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Discussing the Conditional Probability from a Cognitive Psychological Perspective

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Abstract Probability signifies the most obscure and the least achievable content among other mathematics areas; particularly, the conditional probability concept that should not be left behind any standard course of probability from primary to university level. Moreover, understanding the conditional probability becomes more serious when the discussion fits into the field of teacher education, wherein the prospective mathematics teachers need not only to learn it but also to promote their pupils' conditional probability reasoning. From this aspect, the current study aims at exploring the prospective mathematics teachers' conditional probability reasoning from a cognitive psychological perspective that focuses on how their minds work. Accordingly, the generic inductive approach has been employed. Hence, a purposive sample of the university students who study in a four-year mathematics teacher preparation program has been selected. After analyzing the school curriculum of probability, an authentic conditional probability situation has been utilized. Following this, the students' interpretations have been translated, coded, and categorized using NVivo software. As a result, the generalizer (58.8%), conservative (11.8%), correlational (23.5%), and the rational thinker (5.9%) have been defined as the primary models of conditional probability reasoning. Furthermore, while the generalizer and conservative thinkers both share the anchoring and adjustment bias, the illusion of validity has been operated by only the conservatives. Besides, the one-step, availability, causality, and gambler fallacy, were the correlational thinkers' assigned biases. On the other side, the rational thinkers have reached a high level of contextual knowledge, with understanding the idea of sample space reduction.

Keywords: *conditional probability, probabilistic reasoning, preservice mathematics teachers, cognitive psychology*

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1. Introduction

Probability is one unique area of science that helps us to quantify our information regarding unknown phenomena. Hence, it plays an important role in our daily life, where many situations are under uncertainty (e.g., tomorrow's weather, result of a medical surgery, graduation date). Because of such importance, probability has been embedded within the mathematics curriculum from primary school level to teacher education.

While probability is typically discussed from two main interpretations; the objective that focuses on mathematical procedures, and subjective that identifies the probability as a degree of belief (confidence), the significance of the conditional probability concept is that it stands at the border between these two main interpretations [1]. Further to this and about school life, the concept of conditional probability has been emphasized to be a major area of study, not only within the school context but also in teacher education programs. As highlighted by [2], conditional probability signifies an important ingredient of probability that should be included in any standard course at school and university level. Moreover, [3] has included the conditional probability in his list of fundamental stochastic ideas, which can be studied at various degrees of formalization through a spiral curriculum. In particular for university students, studying the conditional probability is crucial, since it is interfering with the understanding of various topics such as Bayesian inference, linear regression, correlation models, and multivariate analysis [4,5,6].

Taken into consideration the notion of the conditional probability, only considering the mathematical facet to understand how the students reason under uncertainty is not adequate to provide a holistic overview regarding students' understanding of probability [2]. More seriously, when those students are the prospective mathematics teachers. While on the one hand, teachers' knowledge plays an important role as far as the quality of students' learning is concerned [7,8,9,10,11,12]; numerous difficulties in students' understanding of the conditional probability have been stated in the literature at various levels of learning [13,14,15,16,17,18,19] on the other hand.

In addition to this, as stated by [20], little research has been done on pre-service mathematics teachers' understanding of probability. This lack of research stands obviously when the discussion puts more focus on reasoning as a cognitive process. More precisely, conditional probability reasoning that implies thinking under uncertainty; when some new information or circumstances regarding a random phenomenon become available, the individual tends to change his estimation by taking this new information into his consideration. This reasoning is usually employed to make accurate decisions or inferences in serious situations that we encounter continuously, for instance, medical diagnoses. Consequently, [21] described the conditional probability reasoning as a fundamental component of statistical literacy. Nevertheless, the investigation of the reasoning process itself in the case of probability education seems to be underestimated, and it needs to be strengthened through further research. As regarded by [22], mathematics education research literature has, until recently, ignored subsequent research results stemming from the field of cognitive psychology (how the minds work?). Accordingly, the logically fallacious reasoning has been considered to be a new area of investigation in probability education research [23,24].

From this aspect, and to fit the gap of research in this arena, the current study aims at exploring pre-service mathematics teachers' conditional probability reasoning from a cognitive psychological perspective that focuses on the reasoning processes. Taking into account that pre-service mathematics teachers in this study are the university students who study mathematics education program.

2. Literature Review and Theoretical Perspective

2.1. The Conditional Probability Reasoning

Because various studies from different theoretical perspectives and cultures [25,26,27,28,29,30,31,32,33] reveal that learners tend to hold several misconceptions about probability, instruction in probability should provide experiences that help students to confront these misconceptions and to promote understanding based on probabilistic reasoning [34]. Consequently, from the viewpoint of enhancing students' understanding of conditional probability, the current study emphasizes the context of teacher education through investigating the university students' conditional probability reasoning.

According to the school curriculum, the theoretical, experimental, and subjective probabilities represent the three essential interpretations that are usually introduced to the students. Moreover, the conditional probability concept signifies a prerequisite for teaching the subjective interpretation [28].

Since the mathematical definition of the conditional probability $P(A|B)$ reflects the probability of event A occurrence, given that the event B has happened, it can be interpreted within the subjective school as if the individual can update his prediction regarding a particular event (A) when additional information (B) is provided. As described by [4], the importance of constructing knowledge related to conditional probability lies in the fact that it allows us to change our degree of confidence in random events when new information is available. Furthermore, within this subjective view, all probabilities could be conditional probabilities [35], wherein even the unconditional probabilities are conditioned by the sample space [36].

On the one hand, the study of conditional probability reasoning has a well-established form with school students compared to university students. For example, [37] identified four levels of conditional probability thinking for middle school students (subjective, transitional, informal quantitative, numerical). Besides, at the secondary school level, [38] have explored the chronological and causal conception. On the other hand, at the university level, the prominent studies of [17,39] established the essentials for adults' difficulties in understanding the conditional probability. Based on these studies, many researchers have explored similar results, for example, [14] identified that preservice middle grades teachers have some difficulties in defining the conditioning event, recognizing the temporal order of the conditioning event and the target event, and confusing conditionality as causality. Because of that, [40] has raised the following question; "If teachers have the same misconceptions as their students, how can they develop appropriate lessons and tasks to facilitate students' understanding of conditional probability?" (p. 352)

In addition to the previously mentioned studies, conditional probability reasoning has been widely discussed through [4,5,6,41]. However, their investigations were centered on the students who belong to the psychology department. Further to this, they have employed a formal perspective to assess the students understanding and difficulties in conditional probability by utilizing a standardized test of conditional probability reasoning (CPR test). One interesting finding that has been discussed through [5,6] is the uncorrelation between students' formal knowledge of conditional probability and their psychological biases. Once again, in 2008, [41] reported that the psychological biases in conditional probability stay unrelated to conditional probability problem-solving performance. Additionally, even after formal instruction, some of these biases couldn't be overcome (e.g., confusion between causality and conditionality, the fallacy of time axis) [4]. Consequently, to clarify why students' biases in conditional probability reasoning remain regardless of the regular instruction, more research is needed. Particularly, for the prospective mathematics teachers, wherein few studies have been found in this arena [20], besides the need for reinforcing the study of conditional probability at the university level [41].

This argumentation reveals one main difference between [4,5,6,41] studies and the current study, which is the context, wherein the context of teacher education is being emphasized in this study. Again, the pre-service mathematics teachers here denote the university students, those who are being prepared to be mathematics teachers and will be responsible for teaching the notion of conditional probability to their future pupils.

The above-mentioned discussion leads to the following question; if the university students' understanding of conditional probability will be highlighted, what is the theoretical perspective that underpins this study? Although the formal content knowledge of conditional probability is essential to be understood by the prospective mathematics teachers, the psychological biases may arise during the actual teaching, even with substantial content knowledge. This is consistent with what [4,41] suggested that teaching strategy should be modified, in such a way of emphasizing the probabilistic reasoning, instead of merely concentrating on algorithmic aspects of computing conditional probabilities. Moreover, while the probabilistic reasoning goes beyond thinking in mathematical models, it is recommended to discuss the concept of probability from a non-mathematical perspective [2], and to investigate the logically fallacious reasoning that represents an area of future research [23,24]. Therefore, to respond to this demand of research, the current study aims at answering the following three inquires:

- How do the university students reason in a conditional probability situation?
- What are the students' shared conceptions and biases in conditional probability reasoning?
- Why do the students still have some psychological biases in conditional probability reasoning, even after studying the formal concepts?

As discussed earlier, a non-mathematical perspective that differs from using a formal computational test needs to be employed to answer the study inquiries. This perspective has been noted by the term "a cognitive psychological perspective" that is presented in the following section.

2.2. The Cognitive Psychological Perspective

Since this study focuses on university students' conditional probability reasoning, the adults' development theory reflects fundamental principles underpinning it. Based on Jean Piaget's theory of child cognitive development, several models that reflect the cognitive development of adults have been expanded, for example, Baxter Magolda's epistemological reflection model [42]. According to this model, there are four developmental levels of cognition, which are;

Absolute Knowledge: In this level, the learner obtains knowledge directly from the instructor, since he thinks that the provided information is certain or absolute.

Transitional knowledge: In this level, the learner understands knowledge as if it is partially certain and partially uncertain.

Independent knowledge: In this stage, the learner creates his perspective to understand knowledge, as he considers the uncertainty of the presented information and the differences in individuals' beliefs.

Contextual knowledge: This represents the highest level of adults' cognitive development. In this level, the learner compares perspectives, thinks through problems, integrates and applies knowledge within its context.

Based on this hieratical order of adults' cognition, the prospective mathematics teachers need to go beyond the acquired knowledge, further, to understand this knowledge within a context. Consequently, to investigate how the university students reason in conditional probability situation, the current study has taken a cognitive psychological perspective through implementing an authentic probabilistic context. While not only mathematical knowledge of probability can be emerged but also understanding this knowledge within a context, which is crucial, particularly for prospective teachers [43].

Cognitive psychology is a branch of psychology that deals with how people perceive, learn, remember, and think about information [44]. It admits the possibility to study these complex cognitive skills of human beings, which were neglected in behaviorism because they are unobservable. This indicates that cognitive psychology research places the focus on the individual's mental processes. Although behaviorists claim that such processes cannot be studied because they are neither directly observable nor measurable, the cognitive psychologists emphasize studying them because they reveal how people think [45]. Hence, to accomplish the study purposes that focus on university students' reasoning, this perspective has been adopted.

One possibility to employ the cognitive psychological perspective in such a way that helps to respond to the research questions is utilizing an authentic context (i.e., a phenomenon that is relevant to our daily life situations). This idea was not determined by accident, rather developed in the light of various recommendations regarding the value of implementing a real context to reason under uncertainty.

In statistics education, wherein the context of the problem is essential, students need to go beyond memorizing useless data and collect real meaningful data [46]. Furthermore, specifically concerning probability, [47] noted that because teaching probability is rarely based on the use of real situations, erroneous reasoning has a free reign. Hence, only the perspective of real practices will make it possible to get beyond the gap between education, inaccurate conceptions regarding school learning, and maintain implicit unrealistic theories that are spread in students' everyday life. Particularly when the discussion rests in the field of subjective probability, the quasi-pedagogic scenarios such as tossing coins or rolling dice, which we usually employ to introduce the concept of probability, have well-defined quantifiable spaces. Consequently, they cannot provide an adequate basis for understanding the subjective facet of probability, where there is no well-defined outcomes-space, which is self-evident [48]. In addition to this, [49] have strengthened the value of contextualizing probability education, as they clarified by calling on pupils' daily social practices, probability education becomes a powerful tool for giving meaning to statistics in a school setting.

Based on the preceding discussion, the basic premise underpinning this study is that through implementing an authentic daily-life context, the conditional probability reasoning for the university students can be explored. That implicitly involves a transition from studying the formal knowledge of conditional probability towards exploring the conditional probability reasoning.

3. Methodology

To answer the research questions, the generic inductive approach [50] has been employed. This approach allows the findings to emerge from the frequent and significant themes of the data without being constrained by a traditional specified qualitative approach [51]. Hence, it fits the current study, since its objectives have an exploratory nature, with a primary focus on exploring the university students' conditional probability reasoning. Accordingly, the following procedures have been conducted:

A purposive sample [52] of university students who are enrolled in a four-year mathematics education program at the faculty of education, University, Country, has been selected. One fundamental principle to select those participants was their prior knowledge of the conditional probability concept, whether at secondary school or during the teacher preparation program. In addition to this, their willingness to engage the current study been taken into consideration; where only 68 students (32 students in the 2nd year, 23 in the 3rd year, and 13 in the 4th year) have expressed their enthusiasm to participate after they have been informed by the research objectives. Later on, the students have been asked to answer a questionnaire that includes two items. Nevertheless, the study focuses on the students' responses to the second item of this questionnaire that has been developed in light of the following procedures:

While the current study intends to employ an authentic context, the school content of probability (as provided by the textbooks for primary and lower secondary school students) has been analyzed inductively from the perspective of the context. Consequently, the following types of probabilistic problems have been exposed (see Table 1).

Since the main purpose of the study is to explore the university students' conditional probability reasoning, the context of "giving birth" represents an appropriate context among the seven-inferred problems. This context can be formulated as a probabilistic situation under some conditions (i.e., conditional probability situation). Furthermore, it is often proposed by university students when they are asked to give an accessible example for their future pupils.

In light of the prior two steps, the university students have been asked to respond to the following two items (A and B);

Item A; knowing that there is a pregnant woman, how to determine the probability of giving birth to a girl?

Item B; if you knew that this woman gave birth to two boys before, and she will give birth to her third child,

Q1. How to determine the probability of giving birth to a girl in that case?

Q2. Do you think that your expectation in the first situation (i.e., firstborn) is the same as the second one (i.e., third born)? (explain)

While item A denotes a simple random experiment [53], item B represents a diachronic conditional probability situation, in which the problem is transmitted as a series of sequential experiments carried out over time [6,36].

To confirm the understandability of these items, the questionnaire was first discussed with a sample of (10) students who approved that these items are familiar to them. Still, they couldn't interpret how different is the first item from the second in terms of probability theory (i.e., simple vs. conditional). Following this, the main participants (68 university students) have been asked to respond to these items, within a time of 30 minutes, and their responses have been collected, translated, and coded by NVivo.

Later on, the processes of data analysis have been conducted in the light of [51] description of developing categories that summarize the raw data and convey key themes; that is the essence of the generic inductive approach. Consequently, during the coding process, the focus was on defining the major categories that describe the students' conditional probability reasoning. The researcher's experience of working with the same students before, supported this process, as the students' responses were readily understood. However, in some cases, it was not obvious what a student means; hence, a follow-up interview has been conducted to confirm such student's intention. Moreover, to ensure the validity and reliability of the data analysis, the *triangulation process* has been taken into consideration, as follows;

Triangulation refers to a powerful technique that facilitates the validation of data through cross verification from two or more sources, in which the weaknesses of one source can be strengthened by another source [54]. To verify the processes of data analysis and study results, data triangulation has been employed [55]. Because of this, multiple groups of participants have been engaged, in which the data analysis processes were first centered on the 2nd year students' responses, then the same procedures have been carried out to analyze the other two groups (3rd and 4th year students).

Additionally, two rounds of the coding process have been operated to guarantee the issues of trustworthiness [56]. The first round started with utilizing the students' responses to develop representative categories of conditional probability reasoning. Inversely, during the second round, the students' responses have been analyzed in light of the first round's inferred categories. Hence, based on these two rounds' results, the consistency between the results has been approved to ensure the study findings.

Table 1. The probabilistic problems' types as clarified in the curriculum

Problem type	Environmental issues	School life	Gender	Lifetime expectancy	preferences	Manufacturing	Quasi-pedagogical
An example	It is more probable to rain tomorrow	It is a weak possibility to win the handball competition	The probability of giving birth to a girl equals 50%	The probability of living up to 90 equals 40%	Your friend probably prefer science compared to arts	The probability that a lamp produced by the factory is defective equals 3%	When tossing a coin, the probability of getting a head equals 50 %

4. Results and Discussion

Before addressing the study results, it is helpful to mention that the term "students" has been used to denote the university students who have engaged in this research. On the other hand, school students have been defined by the term pupils.

As noted earlier, the data analysis process focuses on students' responses to item B (conditional probability situation). Yet, because of the connection between item A and B, it is necessary to clarify at first how the students' expectations have been varied between keeping the initial estimation (i.e., what they have stated in Item A), and changing it (after knowing that the woman gave birth to two boys before), as follows;

Table 2. The students' expectations in item B compared to item A

	The students who haven't changed their initial estimation	***The students who have changed their initial estimation***	
Typical response	The probability still as same as the first situation in item A	The probability will change to be different from item A	
		The expected value will be lower than the initial estimation	The expected value will be higher than the initial estimation
N = (2^{nd} year, 3^{rd}, 4^{th}) = (32, 23, 13) = 68	***(23, 11, 6) = 40***	***(5, 10, 4) = 19***	***(4, 2, 3) = 9***

Furthermore, and more relevant to answer the research questions, the students' responses to item B have been categorized in terms of their ways of reasoning, as follows;

- **First: The students who have disregarded the given condition**

Based on the analysis of the students' responses, as the similar responses have been combined in one category using NVivo, two essential models of students reasoning in this conditional probability situation have been inferred and coded under the terms; ***generalizer thinkers*** and ***conservative thinkers*** (see Table 3 and 4).

Table 3. Generalizer thinkers' typical responses

	Generalizer thinkers (G)		
	Holistic (HOL)		**Atomistic (A)**
Typical response	**The probability will not change because**		
	we still have the same possibilities of S = {B, G; or B, G, twins; or B, G, BB, BG, GG}	the way of predicting the baby's gender is always the same (probabilistic process), no matter first, second, or third born baby.	the events of this random process should always be independent. Hence, there is no relationship between previous babies' gender and newborn.
N = (23, 10, 7) = 40	**(17, 6, 4) = 26**	**(2, 2, 2) = 6**	**(4, 2, 1) = 7**
	(19, 8, 6) = 33		

Note that: S refers to the sample space, B refers to a boy, G refers to a girl, and P refers to probability.

The generalizer thinkers have kept their initial estimation, they have agreed that the probability of giving birth to a girl, knowing that the woman gave birth to two boys before, equals the probability of giving birth to a girl without any given conditions. Hence, their typical response was; there is no difference between our expectations or our thinking in both situations.

Although the generalizer thinkers have judged P(G|BB) to be as equal as P (G), their stated reasons were quite different in terms of what they have generalized. While the **holistic thinkers** overgeneralized the randomness notion, the independence concept has been overgeneralized by the **atomistic thinkers.**

This type of reasoning can be interpreted in terms of the Anchoring and adjustment heuristic that has been explained by [57], which implies an adjustment of the initial value to yield the final answer. During utilizing this heuristic, the students first generate a preliminary judgment called the anchor, then in the second stage, they tried to *adjust* that judgment to include the additional information. However, their adjustment was insufficient [58]. About the given context, *generalizer thinkers* have developed their anchors based on the first situation that was provided in item A. Then, when they tried to interpret item B, they tend to perceive it from the previously developed anchor. This result is consistent with what [59] noted regarding the anchoring and adjustment heuristic, in which it can occur without externally afforded anchors, as the individuals tend to generate their anchor and adjust from it (*self-generated anchors*).

As noted earlier, *generalizers* have been classified into the *holistic thinkers* who strengthened the random process itself, and the *atomistic thinkers* who have focused on the outcomes of this process.

On the one side, the *holistic thinkers* have agreed that the process of determining a baby's gender in any situation reflects a random process with the same sample space, no matter there are any given conditions (extra provided information) or not. Furthermore, as they noted, still, the outcomes of this process cannot be predicted with certainty. Therefore, the students thought that the given condition can not affect their estimation, as the sample space of the experiment still has the same expected outcomes (e.g., {B, G, Twins}). On the other side, another way of thinking has been inferred and coded under the term *atomistic thinkers*. Those thinkers put much focus on the outcomes, with the perception that the outcomes of any random process should always be independent. As they clarified, since the given situation represents a random process, hence, its outcomes should be independent (i.e., there is no relationship among the events; firstborn, second born, third born). Consequently, the probability of giving birth to a girl doesn't depend on previous babies' gender. Similarly, the thirdborn's gender cannot be altered either by the first or second born. Because of such reasoning, some students wrote; if a woman gave birth to two boys before, this doesn't guarantee that she will give birth to a girl or even to a boy later.

Such analysis reflects how the generalizer thinkers do reason in a conditional probability situation; too, the shared bias among them, which is the anchoring and adjustment, has also been clarified. Yet, the reason for having such bias (i.e., the answer for the third research question) needs to be described.

Indeed, the proceeding description of the *generalizer thinkers* exposes that the principal reason under the self-generated anchors is the overgeneralization process in a consensus with [58] determination of the anchoring heuristic, which “can be understood as a signature of resource-rational information processing rather than a sign of human irrationality” (p. 29). Hence, the overgeneralization process represents one possible cause for the anchoring bias. It emerged in a couple of forms. For holistic thinkers, they have overgeneralized the notion of randomness. Those thinkers maybe have been influenced by the quasi-pedagogical probabilistic activities (e.g., tossing a coin, drawing a card) that are mostly practiced in both teacher education and the school curricula. Consequently, their inferences were inspired by the classical interpretation that cannot entirely fulfill in such context, particularly after the situation has been enclosed by the given condition. Since the students used to perform these activities, it becomes difficult for them to model a real phenomenon. Likewise, for the atomistic thinkers, the notion of independence has been overgeneralized. As clarified by [1], the concept of *independence* caused much confusion among students and teachers, because of the tendency "to consider just the definition of independence to chronological independent events" (p.1). Furthermore, as they noted, the alternative conceptions of independence persist even in people who had studied the formal concept, which coincides with the study findings.

In addition to the *generalizer thinkers’* category, another category has been deduced through emphasizing the students’ explanations and their stated reasons. This category indicates the **conservative thinkers** (see Table 4).

Table 4. Conservative thinkers’ typical responses

	Conservative thinkers (CON)	
	Socially Conservative (SO.C)	**Subjectively Conservative (SU.C)**
Typical response	The probability will change because some women can give birth to a certain gender only. Hence, maybe this woman always gives birth to boys. Then, the probability of giving birth to a girl will be lower than in the first situation.	The probability could or couldn’t change, but still, it’s a matter of ALLAH willing.
N= (5, 1, 2) = 8	**(1, 1, 1) = 3**	**(4, 0, 1) = 5**

As clarified principal, the common characteristic between the *generalizer* and *conservative thinkers* is the condition’s exclusion from the situation’s analysis. The *generalizers* have this exclusion because of the *anchoring bias* that has been rooted in the overgeneralization process. Nevertheless, the same exclusion has another root beyond the *anchoring bias*, for the *conservative thinkers*.

For more clarification, the **socially conservative thinkers** have adjusted their estimation to be lower than their initial estimation, after knowing that the woman gave birth to two boys before. Still, their reasons were concentrated around the belief that is some women can give birth to only a certain gender. That indicates their inability to overcome such social dogma, as it prevents them from utilizing the given condition. Similarly, the **subjectively conservative *thinkers*** haven’t clarified clearly whether the probability will change or not; rather, they highlighted the concept of Allah willing to explain the situation, in a manner of neglecting the new given condition.

Acknowledging that all the conservatives were aware of the given condition but they couldn’t operate it in their analysis, the cause for such a case is not purely cognitive rather is rooted in some hold beliefs. Thus, this manner of reasoning can be interpreted in light of [57] psychological explanation regarding the *illusion of validity*, that represents “the unwarranted confidence which is produced by a good fit between the predicted outcome and the input information” (p. 1126). Among the *conservatives*, *socially conservatives* think that some women can give birth to a specific gender, while others believe in Allah's willingness to determine the babies' gender (*Subjectively conservatives*). Then, they have decided intentionally to exclude the given condition from their analysis to make a good consistency between the expected outcome (i.e., giving birth to a girl) and their self-input (i.e., what they believe in). Consequently, because of the illusion of validity, those conservative thinkers have retrieved their initial beliefs to interpret the new situation.

This implicitly explains why the generalizers don’t possess the illusion of validity. As their knowledge of the classical probability (Laplace theory) was inadequate, they have generalized this interpretation and anchored their estimation, without awareness of such factors that may limit their generalization. On the contrary, the *conservatives* have modified their estimation, in which they are aware of the difference between item A and B. Hence, they share the illusion of validity that “persists even when the judge is aware of the factors that limit the accuracy of his predictions” [57], (p. 1126). Accordingly, although both *generalizers* and *conservatives* hold the anchoring bias, the source of this bias stays different. This bias appeared because of the overgeneralization process, wherein the cognitive theme strongly arose in *generalizers'* responses. However, the illusion of validity that is more relevant to the *conservatives*' beliefs was the origin of the anchoring bias.

- **Second: The students who have taken the given condition into their consideration**

Two other categories have been interpreted to reflect the students who employed the given condition in their analysis. The description of the students' conditional probability reasoning in these categories has been clarified in Table 5;

Although all students who belong to the above-mentioned categories have employed the given condition that is the woman gave birth to two boys before, they perceived this condition in various ways. Hence, they have been assigned to different categories of conditional probability reasoning that are explained in detail as follows;

The first inferred category is **correlational thinker**s. It represents the students who think of the relationship among the various outcomes that have been mentioned in the new situation (i.e., B, B, G). Moreover, this category comprises both **horizontal** and **vertical thinkers** whose reasoning has the following characteristics:

Table 5. Correlational and rational thinkers' typical responses

<table>
<tr><th rowspan="2"></th><th colspan="2">Correlational thinkers (COR)</th><th colspan="2" rowspan="2">Rational thinkers (R)</th></tr>
<tr><th>Horizontal (HOR)</th><th>Vertical (V)</th></tr>
<tr><td rowspan="2">Typical response</td><td colspan="4">The probability will change because</td></tr>
<tr><td>in this situation, S has three outcomes {B, B, G}. Hence, P (G) = 1/3.</td><td>the woman gave birth to two boys before. Hence, the probability of giving birth to a boy will be higher than a girl. Inversely, some other students clarified, since the woman gave birth to two boys, then she is more probable to give birth to a girl.</td><td>this situation doesn't look very different than in the first situation. However, if the number of boys continues to increase compared to girls, this could be indicative of a genetic or biological issue.</td><td>one of the previously considered conditions, which is the woman may have a miscarriage, doesn't exist anymore. Rather, in this situation, we already understood that the woman has a high probability of giving birth to a child, whatever boy or girl.</td></tr>
<tr><td rowspan="2">N = (4, 12, 4) = 20</td><td rowspan="2">(0, 1, 2) = 3</td><td rowspan="2">(2, 10, 1) = 13</td><td>(1,0, 1) = 2</td><td>(1, 1, 0) = 2</td></tr>
<tr><td colspan="2">(2, 1, 1) = 4</td></tr>
</table>

The horizontal thinkers have interpreted the situation, not as a diachronic that incorporate a sequence of events, in which first and second outcome already known, but the third outcome is still uncertain; rather, as a static one stage situation. They assumed that this situation involves three equally likely outcomes {B, B, G}, and the matter is deciding the probability of one possible event among those outcomes. Those students tried to overcome the complexity of the given situation by changing it into one stage, which is easier for them to calculate the probability. Consequently, they have reduced the given conditional probability problem that includes compound events to be a simple experiment, since they modeled the situation as if S = {B, B, G}, then P (G) = 1/3. This way of reasoning is similar to what [53] reported during his investigation for prospective lower secondary school teachers' understanding of simple and compound events, in which the *one-step orientated heuristic* has been identified. It appeared when the students found the answer "by simply transforming a two-step problem into a one-step problem or simple trial." (p. 2).

In addition to the one-step bias, the horizontal thinkers share the availability bias**,** in which the individual estimates the likelihood of an event by the ease with which the relevant mental operations of retrieval, construction, or association can be performed [57]. Consequently, the shared availability bias among the horizontals has emerged as a manner of transforming the given diachronic situation into a simple one-step, which is easy for them to calculate or probably more available in their experience. Again, why do those horizontals still have the availability bias? The answer to such a question could be a couple of plausible causes, which are the *retrievability of instances* or the *imaginability bias*. On the one hand, the case of recent activities or tasks that students have performed, or the majority of the examples that they used to solve, maybe the source for *retrievability of instances*. On the other hand, the *imaginability bias* has a more cognitive structure, in which the mind tries to reduce the load of the complicated computational rules (i.e., P(G|BB)), and instead generate a simpler formula (i.e., P (G)).

On the other side of the *horizontal thinkers*, the "vertical thinkers" represents the other sub-category of the *correlational thinkers*' category. The *verticals* symbolize the students who correlate the previous babies' gender with the third born gender. Some *vertical thinkers* have decreased their initial estimation as they though if the woman gave birth to two boys before, then she is more likely to give birth to a similar gender, then to a boy. Inversely, the other *verticals* supposed that if the woman gave birth to two boys before, then she is more likely to give birth to a girl at that time; hence, they have increased their estimation.

The vertical thinkers understood the given conditional probability situation (P(G|BB) as a causal relationship. They have perceived the conditioning event (previous two boys) as a cause, and the consequence is the new event (girl). This bias has widely been stated in different studies under the term causal conception that is considered a cognitive more than being provoked by teaching. It hides the reversible character of conditional probability, which is needed to understand Bayes' theorem and statistical inference [6,13,19,36,38,60]. As declared by [61], although causation is intuitively perceived by human beings because most of our knowledge was created by causes and effects, our conceptions about causation are sometimes biased, and other times there is a confusion between causality and conditionality. Besides this, in a consensus with the study findings, "People tend to overestimate causally perceived conditional probabilities while they ignore diagnostic conditional probabilities" [41], (p. 2).

To argue the possible grounds for such *causal conception*, it is worthwhile to mention that many textbooks discuss the independence notion from a mathematical point of view, which is if A and B are two independent events, then $P(A \cap B) = P(A)*P(B)$. Furthermore, the classroom discussion itself regularly strengthens the idea of "effect", wherein if A does not affect B, then A and B are two independent events. Nonetheless, as noted by [36], in defiance of the well-established view that independence neither confirms nor denies cause and effect.

Besides the *causal conception*, some *verticals* share the gambler fallacy, which has been defined as the belief that after a long run of the same result in a random process, the probability of the same event occurring in the following trial is lower [13,53,62]. Although all *vertical thinkers* have considered the previous two boys as a cause to reflect the probability of giving birth to a girl, some of them thought more theoretically in a similar way of interpreting the situation of tossing a coin, which is; if the first outcome was B and the second also B, then it's more likely to have G as the following outcome.

As clarified earlier in Table 5, in addition to the *correlational thinkers'* category, another way of conditional probability reasoning coded under the term **rational thinkers**. Those thinkers acknowledged that the new situation remains different from the first situation. Moreover, they understood the given condition as new knowledge that helps them to adjust and revise their initial prediction. Hence, they have affirmed that although the probability will change, this doesn't imply that giving birth to two boys before leads to increase or decrease the estimation (as the *correlational thinkers*). Rather, there are some other factors (determinants) that may contribute to change the probability.

More specifically, the *rational thinkers* perceived the provided knowledge regarding the previous babies' gender as an indicator of the woman's biological status, in which if the woman stayed to give birth to boys, the probability of giving birth to a girl would be lower than a boy. On the other side, the other *rational thinkers* have grasped the given condition as a sign of the woman's ability to deliver her baby. Accordingly, if the miscarriage was one possible outcome in the first situation, it is less likely to occur in the second situation. As they explained, because the number of outcomes in the sample space has been decreased (after eliminating the miscarriage outcome), then the probability of giving birth has to increase (no matter boy or girl).

Indeed, although the study has taken a cognitive psychological perspective with a complete focus on the process of thinking itself, it is meaningful to define this *rational thinking* as a high level of conditional probability reasoning. On the one hand, the *rational thinkers* have attained the contextual knowledge level, they compared both situations and interpreted the given condition within its context (without concentrating at the normative standard answer; higher or lower probability). On the other hand, their ability to understand the reduction of sample space has been identified to be a typically high level of understanding conditional probability [37]; particularly, in a chronological situation. As [38] noted, the majority of conditional probability situations can be thought of as a reduction of sample space. Still, this reduction is not easy to realize in chronological events where a series of experiments occur.

5. Conclusion

Based on what has been raised in the study results, and in relation to the research questions, the conditional probability reasoning for the university students has been modeled by the following principal categories; *generalizer* (holistic and atomistic), *conservative* (socially and subjectively), *correlational* (horizontal and vertical), and *rational* thinkers. Although the given term to signify each type seems little distinct from literature, the description of the students' reasoning in each category corresponds to the previous studies. The following figure summarizes how these types of reasoning were widespread among the students;

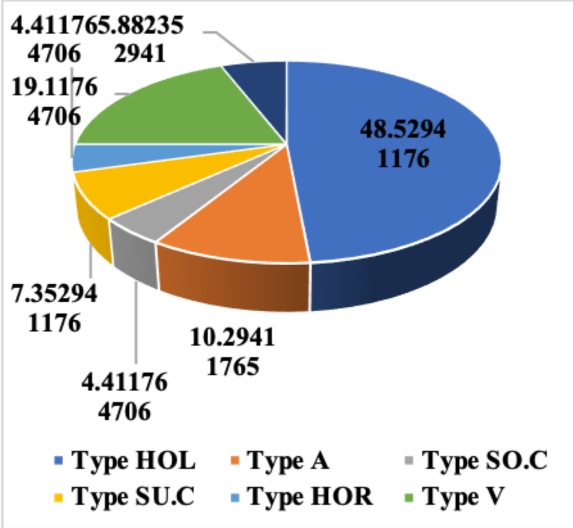

Figure 1. The distribution of conditional probability reasoning among the students

Still, within each type of conditional probability reasoning, there are some biases, heuristics, or conceptions being shared by the students, because of various factors. For the *generalizer thinkers*, they all shared the *anchoring and adjustment bias*. However, the reason for such bias differs between overgeneralizing; the notion of randomness for the *holistic thinkers* compared to the independence concept for the *atomistic thinkers.*

On the other side, and similar to the *generalizers* in dropping the given condition, there are *conservative thinkers*. Those *conservatives* also maintained the *anchoring and adjustment bias* that has been emerged as retrieval of their initial estimation. Nonetheless, the origin of this bias stays distinct. That is the *illusion of validity*. It prevented *socially conservative thinkers* from overcoming some socially shared beliefs, besides causing stress on the concept of *Allah willing* for the *subjective conservatives*. This indicates an essential result of the current study, which is the source of the anchoring and adjustment heuristic. While this bias is mostly described in the literature as a cognitive heuristic, the study revealed that students' beliefs could be another possible origin of such bias.

The other two conditional probability reasoning categories are the *correlational* and *rational thinkers*, who have understood the conditionality through employing the given condition into their analysis. Regarding the *correlational thinkers*, some of them belong to the *horizontal thinkers* who share *the one-step* and *the availability bias*. From a psychological point of view, these biases have been provoked by the retrievability of instances or imaginability bias. The types of activities that students used to practice could be one possible foundation for the *instances' retrievability bias*. On the other side, the emergence of *imageability bias* implies a cognitive nature, wherein the mind works towards reducing the load of the complicated computational rules. The other correlational thinkers belong to the vertical thinkers' category. They maintain the *causal conception*, and some of them hold the *gambler fallacy*. These biases have been rested in students' reasoning because of the conventional teaching that strengthens operating the quasi-pedagogical activities, as well as the ignorance of the subjective probability that is widely used today in the applications of statistics [63].

Interestingly, only about 6% of the students have reached the contextual knowledge level. Those students are *rational thinkers* who understood the idea of sample space reduction when judging under uncertainty. They have interpreted the conditional probability situation taking into consideration employing the given condition to revise their estimation.

The provided findings in this study declare that studying the conditional probability from a cognitive psychological perspective helps not only to explore the students' reasoning but also to investigate their intuitions. As explained by [2], this world of personal intuitions implies another source for success or failure of teaching, particularly in the case of probability that is strongly connected with individuals' emotions towards random phenomena. Thus, through studying university students' conditional probability reasoning, we can interpret whether those prospective teachers accept or ignore what they learn during their preparation program, and how this reasoning may later influence their teaching. Moreover, these results constitute a further contribution to the field of teachers' professional knowledge for teaching probability, since the adequate knowledge teachers need to teach the probability content is not clear [64]. Hence, it will be useful for those who are responsible for prospective mathematics teachers, especially when the reasoning process is taken into consideration.

Yet, the current study has some limitations that should be regarded when interpreting its results. Since subjective probability is strongly connected with individuals' intuitions and social customs, different ways of reasoning may emerge in another culture. Particularly, with the study's small sample size, it is challenging to generalize its results. Furthermore, while the study has modeled the conditional probability reasoning from a cognitive psychology perspective, the scientific genetic analysis of the given situation hasn't been emphasized. As mentioned by [41], the dilemma of such subjective interpretation is that the amount of evidence might vary from one person to another. Besides, sometimes it's difficult to confirm the adequacy of that proof objectively.

Finally, it is recommended to change the way of approaching the conditional probability from teaching the procedures towards teaching the process that involves probabilistic reasoning. Specifically, for the prospective mathematics teachers, it would be better if the conditional probability has been interpreted within various authentic circumstances, in which their psychological biases and conceptions can be explored. Hence, the didactical activities, in teacher preparation programs, can be reconstructed to confront the prospective teachers' biases.

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