nuclei with the different numbers of participating nucleons.

In particular, for the inclusive energy spectra of the k-th final fragment it is possible to use the following expression

$$\sigma_{inc,k}\left(E_k\right) = \left|\sum_{n=1}^{2} C_n \exp[(it_n - \tau_n/2)E_k/\hbar]\right|^2$$
$$= \sum_{n=1}^{2} \left|C_n\right|^2 \exp(-E_k\tau_n/\hbar) \qquad (101)$$
$$+ 2\,\mathrm{Re}\, C_1^* C_2 \exp\{[i(t_2 - t_1) - (\tau_1 + \tau_2)/2]E_k/\hbar\}.$$

The comparison with the experimental data. For the analysis of the observed experimental spectra of a single final fragment it is necessary to sum (or average) the expressions like (93) or (101) over the subfamilies of the final states (with various quantum numbers JSΠ, where J,L,S and Π are quantum numbers of the total momentum, orbital momentum, spin and parity, respectively) and channels, sometimes coherently and sometimes incoherently. And for inclusive energy spectrum of k-th final fragment we shall use the expression (101).

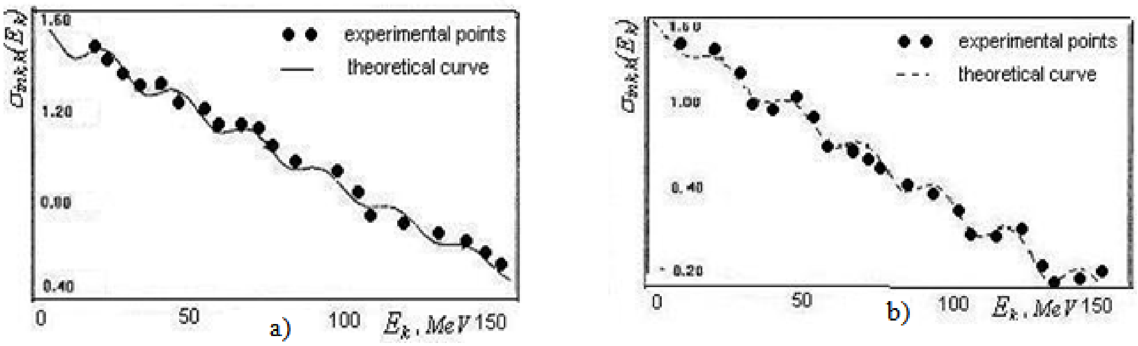

Figure 5. The inclusive process $p + C \to {}^7Be + X$ (protons of 2.1 GeV), experimental data are taken from [38]: a) C_1= 0.04, C_2= 0.36 (θ = 90°); b) C_1= 0.35, C_2= 0.05 (θ = 160°)

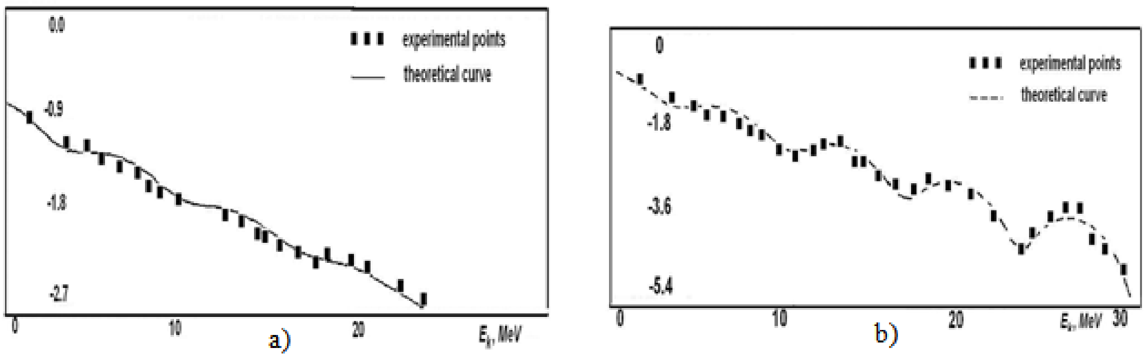

Figure 6. The inclusive process ${}^4He + Ta \to t + X$ (720 MeV/nucleon), experimental data are taken from [39]: a) C_1=0.18, C_2=1.02 (θ=60°); b) C_1=1.13, C_2=0.07 (θ=90°)

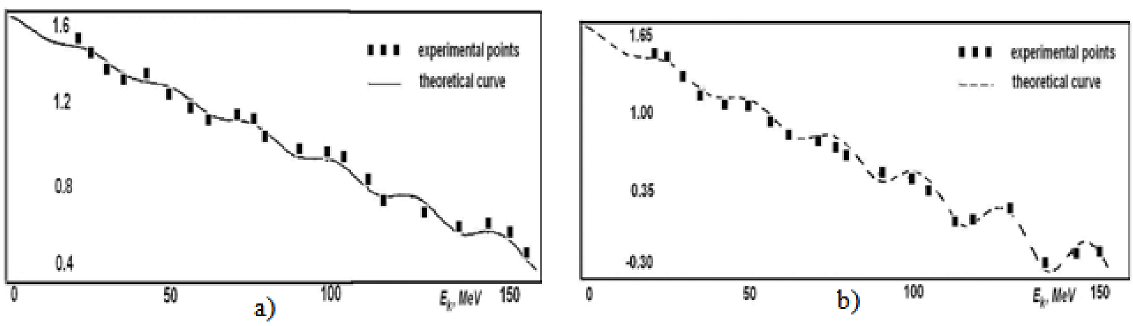

Figure 7. The inclusive process ${}^{20}Ne + U \to p + X$ (1045 MeV/nucleon), experimental data are taken from [39]: a) C_1=0.35, C_2=5.65 (θ=90°); b) C_1=5.65, C_2=0.35 (θ=150°)

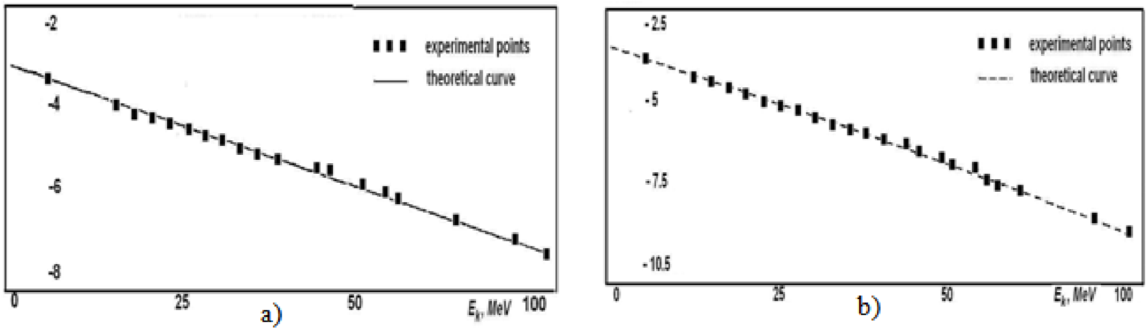

Figure 8. The inclusive process ^{40}Ar + $^{51}V \rightarrow p + X$ (41 MeV/nucleon); experimental data are taken from [36]: a) C_1=0.002, C_2=0.03 (θ=97°); b) C_1=0.03, C_2=0.022 (θ=129°)

In Fgure 5 - Fgure 8 are represented some calculated inclusive energy spectra $\sigma_{inc,k}(E_k)$ in the semi-logarithmic scale in compare with the experimental data from [36,38,39].

In Figure 5 - Figure 8, θ is the detected angle of k-th fragment in emission. The values of τ_1, τ_2 and $t_2 - t_1$, which were found in [39] from the fitting of theoretical curves to the experimental data, are written in Table 1.

Table 1. Parameters of time resonances for some inclusive spвctra

Reaction	Energy of bomb. particle GeV/nucleon	τ_1, 10^{-23} sec	τ_2, 10^{-23} sec	$t_2 - t_1$, 10^{-22} sec
$P + C \rightarrow {}^7Be + X$	2.1	10.45	17.0	5.95
$^{20}Ne + Al \rightarrow p + X$	0.393	0.1	0.99	1.7
$^{4}He + Ta \rightarrow t + X$	0.72	1.72	3.15	1.22
$^{20}Ne + U \rightarrow p + X$	1.045	0.92	1.7	1.72
$^{20}Ar + V \rightarrow p + X$	0.041	7.5	9.0	0.20
$^{132}Xe + Au \rightarrow p + X$	0.044	6.0	7.0	1.0
$^{20}Ne + U \rightarrow p + X$	0.4	1.7	2.2	0.10
$^{20}Ne + U \rightarrow d + X$	0.25	4.2	7.2	0.10

Since the inclination of energy spectra is essentially increases with the angle increasing, it signifies that the increasing contribution of the compound-nucleus states with larger values of t_n and τ_n is connected with the formation of more heavy compound nuclei at the lesser velocity in L-system. It agrees with the observed in [33,35,39] phenomena of more clear oscillations for the intermediate emission angles.

It is possible that for the most easy compound system (p+C), represented here, there is a superposition of the direct process (i.e. n = 0 instead of n = 1) and the time resonance (n = 2), since the difference $t_2 - t_{1(0)}$ is *noticeably larger than usually.*

Later there were performed new calculations in [42] and their comparison with the experimental data from [43,44]. They are represented in Figure 9 - Figure 10.

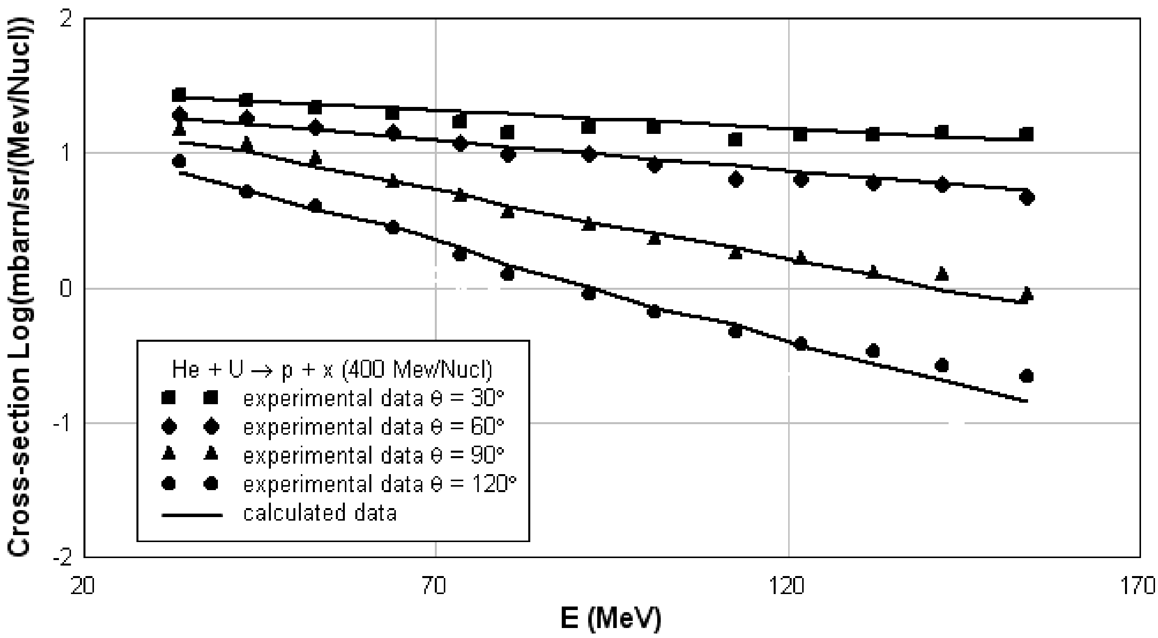

Figure 9. Inclusive energy spectrum of $^4He + U \rightarrow p + X$, of 400 MeV/nucleon, experimental data are taken from [43]

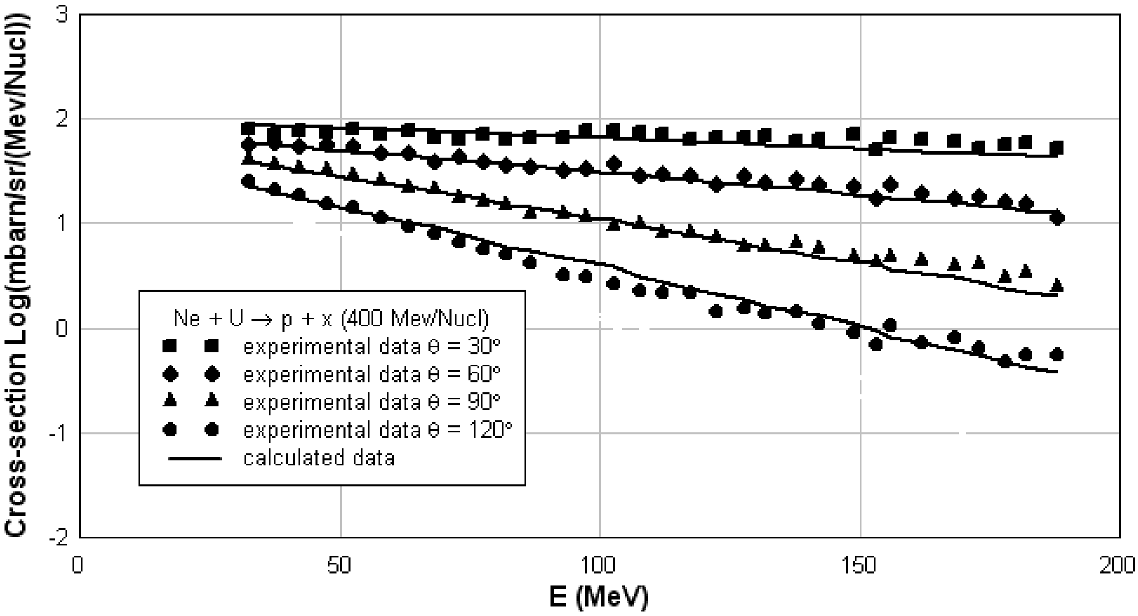

Figure 10. Inclusive energy spectrum of $^{20}Ne + U \rightarrow p + X$, of 400 MeV/nucleon, experimental data are taken from [44]

The values of τ_1, τ_2 and $t_2 - t_1$ in sec, which were found in [42] from the agreement of theoretical curves with the experimental data, are represented in Table 2 and Table 3 for Figure 9 and Figure 10, respectively

Table 2.

θ	$\tau_1(10^{-23}s)$	$\tau_2(10^{-23}s)$	t_2-$t_1(10^{-23}s)$	c_1	c_2
30^0	0.38	0.38	0.25	2.8	2.8
60^0	0.64	0.64	0.25	2.6	2.6
90^0	1.5	1.5	0.25	2.5	2.5
120^0	2.1	2.1	0.25	2.3	2.3

Table 3.

θ	$\tau_1(10^{-23}s)$	$\tau_2(10^{-23}s)$	t_2-$t_1(10^{-23}s)$	c_1	c_2
30^0	0.25	0.25	0.25	5	5
60^0	0.6	0.6	0.25	4.5	4.5
90^0	1.2	1.2	0.25	4.2	4.2
120^0	1.7	1.7	0.25	3.6	3.6

The explanation of the time-resonances structure in the cross sections of high-energy nuclear reactions in the region of the densely situated strongly overlapped energy resonances. How is it possible to explain the manipulations with relatively smooth energy behavior of the expressions (91) and (93) for the cross sections or the expressions (81) and (92) for $T_{\alpha\beta}$ or $f_{\alpha\beta}$, which correspond to time resonances and simultaneously to the experimental data on cross sections, although really the amplitudes have to fluctuate strongly with energy in the region of strongly overlapped energy resonances for extremely high energies? At first sight, in the region of high energies the structure of energy resonances has to vanish not only due to the "smoothing" by energy spreads (since $\Delta E >> \Gamma_{JS\Pi}, \rho_{JS\Pi}^{-1}$), but also de facto due to the strong decreasing of the probability of the formation of the intermediate long-living many-nucleon states. The density of the compound-resonances is quickly increases, beginning from the low-energy well resolved energy resonances where the various versions of the Fermi-gas model with the shell-model and collective-model corrections work rather successfully work. Only near 30-40 MeV/nucleon in the compound system it is possible to expect the saturation effects and the further strong decreasing of the densities. However namely for these energies the resonances of another structure can appear. These resonances are connected with the local excitations of long-living intermediate many-quark-gluon states of the baryon subsystems (see [45]).

Let us consider the possibility of the abovementioned explanation of the structure of time resonances more attentively, limited ourselves only by the partial JSΠ-amplitudes $T^{JS\Pi}{}_{\alpha\beta} = \delta_{\alpha\beta} - S^{JS\Pi}{}_{\alpha\beta}$, where $S^{JS\Pi}{}_{\alpha\beta}$ is the element of the S-matrix.

As it was said above, for the sufficiently high energies if we neglect bound and virtual states and the threshold particularities we can describe the S-matrix by the analytic expression (67) when the indexes JSΠ (and now even J) were omitted for the simplicity, the unitary background (non-resonance) matrix $\hat{U}$ and the projection resonance matrix $\hat{P}_n$ ($\hat{P}_n = \hat{P}_n{}^{+} = \hat{P}_n{}^2$, Trace $\hat{P}_n$ =1), slowly changing with the total energy ε or almost did not depend on ε, $\hat{U}^T$ is the matrix, transposed to $\hat{U}$. For the simplest Baz'-Newton conditions (see [27] and also [28]), when the fluctuations of $\hat{P}_n$ can be neglected ($\hat{P}_n =< \hat{P} >$), the Simonius S-matrix acquires such a form:

$$\hat{S} = \hat{S}_b - \hat{a}\left(1 - \prod_x \frac{\varepsilon - \varepsilon_n - i\Gamma_n/2}{\varepsilon - \varepsilon_n + i\Gamma_n/2}\right) \qquad (102a)$$

where $\hat{S}_b = \hat{U}\hat{U}^T$ and $\hat{a} = \hat{U} < \hat{P} > \hat{U}^T$. The averaged on energy the S-matrix $< \hat{S} >_{\Delta\varepsilon}$ in this case in accordance with [28]:

$$< \hat{S} >_{\Delta\varepsilon} = \hat{S}_b - \hat{a}\left[1 - \exp\left(-\pi\,\Gamma/\rho\right)\right]$$

for unresolved resonances in ($\Delta E >> \rho, \Gamma$) and the fluctuating S-матрица $\hat{S}^c$ (or S-matrix of the compound nucleus) is equal

$$\hat{S}^c = S - \langle S \rangle_{\Delta\varepsilon} = \hat{a}\left[\prod_n \frac{\varepsilon - \varepsilon_n - i\Gamma_n/2}{\varepsilon - \varepsilon_n + i\Gamma_n/2} - \exp(-\pi\Gamma\rho)\right]. \qquad (103)$$

We repeat that $\hat{S}_b$ and $\hat{a}$ almost do not depend on energy (slowly change with energy). For the strongly overlapped resonances when $\pi\Gamma\rho >>1$

$$\hat{S}^c \to \hat{a}\prod_n \left(\frac{\varepsilon-\varepsilon_n-i\Gamma_n/2}{\varepsilon-\varepsilon_n+i\Gamma_n/2}\right) \qquad (103a)$$

and the averaged over energy cross section of the processes, going through the step of formation of compound nucleus $<\sigma^c{}_{\alpha\beta}>_{\Delta\varepsilon}$, is evidently proportional to $|a_{\alpha\beta}|^2$:

$$<\sigma^c{}_{\alpha\beta}>\sim<\left|S^c{}_{\alpha\beta}\right|^2>_{\Delta\varepsilon}=\left|a_{\alpha\beta}\right|^2 \qquad (104)$$

(here and below we continue to omit the indexes JSΠ). If the initial energy of bombarding particles is fixed and therefore the total energy ε is also fixed (within $\Delta\varepsilon$), the cross section (104) can be re-written in the form

$$<\sigma^c{}_{\alpha\beta}>_{\Delta\varepsilon}\sim<\left|S^c{}_{\alpha\beta}\right|^2>_{\Delta\varepsilon}=<\left|a_{\alpha\beta}\right|^2>_{\Delta\varepsilon}\cong\left|a_{\alpha\beta}\right|^2, \qquad (104a)$$

where ΔE is defined by $\Delta\varepsilon$ and the energy resolution of the detector of final fragments.

From [28,46] we can see that the averaged over energy time delay of compound nucleus and the variance of the time-delay-of-compound-nucleus distributions are defined by such general relations

$$<\tau^c{}_{\alpha\beta}>=<\left|S^c{}_{\alpha\beta}\right|^2\hbar\partial\arg S^c{}_{\alpha\beta}/\partial E>/<\left|S^c{}_{\alpha\beta}\right|^2> \qquad (105)$$

and

$$D\tau^c{}_{\alpha\beta}=\frac{\hbar^2<(\partial\left|S^c{}_{\alpha\beta}\right|/\partial E)^2>_{\Delta E}}{<\left|S^c{}_{\alpha\beta}\right|^2>_{\Delta E}}+\frac{\hbar^2<\left|S^c{}_{\alpha\beta}\right|^2(\partial\arg S^c{}_{\alpha\beta}/\partial E)^2>_{\Delta E}}{<\left|S^c{}_{\alpha\beta}\right|^2>_{\Delta E}}-<\tau^c{}_{\alpha\beta}>^2, \qquad (106)$$

respectively (energy E is the kinetic energy of final fragment). For the quantities, averaged over energy, in the approximation of continuum ($\sum_n \to \int\rho d\varepsilon$) we easily derive, utilizing [28,41,46], that the mean time delay, averaged over all channels, is equal

$$<\tau^c>=<\sum_n\frac{\hbar\Gamma_n}{(\varepsilon-\varepsilon_n)^2+\Gamma_n{}^2/4}>_{\Delta E}=2\pi\hbar\rho. \qquad (107)$$

And $D\tau^c{}_{\alpha\beta}$ in the same continuum approximation

$$D\tau^c{}_{\alpha\beta}=\frac{\hbar^2<(\partial\left|a_{\alpha\beta}\right|/\partial E)^2>_{\Delta E}}{<\left|a_{\alpha\beta}\right|^2>_{\Delta E}}, \qquad (108)$$

if

$$<\tau^c>^2,\exp(-\pi\rho\Gamma)<<\frac{\hbar^2<(\partial\left(a_{\alpha\beta}\right(/\partial E)^2>_{\Delta E}}{<\left(a_{\alpha\beta}\right(^2>_{\Delta E}}. \qquad (109)$$

Now it is possible to see the mathematical similarity (even coincidence) between the cross section of compound nucleus (104) under above-mentioned conditions (the Baz'-Newton condition for $\hat{P}_n=<\hat{P}>$ and for the strong resonance overlapping when $\pi\Gamma\rho>>1$) and the time-resonance cross section (91) for a short time resonance. Therefore, returning to the expression $T_{\alpha\beta}=\delta_{\alpha\beta}-S_{\alpha\beta}$ with $S_{\alpha\beta}\to S^c{}_{\alpha\beta}$, defined by (103a) for strongly overlapped resonances (with $\pi\Gamma\rho>>1$), we can re-write (104) approximately in such form

$$\sigma^{c(n)}{}_{\alpha\beta}\sim\left|a^{(n)}{}_{\alpha\beta}\right|^2\sim\exp\left(-E\tau_n/\hbar\right) \qquad (110)$$

(if $\hbar/\tau_n>>\Delta E$ for small ΔE). And under the same conditions that in (102)-(103)

$$D\tau^c\cong\tau_n{}^2/4 \qquad (111)$$

(here and further we write $D\tau^c$ without indexes $\alpha\beta$).

If $\tau_n<<2\pi\hbar\rho$ (pt is possible when $\Delta E>>\rho^{-1}$), then $D\tau^c\cong\tau_n{}^2/4<<(<\tau^c>)^2$ and we have a narrow *time resonance* (*explosion*) of the compound nucleus.

When there are some independent non-fluctuating projectors $\hat{P}_\nu=<\hat{P}^{(\nu)}>$, ν = 1, 2… η (η is much lesser of the resonance number), it is possible to obtain at the same reasoning the result like (101) for $\sigma_{\alpha\beta}$ with oscillative terms.

Under more realistic Lyuboshitz conditions of the statistically equivalent channels of the compound-nucleus decay [26] (see also [28,46]), when the fluctuations of $\hat{P}_n$ are the same in all open channels, it is possible to show that

$$<\tau^c>=2\pi\hbar\rho/NT \qquad (112)$$

where $T=1-\exp(-2\pi\rho\Gamma/N)$ and the sum of last two terms in the right part of relation (86) for $D\tau^c$ can be neglected in the continuum approximation. From (112) it is clear that for strongly overlapped resonances when $\pi\rho\Gamma/N>>1$ and $T\to1$, we have:

$$<\tau^c>=2\pi\hbar\rho/N \qquad (112a)$$

In [39,46] it was shown that under the same conditions and when S_b can be considered as independent from energy ε (and E), $D\tau^c<<<\tau^c>^2$. If then one extend the Hauser-Feshbach formula for the compound-nuclear-reactions cross sections $<\sigma^c{}_{\alpha\beta}>$ into the region of high energies, then under the same conditions it is possible to be easily convinced in such behavior of $<\sigma^c{}_{\alpha\beta}>\cong N^{-1}\cong\exp(-E\tau_n/\hbar)$. Under the Lyuboshitz conditions for the strongly overlapped resonances, $D\tau^c\cong\tau^2{}_n/4<<<\tau^c>^2$ – and the exponential decreasing of energy spectra of final fragments corresponds to the narrow *time resonance* (*explosion*) of the compound nucleus. There are possible also the cases when we can observe one or several *time resonances* (*explosions*) in cross sections.

Real time resonances (explosions) of compound nuclei can be only for the resonance high densities ($\Delta E>>\rho^{-1}$) and strongly overlapped resonances ($\pi\Gamma\rho>>1$ or even $\pi\Gamma\rho/N>>1$).

2.9. Connection of Analytic Properties of the *S*-matrix with Duration of the Partial-Wave Scattering and Orthodox Causality

Let us clarify how obtained results on analytic properties of the *S*-matrix agree with orthodox causality. Following [28], we define the mean duration of the *l*-partial-wave scattering as the difference between mean time moments averaged over outgoing and ingoing wave-packer durations through the sphere surface with radius $r \geq a$ according to

$$< \tau_l(\gamma, r) > = \frac{\int_{-\infty}^{\infty} dttj_{l,out}}{\int_{-\infty}^{\infty} dtj_{l,out}} - \frac{\int_{-\infty}^{\infty} dttj_{l,in}}{\int_{-\infty}^{\infty} dtj_{l,in}}, \qquad (103)$$

where $j_{l,in}$ and $j_{l,out}$ are the probability flux densities, corresponding to wave packets

$$r\phi_{l,in}(r,t) = \int_0^{\infty} dkA(k)[(i/2)\exp(il\pi/2)]f_{l-}(k,r)\exp(-iEt/\hbar) \qquad (104a)$$

and

$$r\phi_{l,out}(\gamma,r,t) = \int_0^{\infty} dkA(k)[(i/2)\exp(il\pi/2)] f_{l+}(k,r)S_l(\gamma,k)\exp(-iEt/\hbar), \qquad (104b)$$

respectively.

Integrating over dt in (103) with help of the simple technique of Fourier-Laplace transformations similarly to that it was made in [28], we obtain the following final expression:

$$< \tau_l(\gamma,r) > = \frac{\int_0^{\infty} dk\left|A(k)S_l(\gamma,k)f_{l+}(k,r)\right|^2 \hbar(\partial \arg AS_l f_{l+}/\partial E)}{\int_0^{\infty} dk\left|AS_l f_{l+}\right|^2} \qquad (103a)$$

$$- \frac{\int_0^{\infty} dk\left|A(k)f_{l-}(k,r)\right|^2 \hbar(\partial \arg Af_{l-}/\partial E)}{\int_0^{\infty} dk\left|Af_{l-}\right|^2}.$$

We note that, unlike the physical radial wave packet

$$\phi_l^{(+)}(\gamma,r,t) = \phi_{l,in}(r,t) - \phi_{l,out}(\gamma,r,t), \qquad (104c)$$

which is finite at the limit $k \to \infty$, functions $f_{l\pm}(k,r)$ have the pole of the *l*-the order, and so for the finiteness of wave packets (104a) and (104b) it is necessary that wave-packet amplitudes $A(k)$ would have zero in point $k = 0$, at least of the *l*-the order, or would be zero in the finite interval $(0, \kappa)$, $\kappa > 0$. With such limitations for $A(k)$, it is natural to try clear up, at what conditions the orthodox causality is fulfilled, if one formulate it thus: *for any square integrable function A*(k), *with the only above-mentioned limitation, the mean duration* $< \tau(\gamma, r) >$ *of the l-partial-wave scattering for sufficiently large* $r \geq a$ *cannot be negative, i.e.*

$$< \tau_l(\gamma,r) > \geq 0. \qquad (105)$$

Following [28], it is not difficult to check that in the case of unitary $S_l(\gamma,k)$ for the fulfillment of the condition (105) it is necessary and sufficient that eq.

$$\tau_l(\gamma,r) = \hbar\frac{\partial \arg S_l(\gamma,k)[f_{l+}(k,r)/f_{l-}(k,r)]}{\partial E} \geq 0 \qquad (106)$$

were fulfilled. Really, in this case according to (103a)

$$\tau_l(\gamma,r) = [\int_0^{\infty} dk\left|A(k)f_{l+}(k,r)\right|^2 \tau_l(\gamma,r)]/[\int_0^{\infty} dk\left|A(k)f_{l+}(k,r)\right|^2$$

and also in view of non-negative values of k and $\left|A(k)f_{l+}(k,r)\right|^2$ the validity of (105) follows directly and necessarily from (106). And inversely, if one assumes the validity of (105), but in the vicinity of a certain point k_0 the relation $\tau_l(\gamma,r) < 0$ is valid for $r \geq a$, then, choosing $A(k)$ identically equal 0 out of this vicinity, one will violate the condition (105) which contradicts the initial assumption and therefore proves the sufficiency of our theorem.

Let study firstly the validity of the condition (106) in the case when out of the interaction sphere there is only the centrifugal tail. Then

$$\tau_l(\gamma,r) = \hbar\frac{\partial \arg S_l(\gamma,k)[h_l^{(1)}(k,r)/h_l^{(2)}(k,r)]}{\partial E}$$
$$= \hbar\frac{\partial \arg S_l}{\partial E} + 2\hbar\frac{(\partial n_l/\partial k)j_l - (\partial j_l/\partial k)n_l}{(j_l)^2 + (n_l)^2} \qquad (107)$$
$$= \hbar\frac{\partial \arg S_l}{\partial E} + \frac{2r/v}{[krj_l(kr)]^2 + [krn_l(kr)]^2},$$

since $[dn_l(x)/dk]j_l(x) - [dj_l(x)/dx]n_l(x) = x^{-2}$. Here $v = \hbar k/\mu$. Utilizing (66) for calculation of $\frac{\partial \arg S_l}{\partial E}$, we obtain

$$\hbar\frac{\partial \arg S_l}{\partial E} = -\frac{2\alpha}{v} + \frac{1}{v}\sum_{\lambda}\frac{2\chi_\lambda}{k^2 + \chi_\lambda^2}$$
$$+ \frac{1}{v}\sum_s \frac{4\,\mathrm{Im}\,k_s(k^2 + |k_s|^2}{(|k_s|^2 - k^2)^2 + (2k\,\mathrm{Im}\,k_s)^2}, \qquad (108)$$

where $\chi_\lambda = -ik_\lambda$. Since $\alpha \leq a$, the sum $\sum_s$ is always positive, the quantity $[krj_l(kr)]^2 + [krn_l(kr)]^2$ is finite when $k > 0$ and $r > a$ and tends to 1 when $r \to \infty$, and

$$\sum_s \frac{2\chi_\lambda}{k^2 + \chi_\lambda^2} \geq \frac{2}{\chi_1}$$

when χ_1 corresponds to the first bound state (we remind that there is at least one pole, corresponding to zero on the positive imaginary semi-axis between every two adjacent zeros k_λ, located in the order of increasing $|k_\lambda|$ on the negative imaginary semi-axis), then the condition (106) is fulfilled for sufficiently large values of r when

$$\frac{r}{[krj_l(kr)]^2 + [krn_l(kr)]^2} \geq a + \frac{1}{\chi_1}.$$

Inequality

$$\tau_0(\gamma, k, r) \geq 0. \qquad (106a)$$

(in the case when out of the interaction sphere there is only the centrifugal tail) for l = 0 and $r \geq a + 1/\chi_1$ is concordant with the Goebel-Carplus-Ruderman inequality (see, for instance [39]). For $l \neq 0$, $k \neq 0$ and $r \geq a$ the quantity Q_l = $\{[krj_l(kr)]^2 + [krn_l(kr)]^2\}^{-1}$ is positive, finite, tends to 0 as $(kr)^l$ when $kr \to 0$, and monotonically grows, approaching to 1, with increasing kr. And in this last case

$$\tau_l(\gamma, k, r) \geq 0 \quad for \quad r \geq R_l(k), \qquad (106b)$$

where $R_l(k)$ is the largest real solution of equation $rQ_l(kr) = a + 1/\chi_{1l}$.

Let consider what contribution for the time delay (108) would give every separate factor of representation of the type (66) for $S_l(\gamma, k)$. The factor $\exp[-i2\alpha_l k]$, which is typical for the hard repulsive barrier of radius α_l, causes the negative time delay – α_l/v. The factor $\frac{1+k/k_{nl}}{1-k/k_{nl}}$ with $\chi_{nl} = k_{nl}/i > 0$, correspondent to a bound state, causes the negative time delay $-2\chi_{nl}/v[k^2+\chi_{nl}^2]$. The similar negative time delay will be caused by the factor with "redundant" pole. The factor $\frac{1+k/k_{ml}}{1-k/k_{ml}}$ with $\chi_{ml} = k_{ml}/i < 0$, correspondent to a virtual (anti-bound) state, causes the positive time delay. For small k ($k \to 0$) the both formulas (for bound and anti-bound states) are the particular cases of the following expression for time delay – $A/v[1+k^2A^2]$, where A is the scattering length. The factor $\frac{(1+k/k_s)(1-k/k_s^\bullet)}{(1-k/k_s)(1+k/k_s^\bullet)}$ with Im $k_s > 0$, correspondent to a resonance state, causes the positive timer delay $\frac{1}{v}\cdot\frac{4\,\mathrm{Im}\,k_s(k^2+|k_s|^2}{(|k_s|^2-k^2)^2+(2k\,\mathrm{Im}\,k_s)^2}$. For the same every factor the signs of the correspondent scattering l-th partial time delays will be the same, differing from the studied here time delays of the l-partial-wave scattering twice less in absolute value.

In the more general case when at the external region $r > a$, besides the centrifugal barrier, there is a potential, decreasing more rapidly than any exponential function,

$$\tau_l = \hbar\frac{\partial \arg S_l(\gamma,k)[f_{l+}(k,r)/f_{l-}(k,r)]}{\partial E}$$
$$= \hbar\frac{\partial \arg S_l}{\partial E} + \frac{2r}{v} + \frac{2}{v}\mathrm{Im}\frac{\partial\phi_{l+}/\partial k}{\phi_{l)}},$$

where $\phi_{l+}(k,r) = \exp(-ikr)f_{l+}(k,r)$.

Taking into account that the expression 2Re $i\frac{\partial\phi_{l+}/\partial k}{\phi_{l+}}$ tends to 0 when $r \to \infty$, and utilizing the result (108), one can easy to show that also in this case the inequality (105) is valid for sufficiently large values of r.

Finally, in the case when $f_{l-}(k,r)$ has in D^+ singularities of the type (22) for the potentials with the exponential law of decreasing, it is convenient to introduce the function $\tilde{S}_l(\gamma,k,r) = S_l(\gamma,k)\frac{\phi_{l+}(k,r)}{\phi_{l-}(k,r)}$ instead of $S_l(\gamma,k)$. Then, rewriting equations (18) and (19) in the forms:

$$(-1)^l \oint_{k_{nl}} \tilde{S}_l(\gamma,k,r')\frac{\phi_{l-}(k,r')}{\phi_{l+}(k,r')}f_{l+}(k,r)f_{l-}(k,r')dk \qquad (18a)$$
$$= (B_{nl})^2 f_{l+}(k,r)f_{l+}(k,r'),$$

$$(-1)^l \oint_{k_m} \tilde{S}_l(\gamma,k,r')\phi_{l-}(k,r')\exp(ikr')f_{l+}(k,r)dk \qquad (19c)$$
$$= \oint_{k_m} f_{l+}(k,r)f_{l-}(k,r'),$$

one can easily conclude that the function $\tilde{S}_l(\gamma,k,r)$ has the poles of the first order on the upper imaginary semi-axis which correspond to the bound states with the residues

$$(-1)^{l+1}i\frac{(B_{nl})^2}{2\pi}\frac{\phi_{l+}(k_{nl},r')}{\phi_{l-}(k_{nl},r')}$$

and, unlike $S_l(\gamma,k,r)$, it has no "redundant" poles. If one chooses the sufficiently large finite values of r' for which at the fixed l the relation $\frac{\phi_{l+}(k_{nl},r')}{\phi_{l-}(k_{nl},r')}$ will have the same sign independently from k_{nl} (it can be always obtained, because $\phi_{l\pm}(k_{nl},r) \to 1$ when $r \to \infty$, if k_{nl} does not coincide with "redundant" pole; but if such coincidence takes place, the correspondent residue will be 0, since the correspondent pole vanishes!). Then the direct calculation of the quantity

$$\tau_l(\gamma,r) = \hbar\frac{\partial \arg\tilde{S}_l(\gamma,k,r)}{\partial E} + \frac{2r}{v}$$

relative to the scheme (107)-(108) will show the validity of (106) for sufficiently large r also in this case.

The same procedure (8a)-(9c) etc can be repeated also for the case of the presence in the external region $r > a$ of the potential tail of the Yukawa type because of the coincidence of the logarithmic divergence at points $k_\gamma = ib/2$ of the factor $F(k)$ in the expression (40a) for $S_l(\gamma,k,r)$ and of the term $[1+\frac{i\rho}{2k}\ln(1+\frac{2ik}{b})]^{-1}$ in the expression (24a) for the function $f_{l-}(k, r)$ and hence its vanishing in $\tilde{S}_l(\gamma,k,r)$.

The case with the non-unitary $S_l(\gamma,k,r)$ appears to be somewhat more complicated. Let rewrite (103a) in the following form:

$$\tau_l(\gamma,r) = <\hbar\frac{\partial \arg S_l(\gamma,k)f_{l+}(k,r)}{\partial E}>_1$$
$$+ <\hbar\frac{\partial \arg f_{l-}(k,r)}{\partial E}>_2 \qquad (103b)$$
$$+ <\hbar\frac{\partial \arg A(k)}{\partial E}>_1 - <\hbar\frac{\partial \arg A(k)}{\partial E}>_2,$$

where $< >_1$ and $< >_2$ signify average in the momentum space with the weights $|AS_l f_{l+}|^2$ and $|Af_{l-}|^2$, relatively, then choose without the limitation of the generality such $A(k)$ in order that the quantity $\hbar\partial \arg A/\partial E$ would be limited (such choice of $A(k)$ does physically signify that the mean time moment of the incoming-wave entrance into the sphere of radius r around the scatterer, which is equal to $-([v\left|f_{l-}\right|^2]^{-1}>_1+<\hbar\partial \arg A/\partial E>_1$, would be finite). Then, since two last terms in (103b) are finite, and the quantities $<\hbar\partial \arg f_{l\pm}/\partial E>_{1,2}$ are positive and proportional to r for sufficiently large r, one can affirm that $<\tau_l(\gamma, r)> \geq 0$ at least at the range $r >> a$.

Thus, the completeness condition of the type (11) together with the conditions of symmetry and generalized unitarity of $S_l(\gamma,k)$ guarantee the fulfillment of the orthodox causality (105) for sufficiently large values of r but, in general, do not ensure the fulfillment of the micro-causality for $r \geq a$ (mainly because of the influence of the centrifugal barrier and partially because of the distortion of the wave-packet form during scattering).

In the case of non-central or parity-violating interactions the relation

$$\hbar\frac{\partial \arg S^j_{l'l}}{\partial E}=\frac{2}{v}\left[\begin{array}{l}-\alpha-\sum\limits_n \dfrac{2\operatorname{Im}k_{nj}(k^2+\left|k_{nj}\right|^2}{[\left|k_{nj}\right|^2-k^2]^2+[2k\operatorname{Im}k_{nj}]^2}\\ -\sum\limits_m\dfrac{\chi_m}{k^2+\chi_m^2}+\sum\limits_p\dfrac{\chi_p}{k^2+\chi_p^2}\\ +\sum\limits_{t,t'}\dfrac{\operatorname{Im}k_{t,t'}}{(\operatorname{Re}k_{t,t'}-k)^2+(\operatorname{Im}k_{t,t'})^2}\\ -\sum\limits_{s,s'}\dfrac{\operatorname{Im}k_{s,s'}}{(\operatorname{Re}k_{s,s'}-k)^2+(\operatorname{Im}k_{s,s'})^2}\end{array}\right] \qquad (109)$$

must be valid instead of (108) for $S^j_{l'l}$. Here Im $k_{nj} > 0$, $\chi_m = -ik_m < 0$, Im $k_{s,s'} < 0$, and χ_p, Im $k_{t,t'}$ can be not only positive but also negative and, moreover, the numbers of points with χ_p, Im k_{t}, Im $k_{t'}$ can be infinite. Therefore, a causality condition like (106a) and (106b) demands certain restrictions for the topology of zeros and poles of $S^j_{l'l}$, namely

$$\begin{array}{l}-\sum\limits_m\dfrac{\chi_m}{k^2+\chi_m^2}+\sum\limits_p\dfrac{\chi_p}{k^2+\chi_p^2}\\ +\sum\limits_{t,t'}\dfrac{\operatorname{Im}k_{t,t'}}{(\operatorname{Re}k_{t,t'}-k)^2+(\operatorname{Im}k_{t,t'})^2}\\ -\sum\limits_{s,s'}\dfrac{\operatorname{Im}k_{s,s'}}{(\operatorname{Re}k_{s,s'}-k)^2+(\operatorname{Im}k_{s,s'})^2}\geq 0.\end{array} \qquad (110)$$

2.10. Conclusions and perspectives.

In the presented review there are the results of the almost complete study of the non-relativistic S-matrix analytic structure for *unknown* central, non-central (tensor) and parity-violating T-invariant interactions, linear or non-linear, with unknown physical dynamics and kinetics, with possible absorption and/or generation of bombarding particles inside sphere of small radius $r \leq a$, surrounded in the external range ($a < r < \infty$) by centrifugal barrier with possible presence of decreasing (more rapidly than any exponential function, or according to the exponential law, or the Yukawa law etc) potential tails for one-channel and discrete-many-channel scattering. This study was based on some general mathematical assumptions like the possibility of the S-matrix analytic continuation into the regions of complex values of particle wave numbers or kinetic energies and the completeness conditions for external wave functions and on the physical principles like the causality and some kinds of the symmetry for the *S*-matrix.

It is rather curious how the results of a research, based on the well-known cognitive principle “with the least number of assumptions to obtain the most number of results of rather general physical and mathematical character”, can also help to reveal some concrete physical phenomena and effects: (a) the enhancement phenomena caused by parity violations, indicated in Appendix V; (b) the phenomena of time resonances (explosions), formed from the strongly overlapping energy resonances of high-energy many-channel nuclear reactions.

It follows from all totality of the presented results an interesting perspective of future investigations – a research program of concrete tasks, problems and then the continuation, extension and application of the rigorous study of the analytic properties of the S-matrix on the base of general physical principles and general mathematic assumptions together with search of the observable physical manifestations of microscopic quantum collisions:

(1) Between remained important tasks it is possible to propose (a) the study of enhancement phenomena caused by violations of T-invariance, quite similarly to enhancement phenomena caused by parity violations; (b) the study of the S-matrix analytic structure for unknown interactions, enclosed by a centrifugal barrier and a screened Coulomb barrier (the last one is namely the Yukawa-potential type, differing from the Yukawa potential by the positive sign (repulsion instead of attraction) and by the scale.

(2) As an interesting continuation of the presented approach there is remained open a way for the study of other types of many-channel collisions (for instance, collisions with rearrangement of colliding systems, with multiple generation of particles, chain reactions etc), the classes of T-violating interactions, including the interactions with microscopic quantum dissipation (quantum friction), various relativistic collisions, collisions at the presence of external fields, scattering with accompanied processes like bremsstrahlung etc).

(3) And it is appeared a somewhat unexpected perspective – how the rigorous mathematical method or approach can help to reveal the physical phenomena and effects (enhancement phenomena caused by parity violations in Appendix IV or may be by T-invariance violations and time resonances in section 8).

Appendix I

The derivation of the product expansion for $S_l(\gamma,k)$ with the generalized “unitarity condition”(1a) for unknown interactions surrounded by centrifugal barrier and potentials decreasing more rapidly that any exponential function (within approach outlined in [9, 10,12])

First, we consider some intermediate products. Taking the behavior of (20) into account, we can easily see that the factor exp($2i\alpha k$), with $\alpha \le a$, is one of the multipliers of $S(\gamma,k)$.

The product

$$\prod_n \frac{k_{nl}+k}{k_{nl}-k} \qquad (I,1)$$

contains all the poles of $S(\gamma,k)$ in $D^{+,}$ has no other singularities or zeros, satisfies conditions (1a)-(2a) It is bounded for Re k >0 due to the final number of k_{nl} (we remind also that all the eigen values k_{nl} are simple (non-multiple) and are situated outside the real axis k) and, if γ = Re γ, it has the absolute value equal to 1 for Re k >0.

The product

$$\prod_n \frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda \frac{k_\lambda - k}{k_\lambda + k}\prod_s \frac{k_s-k}{k_s+k}\prod_{s'}\frac{k_{s'}-k}{k_{s'}+k} \qquad (I,2)$$

contains all the poles and the zeros of $S_l(\gamma,k)$ in D^+, has no other singularities in D^+, satisfies conditions (1a)-(2a), is regular in D^+ in the case of convergence. It is bounded for Re k >0 due to conditions (10a), (10b) and, if γ = Re γ, it has the absolute value equal to 1 for Re k >0.

The function

$$J_{lN}(k) = \tilde{S}(\gamma,k)\left[\frac{\prod_n \frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda^{N_1} \frac{k_\lambda - k}{k_\lambda + k}}{\prod_s^{N_2} \frac{k_s-k}{k_s+k}\prod_{s'}^{N_3}\frac{k_{s'}-k}{k_{s'}+k}}\right]^{-1}, \qquad (I,3)$$

with $\tilde{S}(\gamma,k)$ = $e^{2i\alpha k}\ S(\gamma,k)$, for finite numbers N = N_1+N_2+N_3 is regular and bounded on the real axis k. If the limit

$$J_l(k) = \lim_{N\to\infty} J_{lN}(k)$$

exists, then it has the same properties, and then

$$S_l(\gamma,k) = e^{-2i\alpha k}\prod_n \frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \cdot\prod_s \frac{k_s-k}{k_s+k}\prod_{s'}\frac{k_{s'}-k}{k_{s'}+k}. \qquad (I,4)$$

To be certain of the validity (correctness) of (I,4), it is necessary to show that three infinite products in (I,4) converge. The condition for their absolute convergence is the convergence of the sum

$$\sum_\lambda [\left|\frac{k_\lambda-k}{k_\lambda+k}-1\right| + \sum_s\left|\frac{k_s-k}{k_s+k}-1\right| + \sum_{s'}\left|\frac{k_{s'}-k}{k_{s'}+k}-1\right| = 2|k|\{\sum\frac{1}{|k_\lambda+k|}+\sum_s\frac{1}{|k_s+k|}+\sum_{ss'}\frac{1}{|k_{s'}+k|}\}. \qquad (I,5)$$

In turn, convergence of (I,5) is determined by the convergence of the sum

$$\sum_\lambda \frac{1}{|k_\lambda|}+\sum_s\frac{1}{|k_s|}+\sum_{s'}\frac{1}{|k_{s'}|} \qquad (I,6)$$

because $|k_\lambda| \to \infty$ as $\lambda \to \infty$, $|k_s| \to \infty$ as $s \to \infty$ and $|k_{s'}| \to \infty$ as $s' \to \infty$. It is easy to see that sum (I,6) converges if the analyticity of the function

$$\tilde{J}_l(k) = \frac{\tilde{S}_l(\gamma,k)}{\prod_n \frac{k_{nl}+k}{k_{nl}-k}} \qquad (I,7)$$

in D^+ is taken into account together with the absence of its zeros above the real axis k and if the following theorem is used.

Theorem [18]. *Let a function f (z) be bounded and analytic for Re z ≥ 0, and let its zeros in the right half-plane z be* $r_1{}^{\theta_1}$, $r_2e^{\theta_2}$,... *Then the series* $\sum_{n=1}^\infty r_n^{-1}\cos\theta_n$ *converges.*

Because $\cos\theta_n = |\cos\theta_n| \ge \varepsilon$, where $\varepsilon \ne 0$, for $\tilde{J}_l(\rho)$ with $\rho = ik$, we have

$$\varepsilon\sum_{n=1}^\infty r_n^{-1} < \sum_{n=1}^\infty r_n^{-1}\cos\theta_n < \infty,$$

which proves that sum (I,6) converges. Hence, the infinite products in (I,4) converge uniformly and give a meromorphic function with poles $-k_\lambda$, $-k_s$ and $-k_{s'}$.

Now let consider the behavior of $S(\gamma,k)$ when $k\to 0$. From the symmetry conditions (2a)-(3a) and the finiteness of the wave function $R_l^{(+)}(\gamma,k,r)$ at point k = 0 (outside the interaction sphere) it does almost evidently follows that $S(\gamma,0)$ =1. Indeed, for rapidly decreasing potential tails ($r > a$) of the type (5) the simple analysis of eq. (8) shows that the behavior of the functions $f_{l\pm}(k,r)/kr$ at $k\to 0$ is defined by the behavior of $h_l^{(1,2)}(kr) = j_l(kr) \pm in_l(kr)$, i.e. by $n_l(kr) \underset{k\to 0}{\longrightarrow} -\frac{(2l-1)!!}{(kr)^{l+1}}$ (to within a constant). So, the behavior of the wave function $R_l^{(+)}(\gamma,k,r) = \frac{i}{2kr}\left[\begin{matrix} f_{l-}(k,r)\exp(il\pi/2) \\ -S_l(\gamma,k)f_{l+}(k,r)\exp(-il\pi/2)\end{matrix}\right]$ at $k\to 0$ is defined by the behavior of the expression $\frac{i}{2}\{-ij_l(kr)[1+S_l(\gamma,k)] - n_l(kr)[1-S_l(\gamma,k)]\}$ from which, taking into account the divergence of $n_l(kr)$ at $k \to 0$, it does immediately follow that $S(\gamma,0)$ =1 (and this does automatically satisfies the symmetry conditions (2a)-(3a) at the point k = 0) and also for small k the evaluation $S(\gamma,k)$ =1+ $o(k^{l+1})$ is, at least, valid to the function $J_l(k)$, we note that it is not only analytical in D^+ but being an entire function without zeros it can be written in the form exp $(u + iv)$ where $u + iv$ is an entire function (see, for example, the relevant theorem in [6]). The real function $u(k)$ must be negative in D^+ due to equality (21) and be positive in D^- due to the conditions (4) and (5). Therefore, according to the Caushy-Rieman equations the condition

$$0 = \partial u/\partial lmk = -\partial v/\partial k, lmk = 0, \qquad (I,8)$$

must be satisfied over the real axis k. From (I,8) it follows that the function $v(k)$ increases monotonically and takes any real value not more than once. Then the function u +

iv takes any imaginary value not more than once and consequently must be a linear function of k :

$$u + iv = 2i\alpha_1 k + \alpha_2. \quad (I,9)$$

Obviously, α_1 = 0 and, due to the equality S_l (0)= 1, α_2 =0.

Thus, we finally obtain:

$$S_l(\gamma,k) = e^{-2i\alpha k}\prod_n \frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda \frac{k_\lambda - k}{k_\lambda + k}\prod_s \frac{k_s - k}{k_s + k}\prod_{s'} \frac{k_{s'} - k}{k_{s'} + k}, \quad (I,10)$$

with $\alpha = a - \alpha_1 \le a$, which is just namely (23a). In a similar way we can obtain (23b).

Appendix II

The translation from the Ukrainian publication [6], dedicated to the memory of Yu.V.Tsekhmistrenko- see also [12]: Necessary and sufficient conditions for the existence of the "redundant" poles $(m/2)ib$ with b>0, m=1,2,…, in $f_0(k,r)$, and hence also in $S_0(k)$.

Using the Jost equation

$$f_0(k,r) = \exp(ikr) + k^{-1}\int_r^\infty \sin k(r'-r)V(r')f_0(k,r')dr' \quad (II,1)$$

and solving it formally by the method of successive approximations, we obtain the sum

$$f_0(k,r) = \sum_{\nu=0}^{\infty} f_{0\nu}(k,r), \quad (II,2)$$

where

$$f_{00}(k,r) = \exp(ikr), \quad (II,3)$$

$$f_{0\nu}(k,r) = [\exp(ikr)](2ikr)^\nu \int_r^\infty \{\exp[2ik(r_1 - r)] - 1\}V(r_1) \int_{r_{\nu-1}}^\infty \{\exp[-2ik(r_\nu - r_{\nu-1})]\}V(r_\nu)dr_\nu ... dr_1, \quad \nu = 1,2,.... \quad (II,4)$$

for the cases when

$$|V| < M / r^{2+\delta}, M < \infty, \text{and } \delta > 0 \text{ [19]}. \quad (II,5)$$

Further, we use the following theorems:

Theorem A [18]. *Let F(k,r) be a function of complex variables k and r which is definite and continuous for all values of k in some domain D and for all values of r over the contour C. Then the function*

$$\Phi(k) = \int_C F(k,r)dr$$

is an analytical function of k in the domain D. When the contour C is infinite, the uniform convergency of the integral is also necessary.

Theorem B [18]. *Let all the functions of the series $u_1(z)$, $u_2(z)$,... be analytic functions of z in the domain D and the sum $\sum_{n=1}^{\infty} u_n(z)$ be uniformly convergent in every domain D′, inside D. Then the function u(z)= $\sum_{n=1}^{\infty} u_n(z)$ is an analytic function of z inside D.*

It follows from (II,4) and theorem A that $f_{0\nu}(k,r)$ (ν=0,1,2,...) are analytical functions of k in the upper half-plane. If Born series (II,2) is uniformly convergent, then by theorem B, the function $f_0(k,r)$ is analytic in the upper half-plane. Similarly, $f_0(-k,r)$ is analytic in the lower half-plane. Moreover, if $f_{0\nu}(k,r)$, $\nu \ge 1$, is analytic in the lower half-plane, then all successive terms are also analytic.

And now show that the following theorems holds.

Theorem 3. *If $f_{0\nu}(k,r)$ (ν > 1) has singular points, then $f_{01}(k,r)$ has to have them also.*

Indeed, let $f_{01}(k,r)$ be analytic everywhere. Then all successive terms are also analytic everywhere. This contradiction proves theorem 3.

We use the analytic structure of $f_{01}(k,r)$ in the upper half-plane. Obviously, the problem can be reduced to studying the analytical structure of the integral

$$I = \int_r^\infty [\exp(-2ikr')]V(r')dr'$$

because all other terms give functions which are analytic in the whole plane.

As shown in [18,19], in the case of a potential

$$V(r) = P_n(r)\exp(-br), \quad (II,6)$$

where P_n (r) is an nth-order polynomial and b>0, the function $f_0(-k,r)$ has poles of an order not higher than n+1 at the points ib/2, ib, 3ib/2,..., and it is analytic at all other points of the complex plane.

We now show that if $f_{01}(-k,r)$ has a pole of the order not higher than n+1 at the point ib/2, then the potential must have a term of type (II,6). It follows from such assertion that the integral

$$I_1 = \int_r^\infty [\exp(-2ikr')]V_1(r')dr', \quad (II,7)$$

where $V_1 = V - V_2$ (where term V_2 does not give poles), can be represented on the real axis k in the form

$$I_1 \equiv \sum_{\mu=0}^{n} \frac{\varphi_\mu(-2ik,r)}{(2ik+b)^\mu}, \quad (II,7a)$$

where φ_μ ($-2ik$, r) is analytic at all the poles and is nonzero at the point ib/2. Further, rewriting I_1 in the form

$$I_1 = \int_r^\infty [\exp(-(2ik+b)r')][V_1(r')\exp(br')]dr'$$

and successively integrating by parts, we can transform the right-hand side of identity (II,7a) into the series:

$$-\exp(-2ikr)V_1 - \frac{1}{2ik+b}\exp(-2ikr)(bV_1 + \frac{dV_1}{dr}) - \frac{1}{(2ik+b)^n}\exp(-2ikr)\,(b^n V_1 + nb^{n-1} + \ldots + \frac{d^n V_1}{dr^n}) \equiv \sum_{\mu=0}^{n}\frac{\varphi_\mu(-2ik,r)}{(2ik+b)^\mu}. \quad (II,8)$$

Comparing the coefficients of equal powers of $2ik + b$, we obtain a successive system of the corresponding equalities. Because there is no term containing $\frac{1}{(2ik+b)^m}$, $m>n$, in the right-hand side of (II,8), we obtain

$$b^{n+1}V_1+(n+1)b^n\frac{dV_1}{dr}+\frac{(n+1)n}{2}$$
$$-b^{n-1}\frac{d^2V_1}{dr^2}+\ldots+\frac{d^{n+1}V_1}{dr^{n+1}}=0,$$

whence follows

$$V=P_n(r)\exp(-br)+V_2, \qquad \text{(II,9)}$$

where V_2 is an arbitrary function which cannot be brought to the form $P_s(r)\exp(-b_s r)$.

The following general theorem can be proved analogously.

Theorem 4. *For $f_0(-k,r)$ to have poles of the order not higher than (n_1+1) at the points $ib_1/2$, ib_1, $3ib_1/2$,..., not higher than (n_2+1) at the points $ib_2/2$, ib_2, $3ib_2/2$,..., not higher than (n_m+1) at the points $ib_m/2$, ib_m, $3ib_m/2$,..., it is necessary and sufficient that the corresponding potential would have a term $\sum_{m,n_m} P_{n_m}(r)\exp(-b_m r)$.*

Obviously, to have *essentially singular points*, it is necessary and sufficient that the corresponding potential contain a term $X(r)\exp(-br)$, where $X(r)$ is an uniformly convergent infinite series of the type $\sum_{n=0}^{\infty}\alpha_n r^n$ not equal to $\exp(\text{const } r^{\alpha})$, $0<\alpha<\infty$.

Investigating the behavior of I on the axis $k=ib/2$ in the case where

$$V(r)=v(r)\exp(-br),$$

where $v(r)$ to be an arbitrary function not having a factor $\exp(\text{const } r^{\alpha})$, $\alpha\ge 1$, one can easily conclude that the branch points can appear on that axis. One of the simplest cases is the potential $[\exp(-br)\sin(cr)]/r^q$. For various $q>0$, this potential can give branch points of different types at $k=ib/2\pm c/2$.

If $I(r)=v(r)\exp(-br^{\alpha})$ and $\alpha>1$ is an integer, then I and hence $f_0(k,r)$ are analytic functions in the whole plane.

Thus, the presence of the factor $\exp(-br)$ leads to the function $f_0(-k,r)$ becoming non-analytic in the upper half-plane.

Appendix III

The generalization of the results of Appendix I for the presence of the centrifugal barrier with $l>0$ outside the unknown interaction.

Let be at the external region (with $r\ge a$) the centrifugal barrier $\frac{\hbar^2 l(l+1)}{2\mu r^2}$, $l>0$, and a potential of the type $V=V_0 P_n(r)\exp(-br)$, where $P_n(r)$ is an n-th-order polynomial and $b>0$. Adopting from [12 and 14], we can write the integral equation

$$f_{l-}(k,r)=f_{0-}(k,r)+l(l+1)\int_r^{\infty}G(k;r,r')(r')^{-2}f_{l-}(k,r')dr', \qquad \text{(III,1)}$$

where $r>a$ and the Green's function has the form

$$G(k;r,r')=(2ik)^{-1}\begin{bmatrix} f_{0-}(k,r)f_{0+}(k,r') \\ -f_{0-}(k,r')f_{0+}(k,r)\end{bmatrix}$$
$$[f_{0+}(k,0)]^{-1}\begin{bmatrix}\Phi(k,r)f_{0+}(k,r') \\ -\Phi(k,r')f_{0+}(k,r)\end{bmatrix}. \qquad \text{(III,2)}$$

Because the function $\Phi(k,r)=\frac{1}{2ik}\begin{bmatrix} f_{0-}(k,0)f_{0+}(k,r) \\ -f_{0+}(k,0)f_{0-}(k,r)\end{bmatrix}$ is regular everywhere, the Green's function in eq.(II,2) has no singularity for any finite k and equation (II,1) allows computing $f_{l-}(k,r)$ from $f_{0-}(k,r)$ in a univalent way. The solution $f_{l-}(k,r)$ of equation (II,1) for any l contains the same isolate singularities at the same points as the function $f_{0-}(k,r)$. For instance, (1) if for $r>a$ there is a centrifugal barrier and a potential decreasing more rapidly than any exponential function, in that case $f_{l\pm}(k,r)$ are analytical in all the plane k except the points $k=0$ and $k=\infty$ as well as $f_{0\pm}(k,r)$; (2) if the external potential tail contains a term $V_0\sum_{m,n_m}P_{n_m}(r)\exp(-b_m r)$, the functions $f_{l\pm}(k,r)$ for any $l\ge 0$ are analytical in all the plane k except the points $k=0$ and $k=\infty$ and also have the poles of the order not higher than (n_1+1) at the points $ib_1/2$, ib_1, $3ib_1/2$,..., not higher than (n_2+1) at the points $ib_2/2$, ib_2, $3ib_2/2$,..., not higher than (n_m+1) at the points $ib_m/2$, ib_m, $3ib_m/2$,...; (3) if the external potential tail contains a term $V_0\,[(br)^{-1}\exp(-br)]$, $V_0>0$, $b^{-1}\sim a$, the functions $f_{l\pm}(k,r)$ for any $l\ge 0$ have the logarithmic singularity at the point $k_\gamma=ib/2$.

Appendix IV

The derivation of the formula (32).

We rewrite expression (24a) for $f_{0-}(k,r)$ inside the circle γ_c around the point $k_\gamma=ib/2$ in the form

$$f_{0-}(k,r)\to\lim_{k\to k_\gamma}\left\{\begin{matrix}1+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1} \\ \int_b^{\infty}db'\frac{\exp(-b'r)}{b'(b'\mp 2ik)}\end{matrix}\right\}\exp(\pm ikr)$$
$$=\lim_{\eta\to 0}\left\{\begin{matrix}1+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1} \\ \int_0^{\infty}d\tilde{b}\frac{\exp[-(\tilde{b}+b)r]}{(\tilde{b}+\eta)(\tilde{b}+b)}\end{matrix}\right\}\exp(-ikr), \qquad \text{(IV,1)}$$

where we introduce the variables $\tilde{b}=b'-b$, $\eta=2ik+b$. We then make the following simple transformations of the right-hand part of (IV,1):

$$f_{0-}(k,r)\to\lim_{\eta\to 0}\left\{\begin{matrix}1+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1} \\ \cdot\int_0^{\infty}d\tilde{b}\frac{\exp(-\tilde{b}r)\exp(2ikr)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)}\end{matrix}\right\}\exp(\pm ikr)$$
$$=\exp(-ikr)+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\cdot\int_b^{\infty}d\tilde{b}\frac{\exp(-\tilde{b}r)}{\tilde{b}(\tilde{b}-2ik)}\exp(ikr) \qquad \text{(IV,2)}$$

$$+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}$$

$$\cdot\lim_{\eta\to 0}\int_0^b d\tilde{b}\frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)}\exp(ikr)$$

$$=\exp(-ikr)+\frac{A_-}{A_+}[f_{0+}(k,r)-\exp(ikr)]$$

$$+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}$$

$$\lim_{\eta\to 0}\int_0^b d\tilde{b}\frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)}\exp(ikr),$$

where $A_{\mp}=[1\pm\frac{i\rho}{2k}\ln(1\pm\frac{2ik}{b})]^{-1}$ and we use definition (24a) for f_{0+} (k,r).

We now analyze the last integral $J=\lim_{\eta\to 0}\int_0^b d\tilde{b}\frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)}$ in the last right-hand part of (IV,2), using formulas (3.352.1) and (8.214.1) from [21]:

$$J=\lim_{\eta\to 0}\int_0^b d\tilde{b}\frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)}$$

$$=\lim_{\eta\to 0}\frac{1}{b-\eta}\int_0^b d\tilde{b}\exp(-\tilde{b}r)[\frac{1}{\tilde{b}+\eta}-\frac{1}{\tilde{b}+b}]$$

$$=\lim_{\eta\to 0}\left\{\begin{array}{l}\frac{1}{b-\eta}\exp(hr)[\mathrm{Ei}(-br-hr)-\mathrm{Ei}(-hr)]\\ -\frac{1}{b-\eta}\exp(br)[\mathrm{Ei}(-2br)-\mathrm{Ei}(-br)]\end{array}\right\}\quad (IV,3)$$

$$=\lim_{\eta\to 0}\frac{1}{b}[-\ln(hr)+X(h,r)],$$

where $X(\eta,r)=\sum_{k=1}^{\infty}\frac{(-br+\eta r)^k}{k\cdot k!}-\sum_{k=1}^{\infty}\frac{(-\eta r)^k}{k\cdot k!}-\exp(br)[\sum_{k=1}^{\infty}\frac{(-2br)^k}{k\cdot k!}-\sum_{k=1}^{\infty}\frac{(-br)^k}{k\cdot k!}]$ is analytic function of $\eta=b+2ik$ at the point $\eta=0$ and in a small circle $|\eta|<b$ and the function Ei is defined in [21]. Further we can obviously rewrite (IV,3) as

$$J=\frac{1}{b}\left\{1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\frac{2ik}{\rho}+Z(k,r)\right\},\quad (IV,3a)$$

where $Z(k,r)=X(\eta,r)-\ln(br)-\frac{2ik}{\rho}$ is an analytical function of k at the point $k=k_\gamma=ib/2$ and in a small circle $|k|<b/2$.

Using (IV,3a), we continue the transformations of (IV,2):

$$f_{0-}(k,r)\lim_{k\to k_\gamma}\exp(-ikr)+\frac{A_-}{A_+}\left[\begin{array}{l}f_{0+}(k,r)\\ -\exp(ikr)\end{array}\right]$$

$$+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\qquad (IV,2a)$$

$$\cdot\lim_{\eta\to 0}\exp(ikr)\frac{1}{b}\left\{\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]\frac{2ik}{\rho}+Z(k,r)\right\}$$

$$=W(k,r)+\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}U(k,r),$$

where $W(k,r)=\exp(-ikr)+\frac{2ik}{b}\exp(ikr)$ and $U(k,r)=[1-\frac{i\rho}{2k}\ln(1-\frac{2ik}{b})]^{-1}f_{0+}(k,r)+\exp(ikr)\{\frac{\rho}{b}Z(k,r)-[1-\frac{i\rho}{2k}\ln(1-\frac{2ik}{b})]\}$ are analytic functions of k at the point $k=k_\gamma=ib/2$ and in a small circle $|k|<b/2$. From (IV,2a) the formula (32) can be obtained.

Appendix V

A possibility of sharp enhancement of $S^j_{l'l}$ ($l'\neq l$) in comparison with $S^j_{ll}(\gamma,k)$ near an isolated resonance.

Let assume that a factor like

$$S^j_{ll'}(\gamma,k)\approx\frac{\delta_{ll'}(E-E_t^{(ll)})+i\Gamma_t^{(l'l)}/2}{E-E_s+i\Gamma_s/2},l',l=0,1,\quad (V,1)$$

where $E_t^{(ll)}=\frac{\hbar^2\left|k_t^{(ll)}\right|^2}{2\mu}$, $E_s=\frac{\hbar^2|k_s|^2}{2\mu}$, $\Gamma_t^{(ll)}=-2ik\,\mathrm{Im}\,k_t^{(ll)}$, $\Gamma_t^{(00)}=-\Gamma_t^{(11)},\Gamma_s=2ik\,\mathrm{Im}\,k_s$, plays an essential role in (64) in some energy region. If $E_t^{(ll)}\approx E_s$, $\Gamma_{1s}^2+\Gamma_{2s}^2=\Gamma_s^2$, where $\Gamma_{1s}^2=(\Gamma_t^{(ll)})^2$, $\Gamma_{2s}^2=(\Gamma_t^{(l'l)})^2$ for $l'\neq l$, (46) is fulfilled and the scattering cross section is

$$\sigma_{el}\approx\frac{\pi}{k^2}\frac{(\Gamma_s-\Gamma_{1s})^2/4+\Gamma_{2s}^2/4}{(E-E_s)^2+\Gamma_s^2/4},\quad \text{small } k.\quad (V,2)$$

When $\Gamma_{1s}<<\Gamma_{2s}\approx\Gamma_s$ it may happen a sharp enhancement of $S^j_{l'l}$ ($l'\neq l$) in comparison with $S^j_{ll}(\gamma,k)$ at the resonance. In the extreme case in which $\Gamma_{1s}=0$, a resonance of $S^j_{l'l}$ ($l'\neq l$) corresponds to an anti-resonance of $S^j_{ll}(\gamma,k)$. Therefore, the influence of non-central or parity-violating interactions in these resonance regions may be essential even if their strength is very small (but, of course, non-zero).

3. To the Space-Time Description of Cross Sections and Durations of Neutron-Nucleus Scattering Near 1-2 Resonances in the C- and L-systems

3.1. The Pre-History of the Problem.

It was found in [1-7] the phenomenon of time advance instead of expected time delay in the C-system. This phenomenon is usually accompanied by a cross section minimum for almost the same energy. Then naturally the question had arisen if this advance manifested also in the L-system?

Then in [8,9,10] it was found that the standard formulas of cross section transformations from the L- to C- system are inapplicable in the cases of two (and more) collision mechanisms. Usually the delay-advance phenomenon appears for nucleon-nucleus elastic scattering near a resonance, distorted by the non-resonant background, in the *C*-system. Usually (see, for instance, [1,2,3]) the amplitude $F_C\ (E,\theta)$ for the elastic scattering of nucleons by spherical nuclei near an isolated resonance in the *C*-system can be written as

$$F_C(E,\theta) = f(E,\theta) + f_{l,\,res}(E,\theta), \quad (1)$$

where

$$f_{l,res}(E,\theta) = (2ik)^{-1}(2l+1)$$
$$P_l(\cos q)[\exp\,(2id_l)\frac{E-E_{res}-i\Gamma/2}{E-E_{res}+i\Gamma/2}-1],$$
$$f(E,\theta) = (2ik)^{-1}\sum_{\lambda\neq l}(2l+1)P_l(\cos\theta)[\exp\,(2id_l)-1],$$

here E , E_{res} and Γ are the excitation energy, the resonance energy and the width of the compound nucleus, respectively; we neglect the spin-orbital interaction and consider a comparatively heavy nucleus.

Rewriting (1) in the form

$$F^C(E,\theta) = [A(E^*-E^*_{res})+iB\Gamma/2]$$
$$(E^*-E^*_{res}+i\Gamma/2)^{-1} \quad (1a)$$

where

$$A = f(E,\theta)+(k)^{-1}(2l+1)P_l(\cos\theta)\exp\,(i\delta_l{}^b)\sin\delta_l{}^b$$
$$B = f(E,\theta)+(ik)^{-1}(2l+1)P_l(\cos\theta)\exp(i\delta_l{}^b)\cos\delta_l{}^b$$

we obtain the following expression for the scattering *duration* $\tau^C(E,\theta)$:

$$t^C(E,\theta) = 2R/v + \hbar\,\vartheta\arg F/\vartheta E \equiv 2R/v + \Delta t^C(E,\theta) \quad (2)$$

in case of the quasi-monochromatic particles which have very small energy spreads $\Delta E<<\Gamma$. Formula (2) was obtained in [1]. In formula (2), $v=\hbar k/\mu$ is the projectile velocity, R is the interaction radius, and $\Delta\tau^C$ is

$$t^C(E,\theta) = -(\hbar\mathrm{Re}\alpha/2)[(E^*-E^*_{res}-\mathrm{Im}\alpha/2)^2$$
$$+(\mathrm{Re}a)^2/4]^{-1}+\Delta\tau_{res}, \quad (3)$$

with

$$t_{res} = (\hbar\Gamma/2)[(E^*-E^*_{res})^2+\Gamma^2/4]^{-1}, a = \Gamma B/A \quad (4)$$

From (3) one can see that, if $0<\mathrm{Re}\alpha<\Gamma$, the quantity $\Delta\tau(E,\theta)$ appears to be *negative* in the energy interval ~ $\mathrm{Re}\alpha$ around the center at the energy $E^*_{res}+\mathrm{Im}\alpha/2$. When $0<\mathrm{Re}\alpha/\Gamma<<1$ the minimal delay time can obtain the value $-2\,\hbar/\mathrm{Re}\alpha<0$. Thus, when $\mathrm{Re}\alpha\to 0^+$, the interference of the resonance and the background scattering can bring to *as much as desired large of the advance* instead of the delay! Such situation is mathematically described by the zero $E^*_{res}+i\alpha/2$, besides the pole $E^*_{res}-i\Gamma/2$, of the amplitude $F^C(E,\theta)$ (or the correspondent *T*-matrix) in the lower unphysical half-plane of the complex values for energy *E*. We should notice that a very large advance can bring to the problem of causality violation (see, for instance the note in [2]). The *delay-advance phenomenon* in the *C*-system was studied in [1-3] for the nucleon-nucleus elastic scattering.

For two overlapped resonances the amplitude for an elastic scattering can be written in center-of-mass system also in form (1):

$$F^C(E,\theta) = f(E,\theta) + f_{l,res}(E,\theta)$$

where

$$f(E,\theta) = f_{coul}(E,\theta)+(2ik)^{-1}$$
$$\sum(2\lambda+1)P_\lambda(\cos\theta)\left[\exp\left(2i\delta^b_\lambda\right)-1\right] \quad (5)$$

and already

$$f_{l,res}(E,\theta) = (2\,ik)^{-1}(2l+1)P_l(\cos\theta)\exp\left(2i\delta^b_l\right)$$
$$\left[\left(\frac{E-E_{res,1}-i\Gamma/2}{E-E_{res,1}+i\Gamma/2}\right)\left(\frac{E-E_{res,2}-i\Gamma/2}{E-E_{res,2}+i\Gamma/2}\right)-1\right] \quad (6)$$

we obtain the following expression for the total scattering *duration* $\tau^C(E,\theta)$

$$t^C(E,\theta) = 2R/v + \hbar\,\vartheta\arg F/\vartheta E \equiv 2R/v + \Delta t^C(E,\theta)$$

for the quasi-monochromatic particles which have very small energy spreads $\Delta E<<\Gamma$, when one can use *the method of stationary phase* for approaching the group velocity of the wave packet.

At Figure 1 and Figure 2 we can see the energy dependence of $\Delta t^C(E,\theta)$ for two couples of overlapped resonances in neutron-nucleus elastic scattering [7].

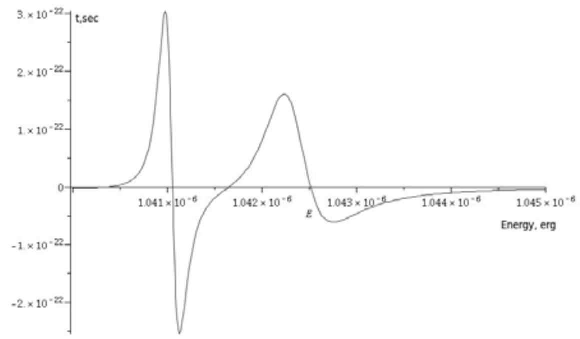

Figure 1. Energy dependence of $\Delta t^C(E,\theta)$ near two overlapped resonances 58Ni $E_1 = 649.8keV; \Gamma_1 = 0.168keV$ and $E_2 = 650.6keV;\ \Gamma_2 = 0.521keV$

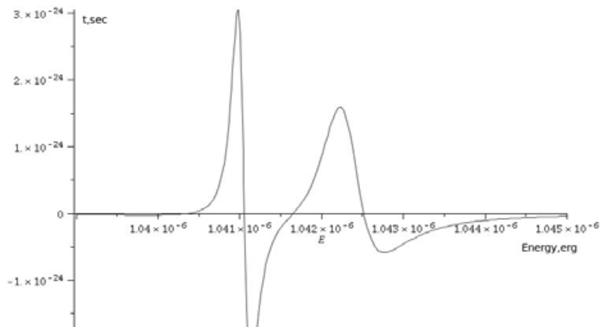

Figure 2. Energy dependence of $\Delta t^C(E,\theta)$ near two overlapped resonances 58Ni $E_3 = 745.6keV; \Gamma_3 = 0.7keV$ and $E_4 = 746{,}5keV; \Gamma_4 = 0{,}8keV$

3.2. The Collision-Process Diagram with 2 Mechanisms (Direct Process and Collision with the Formation of a Compound Nucleus)

In Figure 3 a, b these two processes in the L-system are pictorially presented. They represent a prompt (direct) and a delayed compound-resonance mechanism of the emitting ***y*** particle and ***Y*** nucleus, respectively. The both mechanisms are *macroscopically* schematically indistinguishable but they are *microscopically* different processes:

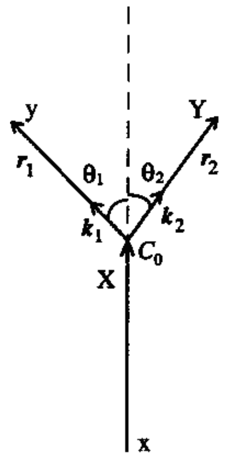

Figure 3. (a) Diagram of the direct process.

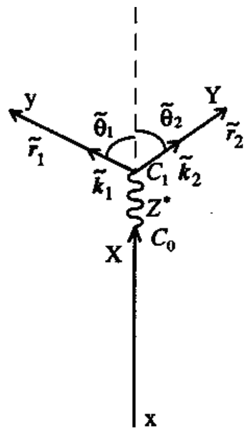

Figure 3. (b) Diagram of process with the compound nucleus

Figure 3 (a) represents the direct process of a prompt emission of the final products from the collision point C_0 with a very small time duration τ_{dir}, while Figure 3 (b) represents the motion of a compound-resonance nucleus Z from point C_0 to point C_1, where it decays by the final products ***y + Y*** after traveling a distance between C_0 and C_1 (which is equal to $\sim V_C \Delta\tau_{res}$) before its decay. Here V_C is the compound-nucleus velocity, equal to the center-of-mass velocity, and $\Delta\tau_{res} = \hbar\Gamma/2) / [(E_Z - E_{res,Z})^2 + \Gamma^2/4]$ is the mean time of the nucleus Z motion before its decay [8,9,10,11] for the case of one compound resonance, the energy spread ΔE of the incident particle ***x*** being very small in comparison with the resonance width Γ, $E_Z = E^*$, $E_{res,Z} = E^*_{res}$. For the clarity of the difference between both processes in time, we impose the evident practical condition

$$t_{dir} << \Delta\tau_{res}(E_Z) \text{ near } (E_Z - E_{res,Z})^2 \approx \Gamma^2/4 \qquad (7)$$

For the *macroscopically* defined cross sections, in the case of very large macroscopic distances r_1 (near the detector of the final particle ***y***) with very small angular and energy resolution ($\Delta\theta_1 << \theta_1$ and $\Delta k_1 << k_1$), the angles θ_1 and $\tilde{\theta}_1$, as well as momentums k_1 and $\tilde{k}_1$, can be considered as practically coincident. Really, $\theta_1 - \tilde{\theta}_1 \sim \Delta r_1/r_1$ and $k_1 - \tilde{k}_1 \sim \Delta r_1/r_1$ with $|\Delta r_1| = |r_1 - \tilde{r}_1|$. Using the *usual macroscopic definition of the cross section* with the help of some transformations for the exit asymptotic wave packet of the system ***y + Y***, in [4] it was obtained the following expression for the cross section σ of reaction (4) in the L-system:

$$\sigma = \sigma_0^{(incoh)} + \sigma_1^{(interf)}, \qquad (8)$$

where

$$\sigma_0^{(incoh)} \cong \left| f_{dir}^{(L)} \right|^2 + \frac{J_{C\to L} \left| \gamma_{Z\bullet}^{(C)} \right|^2}{(E_Z - E_{res,Z})^2 + \Gamma^2/4} \qquad (9)$$

$$f_{dir}^{(C)} = \frac{1}{2ik_1^C} \sum_{l' \neq l} (2i+1) P_{l'}(\cos\theta_1^C)(e^{2i\delta_i} - 1) \qquad (10)$$

$$\sigma_1^{(\text{interf})} = 2 f_{dir}^{(L)} \cdot \frac{J_{C\to L}^{1/2} \gamma_Z^{(C)}}{E_Z - E_{res,Z} + i\Gamma/2} \cos\phi \qquad (11)$$

$$\frac{\gamma_Z^{(L)}(E_1, E_2)}{E_Z - E_{res,Z} + i\Gamma/2} = f_{l,res}(E_1^C, \theta_1^C)$$
$$= \frac{\gamma_Z^{(L)}}{2ik_1^C}(2l+1)P_l(\cos\theta_1^C)\left\{ \exp(2i\delta_l)\frac{E^C - E_{res}^C - i\Gamma/2}{E^C - E_{res}^C + i\Gamma/2} - 1 \right\} \qquad (12)$$

$$\phi = \chi + \beta + \varphi,$$
$$\chi = \arg(J_{C\to L}^{1/2} \gamma_{Z\bullet}^{(L)}) - \arg(f_{dir}^{(L)}), \qquad (13)$$
$$\beta = \arg(E_Z - E_{res.Z}) + i\Gamma/2)^{-1}$$
$$\varphi = k_1 \Delta r_1 + k_2 \Delta r_2, \Delta r_{1,2} = V_{\perp(1,2)} \Delta\tau_{res},$$

$V_{1,2}$ is the projection of the Z-nucleus velocity to the direction of $\vec{k}_{1,2}$, δ_l is the l-wave scattering background phase shift. Formulas (8)-(11) were obtained for a quasi-monochromatic incident beam ($\Delta E << E$) and a very small angular and energy resolution ($\Delta\theta_1 << \theta_1$, $\Delta E << \Gamma$) of the final-particle detector.

For the simplicity we neglect here the spin-orbital coupling and we suppose also that the absolute values of all differences $r_n/v_n - r_p/v_p$ ($n \neq p = 1,2$) are much less than the time resolutions. Here $J_{C\to L}$ is the standard Jacobian of pure cinematic transformations from the C-system to the L-system.

We underline that formulas (8)-(13) for the cross section σ, obtained in [8,9,10,11] and defined by the usual *macroscopic* way, take into account a real *microscopic* motion of the compound nucleus. So, the formulas (8)-(13) differ from the standard kinematical transformation of $\sigma^C(E,\theta) = |F^C(E,\theta)|^2$ from the C-system into the L-system, considering only the kinematical transformations of the energies and angles from the C-system (with $\varphi = 0$) to the L-system. Such difference arises because the formal expression for $\sigma^C(E,\theta)$ as taken *without consideration* of the microscopic difference between the processes in Figure 3a and Figure 3b, and thus *without consideration* of the parameter $\varphi = k_1\Delta r_1 + k_2\Delta r_2$, $\Delta r_{1,2} = V_{\perp(1,2)}\Delta\tau_{res}$.

3.3. The lack of time advance near compound-resonances in the L-system

We underline that formulas (8)-(13) for the cross section σ, obtained here, are defined by the usual *macroscopic* way and also consider the real *microscopic* motion of the compound nucleus which strongly differ them from the standard cinematic transformation $\sigma^C(E,\theta)=|F^C(E,\theta)|^2$ from *C*-system into L-system namely by the interference of the amplitudes $f_{dir}^{(L)}$ and $\frac{J_{C\to L}^{1/2}\gamma_{Z_\bullet}^{(C)}}{E_Z - E_{res,Z} + i\Gamma/2}\cdot\exp(i\varphi)$, $\varphi = k_1\Delta r_1 + k_2\Delta r_2$ (where $\Delta r_{1,2} = V_{proj1,2}\,\Delta\tau_{res}$). The parameter φ reflects the influence of the compound-nucleus motion.

In the first my works (for instance, in [1-3]) usually the analysis of the amplitudes, cross sections and durations of the elastic scattering performed on the base of formulas (1) $\to$ (1a) in C-system, in which the compound-nucleus motion in L-system did not taken into account. But taking in account the motion of the decaying compound nucleus in L-system, the expressions for the amplitude of the collision process, which is going on with the formation of excited compound nucleus in the region of a resonance in C- and L-systems, differ not only by the standard cinematic transformations $\{E^C,\theta^C\}\leftrightarrow\{E^L,\theta^L\}$. It is necessary take into account also the motion of the decaying compound nucleus along the distance $V_C\Delta\tau_{res}$, as it was shown in Figure. 3 a, b. In [1,2,3] formulas (1) and (1a) were written in C-system and are described the coherent sum of the interfering terms for the both of cross section $\sigma^C(E,\theta) = |F^C(E,\theta)|^2$ and the time delay $\Delta\tau^C(E,\theta)$ without the microscopic motion of the decaying compound nucleus from point C_0 till point C_1. It is possible to evaluate the general duration of collision in L-system, taking the superposition of the wave packets of the direct scattering and of the scattering, going on with the formation of the intermediate compound nucleus (in the correspondence with diagrams 1a and 1b, respectively), which was obtained in [8], and in the asymptotic range (for $r\to\infty$) after all the simplifications, considering the conservation of energy-impulse, receives the form

$$\Psi^{r_1\to\infty} \approx \mathrm{const}\times\exp(-\mathrm{i}E_f^0 t/\hbar)\times\exp(\mathrm{i}k_1^0 r_1 + \mathrm{i}k_2^0 r_2)$$
$$\times\left\{\begin{aligned} & f_{dir}^{(L)}\times\exp\left[-\Delta E\left[\left(t-t_i-\frac{r_1}{V_1^0}\right)+\left(t-t_i-\frac{r_2}{V_2^0}\right)\right]/\hbar\right] \\ & +\frac{J_{C\to L}^{1/2}}{E_Z^\bullet - E_{res,Z} + i\Gamma_Z/2} \\ & \times\exp\begin{bmatrix} -\Delta E\left[\left(t-t_i-\Delta\tau_{\mathrm{res}}-\frac{\tilde r_1}{V_1^0}\right)\right. \\ \left.+\left(t-t_i-\Delta\tau_{\mathrm{res}}-\frac{\tilde r_2}{V_2^0}\right)\right]/\hbar \end{bmatrix} \\ & \times\exp\left[\mathrm{i}k_1^0\Delta r_1 + \mathrm{i}k_2^0\Delta r_2\right] \end{aligned}\right\} \quad (14)$$

$$for\ \left\{\begin{aligned} & t > t_i + r_1/V_1^0 \\ & t > t_i + \Delta t_{res} + \tilde r_1/V_1^0 \end{aligned}\right\}$$

where $V_{1,2}^0 = \hbar k_{1,2}^0/m_{1,2}$, $\Delta r_{1,2} = V_{\perp(1,2)}\,\Delta\tau_{res}$, $V_{\perp(1,2)}$ is the projection of the nucleus Z^* motion velocity on the $\boldsymbol{k}_{1,2}$ direction, t_i is the initial time moment, defined by the amplitude phase of the initial weight factor g_i, chosen for the simplicity in the Lorentzian form [const/($E_1 - E_1^0 + i\Delta E$)] with the very small of the energy spread $\Delta E <<\Gamma$; $E_l = \hbar^2 k_l^2/2m_l$ is the kinetic energy of the l-th particle with mass m_l (l=1,2), correspondent to particles y and Y, respectively. Then, utilizing the general approach from [12] for the mean collision duration

$$<\tau_{\mathbf{general}}> = \frac{\int_{t_{\min}}^{\infty} t\Psi^\bullet_{r_1\to\infty}\hat j_1\Psi_{r_1\to\infty}dt}{\int_{t_{\min}}^{\infty}\Psi^\bullet_{r_1\to\infty}\hat j_1\Psi_{r_1\to\infty}dt} - < t_{\mathbf{initial}} > \approx \hbar/2\Delta E \quad (15)$$

(with $<t_{initial}> \approx t_i$ for quasi-monochromatic particles), we obtain after all the simplifications, mentioned in [8] and utilized here, the result, which consists in that, that the general time delay corresponds to the time-energy uncertainty relation $<\tau_{general}>\Delta E\sim\hbar$ for quasi-monochromatic particles (for which $\Delta E <<\Gamma$ and $\Delta\tau_{res}\,\Delta E<<1$).

Thus, we obtain the trivial mean time delay in the approximation $\Delta E <<\Gamma$ and $\Delta\tau_{res}\cdot\Delta E<<1$ for *L*-system *without any advance, caused by "virtual unmoving" compound nucleus in C*-system. Formulas (8)-(13) are the result of the self-consistent approach to the realistic analyze of the experimental data on the cross sections of nucleon-nucleus scattering in L-system. And any attempt to describe the experimental data of the nucleon-nucleus-scattering cross sections near an isolated resonance, distorted by the non-resonance background, in L-system on the simple base of formula (1) in C-system with the further use of the standard cinematic relations $\{E^C,\theta^C\}\leftrightarrow\{E^L,\theta^L\}$ in L-system does not have any practical physical sense. And the reason of it is connected with that we neglect the real motion of the compound nucleus.

For the case of two overlapped resonances [13] we have to calculate the wave function quite similarly to the case of one resonance before:

$$\Psi_{r_1\to\infty}\approx 0,\ when\ t < t_i + \frac{r_1}{V_1^0}, t < t_i + \tau + \frac{\tilde r_1}{V_1^0},$$

$$\Psi_{r_1\to\infty}\approx const\cdot e^{-iE_f t/\hbar}e^{i\left(k_1^0 r_1 + k_2^0 r_2\right)}$$
$$*\left(\begin{aligned} & f_{dir}^L\exp\left[-\Delta E\left(\left(t-t_i-\frac{r_1}{V_1^0}\right)+\left(t-t_i-\frac{r_2}{V_2^0}\right)\right)/\hbar\right] \\ & +\frac{J_{C\to L}^{1/2}\gamma_Z}{\left(E_Z - E_{res,Z1} + i\Gamma_{Z1}/2\right)\left(E_Z - E_{res,Z2} + i\Gamma_{Z2}/2\right)}, \\ & \exp\left[-\Delta E\left[\left(t-t_i-\tau-\frac{\tilde r_1}{V_1^0}\right)+\left(t-t_i-\tau-\frac{\tilde r_2}{V_2^0}\right)\right]/\hbar\right] \\ & \exp\left[ik_1^0\Delta r_1 + ik_2^0\Delta r_2\right] \end{aligned}\right) \quad (16)$$

when

$$t > t_i + \frac{r_1}{V_1^0}, t > t_i + \tau + \frac{\tilde{r}_1}{V_1^0}$$

Here $V_{1,2}^0 = \hbar k_{1,2}^0 / m_{1,2}$, $\Delta r_{1,2} = V_{1,2}\, \Delta\tau_{res}$, where $V_{1,2}$ is the projection of the speed of nucleus Z^* on the vectors $\vec{k}_{1,2}$, t_i is initial moment of time.

To calculate the time of delay in the L-system we have to use this formula:

$$\left\langle \tau_{general} \right\rangle = \frac{\int_{t_{min}}^{\infty} t j_i dt}{\int_{t_{min}}^{\infty} j_i dt} - t_{initial} \approx \frac{\hbar}{4\Delta E} \quad (17)$$

where $j_i = \mathrm{Re}\left[\psi^+ \frac{\hbar}{im} \frac{\partial \psi}{\partial x} \right]$ is the initial current. So, if we will take into account the movement of the compound-nucleus *the advanced time vanishes* also here.

3.4. On Cross Sections of Neutron-Nucleus Scattering near a couple of Overlapped Compound-Nucleus Resonances in the C- and the L-system

We have calculated the excitation functions $\sigma(E)$ for the low-energy elastic scattering of neutrons by nuclei ^{52}Cr and ^{56}Fe and in the region of distorted isolated resonances E_{res}=50,5444 keV and Γ=1,81 keV, E_{res}=27.9179 keV and 0.71 keV, respectively. The values of the parameters for the amplitudes of the direct and resonance scattering *separately* in *C*-system for *l*=0 (and, naturally, without the Coulomb phases) in formulas (8)-(13) were selected with the help of the standard procedure. The fitting parameter χ was chosen to be equal to 0.68π or 0.948 π or 0.956 π or π, respectively.

The calculation results were obtained with the help of formulas (8)-(13) in the comparison with the experimental data, given from [14]. They are represented in Figure 4-Figure 7, respectively. And the results of calculations performed by the *standard cinematic formulas* from C- into L-system (i.e. by the formulas (8)-(13) but with $\varphi \equiv 0$, that is without diagram, depicted in Figure 3 b) are represented in Figure 4 a- Figure 7 a. One can see that for $\varphi \equiv 0$ the minima are not totally filled.

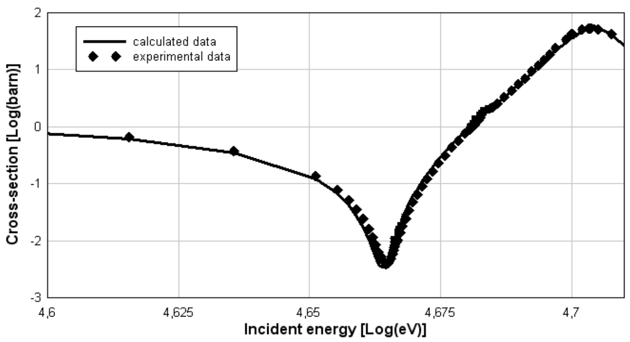

Figure 4. The excitation function for $^{52}Cr(n,n)$.

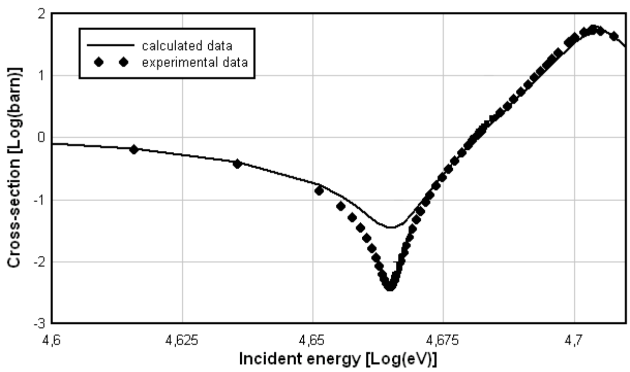

Figure 4. a. The excitation function for $^{52}Cr(n,n)$ with $\varphi \equiv 0$

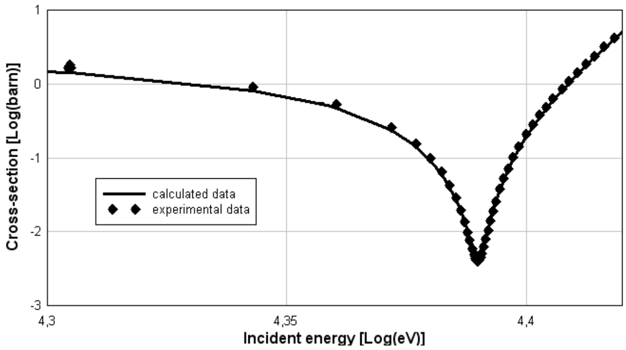

Figure 5. The excitation function for $^{56}Fe(n,n)$.

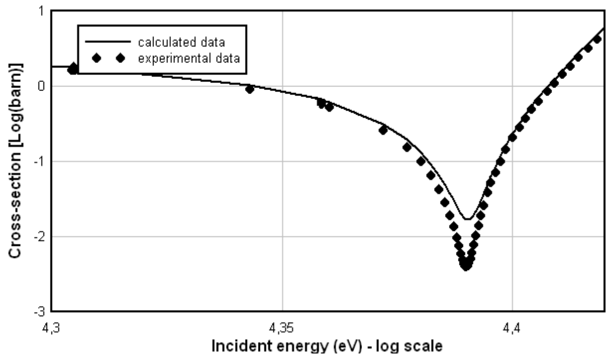

Figure 5. a. The excitation function for $^{56}Fe(n,n)$ with $\varphi \equiv 0$.

3.5. The Cross Sections of the Neutron-Nucleus Scattering with Two Overlapped Resonances

If we want to take into consideration the moving of the compound nucleus, we have to use another formula for cross section:

$$\begin{aligned} \sigma(\theta) &= \int dt \int dr_2 \psi^+_{r_1 \to \infty} \hat{j}_1 \psi_{r_1 \to \infty} \\ &\approx \int dt \int dr_2 \left(\psi_{r_1 \to \infty} \right)^2 \\ &= \sigma_{0(incoh)} + \sigma_{1(interf)} \end{aligned} \quad (18)$$

where

$$\sigma_0 = \left(f_{dir}^{(L)} \right)^2 + \frac{J_{C \to L} \left(\gamma_{Z^+}^{(C)} \right)^2}{\left(\begin{array}{c} \left(E_Z^+ - E_{res,Z1} \right)^2 \\ +\Gamma_{Z1}^2 / 4 \end{array} \right) \left(\begin{array}{c} \left(E_Z^+ - E_{res,Z2} \right)^2 \\ +\Gamma_{Z2}^2 / 4 \end{array} \right)} \quad (19)$$

$$\sigma_1 = 2\left(f_{dir}^{(L)} \frac{J_{C\to L}^{1/2}\gamma_{Z^+}^{(C)}}{\left(E_Z^+ - E_{res,Z1} + i\Gamma_{Z1}/2\right)\left(E_Z^+ - E_{res,Z2} + i\Gamma_{Z2}/2\right)} \right)\cos\Phi \quad (20)$$

We can calculate phase Φ the same way, as in the case with the one resonance.

Other values can be found this way:

$$f_{dir}^{(L)} = \sqrt{J_{C\to L}}\, f_{dir}^{(C)} = \sqrt{J_{C\to L}}\, f_b\left(E_1^C, \theta_1^C\right) \quad (21)$$

$$\frac{\gamma_{Z^+}^{(C)}\left(E_1, E_2\right)}{\left(E_Z^+ - E_{res,Z1} + i\Gamma_{Z1}/2\right)} = f_{l,res}\left(E_1^C, \theta_1^C\right) \quad (22)$$

$$\left(E_Z^+ - E_{res,Z1} + i\Gamma_{Z1}/2\right)$$

At Figure 6, 6a we can see theoretical function according to (18)-(22) and experimental data. The method of least squares was used to fit the function and experimental data. Experimental data where taken from [15]. After approximation we had such values of the parameters $\delta_i : \delta_0 = 2.88$, $\delta_1 = 5.59$, $\delta_3 = 4.1$, $\delta_4 = 2.34$, $\delta_5 = 2.6$, $\delta_6 = 4.75$.

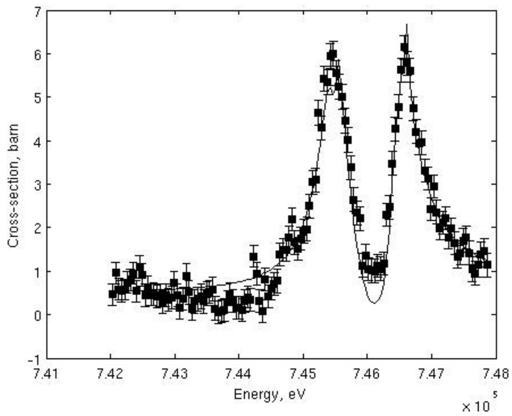

Figure 6. The excitation function for 58Ni near two overlapped resonances with $E_3 = 745.6keV; \Gamma_3 = 0.7keV$ and $E_4 = 746{,}5keV; \Gamma_4 = 0{,}8keV$

After approximation we had such values of the parameters δ_i : $\delta_0 = 3.72$, $\delta_1 = 0.51$, $\delta_2 = 3.01$, $\delta_3 = 3.13$, $\delta_4 = 3.17$, $\delta_5 = 0.43$, $\delta_6 = 3.13$.

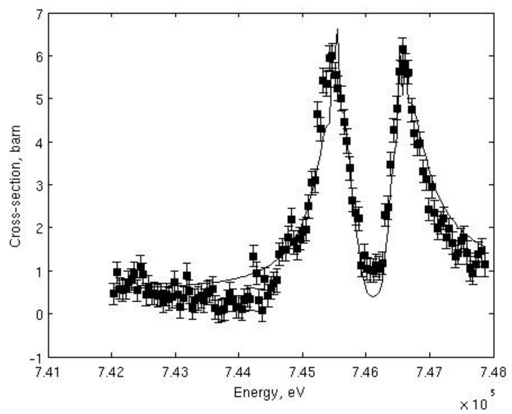

Figure 6. a. The excitation function for ^{58}Ni with $\varphi=0$ near two overlapped resonances with $E_3 = 745.6keV; \Gamma_3 = 0.7keV$ and $E_4 = 746{,}5keV; \Gamma_4 = 0{,}8keV$.

After approximation we had such values of the parameters δ_i : $\delta_0 = 3.72$, $\delta_1 = 0.51$, $\delta_2 = 3.01$, $\delta_3 = 3.13$, $\delta_4 = 3.17$, $\delta_5 = 0.43$, $\delta_6 = 3.13$.

3.6. The Space-Time Description of Direct and Sequential (Via Compound-Nucleus) Processes in the Laboratory System of Nuclear Reactions with 3 Particles in the Final Channel

We shall study the interference phenomena in the laboratory system when two particles are simultaneously detected (in a sense that will be specified below) in the nuclear reactions with three nuclei (particles) in the final channel.

The original idea was presented by Podgoretskij and Kopylov [19] for the two-particle emission (evaporation) from heavy nuclei. Here we consider the interference between prompt direct and delayed resonance processes in reaction of the type

$$x + X \to y + z + U \quad (23)$$

In Figure 7 a, b two possible mechanisms for reaction (1) are pictorially represented

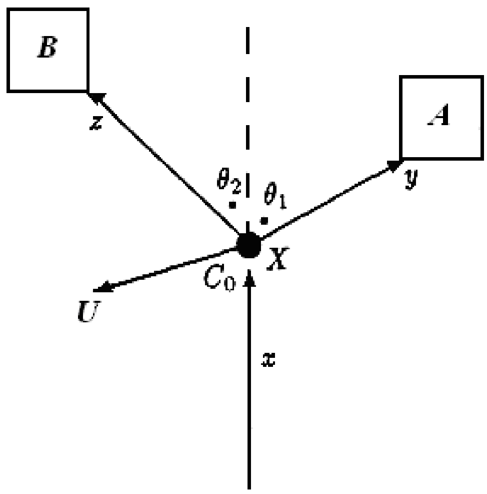

Figure 7. a. Direct process reaction channel

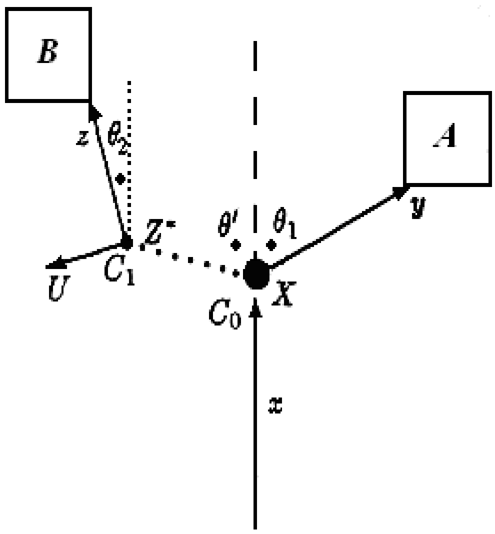

Figure 7. b. Sequential process reaction channel

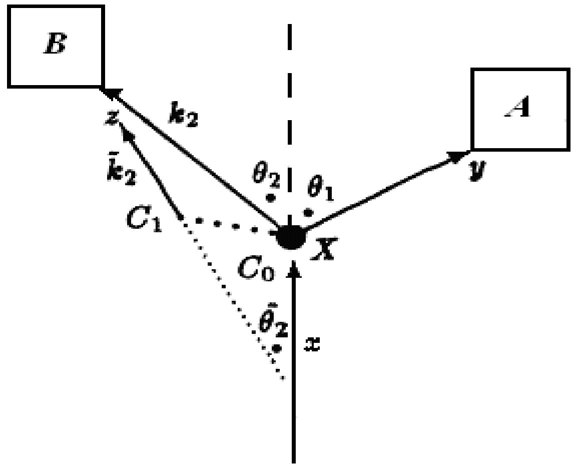

Figure 7. c. Simultaneous representation of direct and sequential processes

The symbols ***A*** and ***B*** enclosed in boxes stand for detectors located at macroscopic distances r_1 and r_2 from the scattering point C_0. In Figure 1a the direct (like quasi-free or so called one and two step direct) process of simultaneous prompt emission at point C_0 of all the three final particles is described. Figure 1 b presents delayed successive decay process with emission of particle *y* and formation of an intermediate excited nucleus *Z** which subsequently decays into *z* and *U* at point C_1, according the reaction

$$x+X \to y+Z^*, Z^* \to z+U \tag{24}$$

In Figure 8 c the superposition of the direct and the sequential emission of one from the final particle is displayed in the same picture. For macroscopic distances and under the condition specified below angles θ_2 and $\tilde{\theta}_2$ as well as impulses ***k*** and ***k*₂** can be considered practically coincident.

The asymptotic wave packet, near detectors ***A*** and ***B*** can be described by the following expression:

$$\begin{aligned}&\Psi_{ab}(r_1,r_2\to\infty)\\&\to C\times\int d\mathrm{k_x} g_i(\mathrm{k}_x)\int d\mathrm{k}_2 g_{f,2}(\mathrm{k}_2)\int d\mathrm{k}_1 f_{f,1}(\mathrm{k}_1)\\&\times\int d\mathrm{k}3\delta(E_i-E_f)\delta(\mathrm{K}_i-\mathrm{K}_f)\\&\left[\begin{array}{l} f_{dir}^{(L)}(E_1,E_2,E_3,\theta_1,\theta_2,\theta_3)e^{(\sum_{j=1}^{j=3} ik_j r_{jC_0}}+\\ \dfrac{f_{Z^*}^{(L)}(E_1,E_2,E_3,\theta_1,\theta_2,\theta_3)}{\varepsilon_Z^*-\varepsilon_{res,Z}+i\Gamma_Z/2}e^{(ik_1 r_{C_0}+\sum_{j=2}^{j=3} ikr_{jC_1})}\end{array}\right]e^{-iE_j i/\hbar}.\end{aligned}\tag{25}$$

In this equation *C* is a normalization constant, g_i, $g_{f,1}$, $g_{f,2}$ are amplitude weight factors describing the impulse spread of the incident particle x and that of the final particles y and z due to detectors resolution,

$$f_{dir}^{(L)}=\sqrt{J_{C\to L}}\, f_{dir}^{(C)} \tag{26}$$

and

$$f_{Z^*}^{(C)}=\sqrt{J_{R\to C}J_{C\to L}}\, f_{xy}^{(C)}\Gamma_Z^{(C)} \tag{27}$$

are the amplitudes for direct and sequential processes (the subscriptions *L* and *C* refer to laboratory and center of mass system, respectively), $f_{xy}^{(C)}$ and $\Gamma_Z^{(C)}$ being the amplitude of the first step direct process $x + X \to y + Z^*$ and the reduced-width amplitude of the decay process $Z^* \to z + U$ respectively; ε_Z^*, $\varepsilon_{res,Z}$ and Γ_Z are the excitation energy, the energy and total width of the resonant state of the nucleus Z^*; $J_{R\to C}$ and $J_{C\to L}$ are the Jacobians of the coordinate transformations from the Recoil system to the C-system and from the C-system to the L-system, respectively; r_{km} are the distances from points *m* ($m = C_0, Ci$) to particles *k* (with *k* = 1,2,3 corresponding to *y, z, U*); E_i, $\boldsymbol{k}_i$ and E_f, $\boldsymbol{k}_f$ the total energies and impulses in the initial and final channels respectively; $E_j = \hbar^2 k_j^2 / 2m_j$ is the kinetic energy of *j*-th particle, θ_j and $\boldsymbol{k}_j$ being the angle of motion (relative to beam, i.e. incident particle *x*, direction) and the wave vector of particle *j*, respectively. In expression (25) $\delta(E_i-E_f)$ and $\delta(\mathrm{K}_i-\mathrm{K}_f)$ take care of energy and impulse conservation. Expression (25) is written on the base of the general formalism described in [20] with application of the asymptotic stationary functions introduced in [16,17] and taking into account particle *U* explicitly. For the sake of simplicity the factor $r_{1C_0}^{-1} r_{2C_0}^{-1} r_{3C_0}^{-1}$ has been omitted as well as spin and internal coordinates.

The factor $e^{-iE_f t/\hbar}$ can be rewritten as

$$e^{-i(E_1+E_2+E_3)\frac{t}{\hbar}}e^{-iE_f'\frac{t}{\hbar}} \tag{28}$$

and the first three factors of the expression (25), combined with the factor (28), can be formally put in the integrals of eq. (25) as follows:

$$\int d\,\mathrm{k}_1 g_{f,1} e^{ik_1 r_{1m}-iE_1\frac{i}{k}}\ldots$$

$$\int d\,\mathrm{k}_2 g_{f,2i} e^{ik_i r_{2m}-iE_2\frac{i}{k}}\ldots$$

$$\int d\,\mathrm{k}_3 g_{f,3i} e^{ik_i r_{3m}-iE_3\frac{i}{k}}\ldots$$

In order to perform the previous integrals a transformation from variables $k_{1,2,3}$ to variables

$$y_{1,2,3}=\left(\frac{i\hbar t}{m_{1,2,3}}\right)^{1/2}\left(\mathrm{k}_{1,2,3}^0-\frac{m_{1,2,3}\mathrm{r}_{1,2,3}}{\hbar t}\right) \tag{29}$$

is useful. Here only projections of $\boldsymbol{k}_{1,2,3}$ over the mean vectors $\mathrm{k}_{1,2,3}^0 \equiv \langle \boldsymbol{k}_{1,2,3} \rangle$ are taken, the components of $\boldsymbol{k}_{1,2,3}$ remaining in other parts of (25). The factor $g_{f1,2}$ can be assumed to have the form

$$g_{f1,2}\approx\frac{c_{1,2}}{E_1-E_{1,2}^0-i\Delta E} \tag{30}$$

and ΔE to be very small ($\Delta E \ll \Gamma_Z$), as well as the energy spread of the incident particle *x*. Using a known result for a similar calculation (see, for instance, [21,22]), the wave function becomes

$$\Psi_{ab} \approx 0 \tag{31}$$

for

$$\begin{gathered} t < t_i + \frac{r_{1C_0}}{v_1^0},\ t < t_i + \frac{r_{2C_0}}{v_2^0},\ t < t_i + \frac{r_{3C_0}}{v_3^0}, \\ t < t_i + \tau + \frac{r_{2C_1}}{v_2^0},\ t < t_i + \tau + \frac{r_{3C_1}}{v_3^0} \end{gathered} \tag{32}$$

and

$$\begin{aligned} &\Psi \propto C \times e^{-iE_f^0 t/\hbar} \times e^{i\sum_j k_j^0 rC_0} \\ &\times \left[f_{dir}^L e^{-\Delta E\left[\begin{array}{l} (t - t_i - \frac{r_{1C_0}}{v_1^0}) \\ + (t - t_i - \frac{r_{2C_0}}{v_2^0}) \\ + (t - t_i - \frac{r_{3C_0}}{v_3^0}) \end{array} \right]/\hbar} + \frac{f_{Z^*}^L}{\varepsilon_Z^* - \varepsilon_{res,Z} + i\Gamma_Z/2} e^{-\Delta E\left[\begin{array}{l} (t - t_i - \frac{r_{1C_0}}{v_1^0}) \\ + (t - t_i - \frac{r_{2C1}}{v_2^0}) \\ + (t - t_i - \frac{r_{3C3}}{v_3^0}) \end{array} \right]/\hbar} \right] \\ &e^{ik_2^0 \Delta r_2 + ik_3^0 \Delta r_3} \end{aligned} \tag{33}$$

for

$$\begin{gathered} t > t_i + \frac{r_{1C_0}}{v_1^0},\ t > t_i + \frac{r_{2C_0}}{v_2^0},\ t > t_i + \frac{r_{3C_0}}{v_3^0}, \\ t > t_i + \tau + \frac{r_{2C_1}}{v_2^0},\ t > t_i + \tau + \frac{r_{3C_1}}{v_3^0} \end{gathered} \tag{34}$$

Here $v_{1,2,3}^0 = \hbar k_{1,2,3}^0 / m_{1,2,3}$, the initial time t_i is defined by the phase of the amplitude weight factor g_i; and the mean time τ of the nucleus Z^* motion before its decay is given by the well known expression:

$$\tau = \frac{\hbar \Gamma_Z / 2}{(\varepsilon_Z^* - \varepsilon_{res,Z})^2 + \Gamma_Z^2 / 4} \tag{35}$$

and

$$\Delta r_{2,3} = V_{\perp(2,3)} \tau, \tag{36}$$

$V_{\perp(2,3)}$ being the projection of the velocity of the nucleus Z^* onto the direction of $\boldsymbol{k}_{2,3}$. The energy spread for particle U is of the order ΔE , according to energy-impulse conservation.

Interference phenomena can occur only in case of simultaneous arrival (within the time resolution of the detectors) of particles y and z on A and B. The coincidence-rate intensity is described by a time integration of $\Psi_{ab}^* \hat{j}_1 \hat{j}_2 \Psi_{ab}$ ($\hat{j}_{1,2}$ being the flux probability density operator for particles y and z) over a time interval ΔT, which is great with respect to the time extension of the wave packets, and a spatial integration over particle U coordinates, i.e.:

$$\begin{aligned} P &\approx \int_{t_{\min}}^{\infty} dt \int_{r_{3\min}}^{r_{3\max}} dr_3 \Psi_{ab}^* \hat{j}_1 \hat{j}_2 \Psi_{ab} \\ &\propto \int_{t_{\min}}^{\infty} dt \int_0^{v_3^0 (t - t_i - \frac{r_{3C_0}}{v_3^0})} dr_3 \left| \Psi_{ab} \right|^2, \end{aligned} \tag{37}$$

where t_{min} is the smallest value among

$$t_i + \frac{r_{1C_0}}{v_1^0},\ t_i + \frac{r_{2C_0}}{v_2^0},\ t_i + \frac{r_{3C_0}}{v_3^0}, t_i + \tau + \frac{r_{2C_1}}{v_2^0}, t_i + \tau + \frac{r_{3C_1}}{v_3^0}$$

r_{3max} is the maximum between $v_3^0 (t - t_i - (\frac{r_{3C_1}}{v_3^0}))$ and $v_3^0 (t - t_i - \tau - \frac{r_{3C_1}}{v_3^0})$, $r_{3min} \to 0$ for ordinary small wave packets.

Under the standard experimental conditions, i.e. when

$$\Delta E \tau / \hbar << 1 \tag{38}$$

and

$$\delta t = \frac{r_l}{v_l^0} - \frac{r_m}{v_m^0} << \Delta T, (l, m = 1, 2, 3, l \neq m) \tag{39}$$

(ΔT is the time resolution of the coincidence scheme), it is possible to write

$$P = P_0 + P_1, \tag{40}$$

$$P_0 = \left| f_{dir}^L \right|^2 + \frac{\left| f_{Z^*}^L \right|^2}{(\varepsilon_Z^* - \varepsilon_{res,Z})^2 + \Gamma_Z^2 / 4} \tag{41}$$

and

$$P_1 = 2 \left| f_{dir}^L \frac{f_{Z^*}^L}{\varepsilon_Z^* - \varepsilon_{res,Z} + i\Gamma_Z / 2} \right| \cos\Phi \tag{42}$$

(in arbitrary units), where

$$\Phi = \delta + \beta + \phi \tag{43}$$

$$\begin{aligned} \delta &= \arg(f_{Z^*}^L) - \arg(f_{dir}^L), \\ \beta &= \arg(\varepsilon_Z^* - \varepsilon_{res,Z} + i\Gamma / 2)^{-1}, \\ \phi &= k_2^0 \Delta r_2 + k_3^0 \Delta r_3 \end{aligned}$$

$\Delta r_{2,3}$ being defined by (36).

The obtained results (40)-(43), with the incoherent sum P_0, the interference term P_1 and the phase Φ, do evidently generalize the results for the L-system, obtained somewhat earlier by us in [9,18] for collisions with two-particle channels. Comparing these results with that obtained in a stationary model [16,17], the latter ones are confirmed by the present self-consistent space-time approach in the limit $E <<\Gamma_{Z.}$ The same conclusion is valid for the cases in which two intermediate excited nuclei are formed, i.e.

$$x+X \to \begin{cases} y+Z^* \to y+z+U \\ z+Y^* \to z+y+U \end{cases},$$

under the conditions $\Delta E << \Gamma_y$, and $\Delta E << \Gamma_Z$.

Conclusions: The results (40)-(43) are firstly obtained in the space-time description of the interference between different (direct and sequential, containing the decaying compound-nucleus) mechanisms with three nuclei in the final channel. They are the clear generalization of the results for nucleon-nucleus and nucleus-nucleus collisions with two-particle channels, presented in [3,8], and can be easily generalized for the cases in which two intermediate excited (compound) nuclei are formed. Moreover, in the limit $\Delta E/\Gamma_Z \to 0$ they factually pass to the correspondent stationary-model results as presented in [16,17].

Finally, it is rather perspective and really topical to develop the much more complete approach to interference phenomena between the direct and various sequential processes in complex nuclear reactions.

3.7. Conclusions and Perspectives

Presented here time analysis of experimental data on nuclear processes permits to make the following conclusions and perspectives:

1. The simple application of time analysis of quasi-monochromatic scattering of neutrons by nuclei in the region of isolated resonances, distorted by the non-resonance background, brings in C-system to the delay-advance paradoxical phenomenon near a resonance in any two-particle channel. Such phenomenon of the time-transfer delay in the time advance is usually connected with a minimum in the cross section, or zero in analytic plane of scattering amplitude (apart from the resonance pole) near the positive semi-axis of kinetic energies in lower non-physical semi-plane of the Riemann surface. Here this paradox is eliminated by the thorough space-time analysis in L-system with moving C-system.

2. Moreover, it is also revealed that the standard formulas of transformations from L-system into C-system are in-suitable in the presence of two (and more) collision mechanisms – quick (direct or potential) process when the center-of-mass is practically not displaced in the collision and the delayed process when the long-living compound nucleus is moving in L-system. And revealed by our group the additional change of the amplitude phase in C $\to$ L transformations now agree with the elimination of the paradox of passing the usual time delay in the time advance. The obtained analytic transformations of the cross section from C-system into L-system are illustrated by the calculations of excitation functions for examples of the elastic scattering of neutrons by nuclei 52Cr, 56Fe and 58Ni near the distorted resonances in L-system.

3. The presented here results of time analysis for quasi-monochromatic nucleon-nucleus scattering near the isolated resonances, distorted by the non-resonance background, can be easily generalized to the scattering nucleons by nuclei near two-three overlapped resonances.

4. Of course, new formulas (8)-(13) and (18)-(22) can be also used for the improvement of the existing general methods of analyzing resonance nuclear data for the two-particle channels in nucleon-nucleus collisions in L-system and, moreover, can be generalized for more complex collisions.

5. Applying time analysis to elastic nucleon-nucleus with 2-3 overlapping compound-resonances, it is possible also to obtain the paradoxical phenomenon of transition decay in advance in C-system. But the behavior of amplitudes and durations can be certainly more complex than for an isolated resonance. Therefore the study of such cases can be more complicated that for an isolated resonance, and it has to be rather interesting and perspective.

7. It is rather interesting the perspective to apply the results of the space-time description of direct and sequential (via compound-nucleus) processes in the L-system of nuclear reactions with 3 particles in the final channel for concrete investigations, elaborations and calculations of many concrete nuclear collisions.

References

For General Introduction (pre-face):

[1] Bogolyubov N.N., private communication: *mathematics now become partly the range of the theoretical physics (for instance, the quantum collision theory).*

[2] Landau L.D., private communication: *a new good method in physics is better than any effect because the good method can bring us to some or even many new effects which can in a new way explain the experimental data*.

[3] V.S.Olkhovsky, *Time as a quantum observable, canonically conjugate to energy. Time analysis of quantum processes of tunneling and collisions (nuclear reactions)* [LAP LAMBERT Academic publishing, 177 pages, 2012].

For the Part I

[1] Heisenberg,W., Die “beobachtbaren Groessen” in der Theorie der Elementarteilchen, Zs.f. Phys. A Hadrons and Nuclei, 120(1943)513, 673-702.

[2] Heisenberg,W., Der mathamatische Rahman der Quantentheorie der Wellnfelder, Zs.Naturforsch. Teil A, 1(1946) 608.

[3] Hu, Ning, Application of the Heisenberg’s S-matrix theory for the resonance scattering and nuclear reactions, Phys.Rev., 74(1948)131-140.

[4] Van Kampen,N., The S-matrix and the causality condition. I. The electromagnetic field., Phys.Rev., 89(1953)1072-1079.

[5] Van Kampen,N., The S-matrix and the causality condition. II. Non-relativistic particles., 91 (1953)1267-1276.

[6] Olkhovsky, V.S. and Tsekhmistrenko,Yu.V., On the analytical structure of the S-function of elastic scattering in the case of infinite potentials, Ukrainian Physical Journal, 6(1961)149-156 (in Ukrainian, abstract in English; partially in English); see also Theor.and Math.Physics, 147(2006)539-551, Appendix 1, pp.547-549.

[7] Olkhovsky,V.S., To the study of properties of the unitary and non-unitary S-matrix on the base of caysality and the completeness conditions for wave functions, Teor.Matem.Fiz, 20(1974) 211-222.

[8] Nikolayev,M.V., and Olkhovsky,V.S., On the analytical Structure of the S-matrix for some classes of potentials, Teor.Mat.Fiz., 31(1977)214.

[9] Nikolayev,M.V., and Olkhovsky,V.S., On Completeness of Radial Equation Solutions and S-matrix Singularities in case of Complex Potentials, Lett.al Nuovo Cimento, 8(1973)703-707.

[10] Olkhovsky,V.S., and Zaichenko,A.K., Analytical properties and resonance structure of the S-matrix in case of non-central and parity-violating interactions, Nuovo Cim., A63(1981)155-170.

[11] Baldo,M., and Olkhovsky,V.S., Analytical properties of the S-matrix for interactions with Yukawa-potential tails, Theor. Mathem.Phys., 147 (2006)539-551

[12] Olkhovsky,V.S., On the analytic properties of the S-matrix for the unknown interactions surrounded by centrifugal and rapidly decreasing potentials, Central Eur. J. Phys., 2011,v.9, N1,13-44.

[13] Saavedra,I., Dispersion relations, “causal” inequalities and the completeness of the set of eigen-states in non-relativistic potential scattering, Nucl.Phys., 29(1962)137-152; Completeness, unitarity, and dispersion relations in potential scattering, Nucl.Phys., 87(1966) 267-272.

[14] R.Newton, Scattering Theory of Waves and Particles, McGRAW-HILL BOOK COMPANY, New Yorl–San Francisco–St.Louis–Toronto–London–Sydney, 1969.

[15] Gasymov, M.G., On the expansion over eigen functions og non-self-adjoint boundary-value problem for the differential equation with a singularity in zero, Doklady AN SSSR, 165(1965)261-264.

[16] Bertero, M., and Dillon,G., An outline of scattering theory for absorptive potentials, Nuovo Cim., A2(1971)1024-1038.

[17] V.De Alfaro, and T.Redge, Potential scattering, North-Holland Publish.Company, Amsterdam, 1965.

[18] E.C. Titchmarsh, The Theory of Functions, Oxford University Press, 1939.

[19] Jost, R., Uber die falschen Nullstellen der Eigenwerte der S-matrix, Helv.Phys.Acta, XX (1947)256-266.

[20] Ma, S., Phys.Rev., On a General Condition of Heisenberg foe the S-matrix, 71(1947) 195-200.

[21] I.S.Gradstein and I.M.Ryzhik: Tables of Integrals, Sums, Series, and Products (Moscow, Fizmatgiz,1963) [in Russian]; English translat.: Tables of Integrals, Sums, Series, and Products, Acad.Press, New York(1965).

[22] A.S.Davydov, Theory of atomic nucleus, Fizmatgiz, Moscow (in Russian), 1958.

[23] Simonius,M., Overlapping resonances and unitarity of the scattering S-matrix, Nucl.Phys., A218(1974)53-60.

[24] Olkhovsky,V.S., Nucl.Phys., A425 (1984)445-457 D’Arrigo,A., Giardina,G., Olkhovsky,V.S., et al, Internat. J. Mod. Phys., E2(1993)809-820.

[25] Smith, F.T., Phys.Rev., 118 (1960) 349; Phys.Rev.,119 (1960) 2098.

[26] Lyuboshitz, V.L., Phys.Lett.,B, 72N1(1977)41; Lyuboshitz, V.L., Sov.J.Nucl.Phys., 27(1978)502; Lyuboshitz, V.L., JETP Lett.,28:1(1978)32;

[27] Baz’, A.I.,JETP Lett.,27:2(1978)132-135}.

[28] V.S.Olkhovsky, Time as a quantum observable, canonically conjugate to energy. Time analysis of quantum processes of tunneling and collisions (nuclear reactions) [LAP LAMBERT Academic publishing, 177 pages, 2012]

[29] Gosset, J., Gutbrod, H., Meyer, W. et al, Phys.Rev.,C16(1977)629.

[30] Westfall, G., Sexstro, R., Poskanzer, A. et al, Phys.Rev.,C17 (1978)1368.

[31] Cumming, J., Haustein, P.,Ruth, T. et al, Phys.Rev., C17(1978) 1632.

[32] Kaufman, S., Steinberg, E., Phys.Rev.,C17(1980)167.

[33] Fortney, D., Porile, N., Phys.Rev.,C21(1980)2511.

[34] Sandoval, A.,Gutbrod, H., Meyer, W. et al, Phys.Rev., C21(1980)1321.

[35] Danielewicz, P., Pan, O., Phys.Rev.,C46(1992)2002.

[36] Alba, R., Coniglione, R., Del Zoppo, A. et al,Phys. Lett., B322 (1994)38.

[37] Chen, K., Fraenkel, Z., Friedlander, G., et al, Phys.Rev., 166 (1968) 949.

[38] Amsden, Antony A., Goldhaber, Alfred S., Harlow, Francis H., and Rayford, Nix J., Phys.Rev., C17 (1978)2080-2096.

[39] Izumo, Ko, Araseki, H., Progr. Theor. Phys., 69 (1983)158-170.

[40] Goto, T., Yamaguchi, K. and Sudo, N., Progr. Theor. Phys., 64 (1980)1-17.

[41] Olkhovsky,V.S., Dolinska,M.D., Omelchenko,S.A., Central Eur. J. Phys., 4(2006)1-18;

[42] Olkhovsky,V.S., Dolinska,M.D., Omelchenko, S.A., Central Europ. J.Phys.,9:N4(2011) 1131-1133.

[43] Westfall, G.D., Gosset, J., Johansen, J. et al, Phys.Rev. Lett., 18: 37(1976) 1202.

[44] Gosset, J., Kapusta, J.I.,Westfall, G.D., Phys.Rev.,C18(1978)844.

[45] Prilepsky, Yu., Shmonin, V.,Acta Phys.Polon.B 18(1987)729.

[46] Olkhovsky, V.S., Sov. J. Part. Nucl.,151(1984)30-148.

For the Part II:

[1] Olkhovsky, V.S., Doroshko, N.L., Europhys. Lett., 18 (1992) 483.

[2] D’ Arrigo, A., Doroshko, N.L., Eremin, N.V., Olkhovsky, V.S. et al, Nucl. hys., A, 549 (1992)375.

[3] D’Arrigo, A., Doroshko, N.L., Eremin, N.V., Olkhovsky, V.S. et al, Nucl. Phys., A, 564 (1993)217.

[4] Kelkar, N.G., J. Phys. G: Nucl. Par. Phys., 29 (2003) L 1-L 8.

[5] Kelkar, N.G., Nowakowski, M. and Khemchandani, K.P., Nucl. Phys., A 724 (2003) 357; Kelkar, N.G., Nowakowski, M. and Khemchandani, K.P. and Jain, B.K., Nucl. Phys., A730 (2004) 121.

[6] Kelkar, N.G., Khemchandani, K.P. and Jain, B.K., J. Phys. G: Nucl. Part. Phys., 32: 3 (2006) L 19.

[7] Prokopets, G.A., Nuclear Physics (Russia), volume 74 (2011), p. 740-746.

[8] Eremin, N.V., Giardina, G., Olkhovsky, V.S., Omelchenko, S.A., Mod. Phys. Lett., A, 9 (1994) 2849.

[9] Olkhovsky, V.S., Dolinska, M.E., Omelchenko, S.A. and Romanyuk, M.V., Internat. J. Mod. Phys. E, 19: 5-6 (2010), pp. 1212-1219.

[10] Olkhovsky, V.S., Dolinska, M.E., Omelchenko, S.A., arXiv. 1101. 5541v1 [nucl-th], Jan. 2011; Appl. Phys. Let., 99 (2011) 244103 (1-3).

[11] Olkhovsky, V.S., Dolinska, M.E., Omelchenko, S.A., On the cross section and duration of the neutron- nucleus scattering with a resonance, distorted by a non-resonant background, in the center-of-mass system and laboratory system, Proc. of the 4-th Internat.Conf. Current Problems in Nuclear Physics and Atomic Energy (NPAE-2012, sept. 3-7, 2012, Kyiv, Ukraine), Kyiv 2013, pp. 198-201.

[12] Olkhovsky, V.S., Sov. J. Particles Nucl.(Engl. Transl.);(United States), 1984, v. 15, (2), p. 130-148; Olkhovsky, V.S., Time as a Quantum Observable, Canonically Conjugated to Energy, and Foundations of Self-Consistent Time Analysis of Quantum Processes, Advances in Math. Phys., vol. 2009 (2009), article ID 859710, 83 pages.

[13] Olkhovsky, V.S., Doroshko, N.L., Lokotko, T.I., On the cross section and duration of the neutron-nucleus scattering with two overlapped resonances in the center-of-mass system and laboratory system, Proc. of the 4-th Internat. Conf. Current Problems in Nuclear Physics and Atomic Energy (NPAE-2012, sept. 3-7, 2012, Kyiv, Ukraine), Kyiv 2013, pp. 192-197.

[14] The JEFF – 3.1.1. Nuclear Data Library, OECD NEA/NEA 2009; EXFOR 13759.002, J.A. Harvey, D.C. Larson, ORNL, 1974.

[15] Brusegan, A., Rohr, G., Shelley, R., Macavero, E., Van Der Vorst, C., Poortmans, F., Mewissen, I., Vanpraet, G., Very high resolution transmission measurements and resonance parameters of Ni 58 and Ni60, 1994.

[16] Olkhovsky, V.S., Zaichenko, A.K., Phys. Lett. B 272 (1991) 183-185.

[17] D’ Arrigo, A., Fazio, G., Giardina, G. et al. Progress of Theor. Phys. 87 (1992) 1359-1365.

[18] Kopylov, G.I., Podgoretskij, M.I., Sov. J. Nucl. Phys. 15 (1972) 219-223.

[19] M.L. Goldberger, K.M. Watson, Collision Theory. Wiley J. & Son Inc., 1964.

[20] A.I. Baz, Ya.B. Zel’ dovich, A.M. Perelomov, Scattering, Reaction in Non-Relativistic Quantum Mechanics. Jerusalem, 1969.

[21] Rosenfeld, L., Nucl. Phys. 70 (1965) 1-27.