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Modern Tesla Coil as a Multidisciplinary Example in STEM Teaching

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Abstract A modern Tesla coil is an excellent multidisciplinary example in undergraduate STEM teaching. It incorporates several concepts from physics and electrical engineering. For example, Ampere's law and Faraday's law are concepts in physics while an *LC* circuit, an *RLC* circuit, and the properties of a transistor are concepts in electrical engineering. A Tesla coil shows the intimate relationship between electricity and magnetism. Ampere's law states that a current induces a magnetic field and Faraday's law states that a changing magnetic flux induces a voltage. In a classical Tesla coil, a spark gap switches the current on and off flowing through the primary coil. Meanwhile, in many modern Tesla coils, a transistor is used instead of a spark gap, since it can switch on and off very quickly using a lower voltage. Several papers described how modern tesla coils work. However, the designs of modern Tesla coils were complicated and the mathematical descriptions for Tesla coils were beyond undergraduate students' level. This paper describes a modern Tesla coil by providing mathematical details that are appropriate for undergraduate students' level. To satisfy the educational purpose of this paper, we also choose the simplest design for a modern Tesla coil. The primary circuit in a modern Tesla coil used here is a parallel *RLC* circuit. We show that it can play the same role as the series *RLC* circuit of the primary circuit in the classical Tesla coil.

Keywords: *Tesla coil, ampere's law, faraday's law, electromagnetic oscillator, transistor*

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1. Introduction

A Tesla coil is one of the devices that has fascinated people for years. Its function ranges from generating a bolt of spectacular lightning to transmitting electric power wirelessly. A Tesla coil uses the intimate relationship between electricity and magnetism (electromagnetism), which were thought to be separate phenomena until the 19th century. Understanding of electromagnetism has contributed greatly to the advancement of many modern electronics technologies.

A Tesla coil can also provide a great educational benefit in physics and electrical engineering education. In introductory physics courses, electricity is generally introduced first and then magnetism is introduced. By showing that a current along a long wire induces a magnetic field around the wire, Ampere's law, we introduce the idea that magnetism is intimately related to electricity. Faraday's experiments further show that a changing magnetic flux induces a voltage. Most of the electrical engineering students learn that an *LC* circuit and an *RLC* circuit show oscillating electromagnetism in an introductory physics course and an electrical engineering course. Later, they will also learn the properties of a transistor in electrical engineering courses.

A modern Tesla coil can be designed to incorporate the concepts of Ampere's law, Faraday's law, an *RLC* circuit, and a transistor. A Tesla coil is a resonant air-core transformer that induces high voltage at very high frequency. It has two coils; a primary coil and a secondary coil that is located inside the primary coil. The secondary coil has a much greater number of turns of wire than the primary coil has. Therefore, much greater voltage is induced in the secondary coil.

The main concept of a Tesla coil is the following. First, an oscillating current is provided in the primary coil. This oscillating current in the primary coil provides an oscillating magnetic flux in the secondary coil. According to Faraday's law, this oscillating magnetic flux induces an oscillating voltage in the secondary coil.

To provide the oscillating current in the primary coil, the classical Tesla coil used a spark gap to turn the current on and off flowing through the primary coil. This spark gap is replaced with a transistor in many modern Tesla coil designs. A transistor turns the current on and off in the primary coil very effectively by turning on and off very quickly. The use of the transistor in a modern Tesla coil is also an excellent opportunity to introduce how a transistor works as a faster acting switch in electronic circuits, which many electrical engineering students will learn and use in their electric circuit courses.

Therefore, a Tesla coil is a great multidisciplinary example in undergraduate physics education that bridges physics and electrical engineering. Since a Tesla coil can be built easily with a few electronic elements, it can be assigned as a student project or be built in the physics lab as a hands-on experience. It can also be introduced in a physic course that is designed for non-engineer major students as a class demonstration to draw students' interest in science.

Several papers showed mathematical descriptions and computer simulation results of Tesla coils [1,2,3,4]. However, the designs of modern Tesla coils were complicated and the mathematical descriptions were beyond undergraduate students' level. This paper will describe a Tesla coil by providing mathematical details that are appropriate for undergraduate students' level. To satisfy the educational purpose of this paper, we also choose the simplest design for a modern Tesla coil.

This paper is organized as follows. First, it will review how a classical Tesla coil works. We will review an *LC* circuit and a series *RLC* circuit, which are the typical examples in an introductory physics course. Next, an ideal classical Tesla coil (two magnetically coupled *LC* circuits) will be mathematically described by solving the second order linear differential equations. Last, it will describe the details of how our modern Tesla coil works.

2. A Classical Tesla Coil with a Spark Gap

This section will review how a classical Tesla coil works. The circuit diagram of a classical Tesla coil is shown in Figure 1. L_1 and C_1 are the inductance and the capacitance in the primary circuit. L_2 and C_2 are the inductance and the capacitance in the secondary circuit. Here, C_2 is not the capacitance of an actual capacitor. It is an effective capacitance between windings of the coils and between the top of the coil and the ground.

When a spark gap is open, the capacitor C_1 is charged by a high voltage source provided by the iron core transformer. When the voltage across the capacitor C_1 reaches the breakdown voltage of the spark gap, the spark gap generates a spark through the air, and this closes the spark gap. Therefore, the primary circuit becomes an *LC* circuit and it is shown in Figure 2, which is one of the typical examples in an introductory physics course.

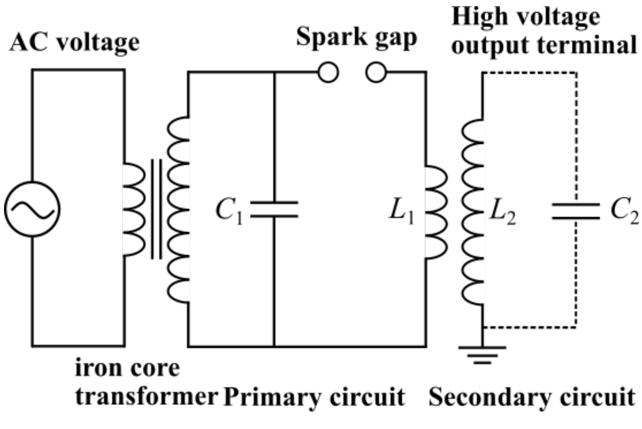

Figure 1. The circuit diagram of a classical Tesla coil with a spark gap

2.1. An *LC* Circuit

Since an *LC* circuit is one of the main components of the Tesla coil, we will review it now. Figure 2 (a) shows the primary *LC* circuit right after the spark gap is closed at $t = 0$. The capacitor C_1 is initially charged to Q_0. When the capacitor is discharging, the current is flowing through the inductor L_1. Figure 2 (b) shows that the capacitor is charged to Q_1 and the current I_1 is flowing through the inductor at a time t.

To find the current, we apply Kirchhoff's loop rule, which states that the sum of voltage changes around a closed loop is zero. When we follow the current around the loop, there is a voltage drop across the inductor and a voltage gain across the capacitor, therefore the sum of voltage changes is [5]

$$-L_1\frac{dI_1}{dt}+\frac{1}{C_1}Q_1=0 \tag{1}$$

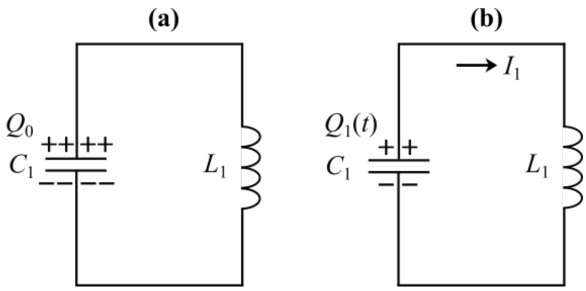

Figure 2. An *LC* circuit **(a)** The capacitor C_1 is initially charged to Q_0 at $t = 0$. **(b)** The capacitor C_1 is charged to Q_1 and the current I_1 is flowing through the inductor at a time *t*.

Since the current is the decrease of charge on the positive plate of the capacitor, the current is $I_1=-dQ_1/dt$, Therefore, equation (1) is written as follows,

$$\frac{d^2Q_1}{dt^2}+\omega_1^2Q_1=0,\ \omega_1=\frac{1}{\sqrt{L_1C_1}} \tag{2}$$

This is a second order linear differential equation, which can be easily solved by a method found in any differential equation textbook. A general method is shown in Appendix A for students who know complex numbers. Here, we show a different way to solve equation (2) for students in an introductory physics course who may not know complex numbers yet. When we look at equation (2), we notice that when Q_1 is differentiated twice, it becomes $-Q_1$ multiplied by a constant. Students learned in the calculus class that cosine function and sine function change their sign when they are differentiated twice. Since both $\cos\omega_1 t$ and $\sin\omega_1 t$ satisfy equation (2), the general solution is a linear combination of $\cos\omega_1 t$ and $\sin\omega_1 t$.

$$Q_1(t)=A\cos\omega_1 t+B\sin\omega_1 t \tag{3}$$

The constants *A* and *B* can be determined by the initial conditions. The initial conditions for the charged capacitor to Q_0 and no current through the inductor are

$$\begin{aligned} &Q_1(0)=Q_0\\ &I_1(0)=-\frac{dQ_1(0)}{dt}=0 \end{aligned} \tag{4}$$

From the initial conditions above, we obtain $A = Q_0$ and $B = 0$. Therefore, the charge and the current are written as follows,

$$Q_1(t) = Q_0 \cos \omega_1 t$$
$$I_1(t) = -\frac{dQ_1(t)}{dt} = \omega_1 Q_0 \sin \omega_1 t \qquad (5)$$

Equation (5) shows that the charge on the capacitor oscillates at the angular frequency ω_1. This means that the energy stored in the capacitor in the form of electric field oscillates. It also shows that the current flowing through the inductor oscillates, meaning that the energy stored in the inductor in the form of magnetic field also oscillates. Let us calculate the total energy stored in the capacitor and the inductor. Using equation (5), the energies stored in the capacitor and the inductor respectively, at a time t are

$$U_E(t) = \frac{Q_1^2}{2C_1} = \frac{Q_0^2}{2C_1}\cos^2 \omega_1 t$$
$$U_B(t) = \frac{1}{2}L_1 I_1^2 = \frac{1}{2}L_1\omega_1^2 Q_0^2 \sin^2 \omega_1 t = \frac{Q_0^2}{2C_1}\sin^2 \omega_1 t \qquad (6)$$

Thus, the total energy is

$$U(t) = U_E(t) + U_B(t)$$
$$= \frac{Q_0^2}{2C_1}\cos^2 \omega_1 t + \frac{Q_0^2}{2C_1}\sin^2 \omega_1 t = \frac{Q_0^2}{2C_1} \qquad (7)$$

Equation (7) shows that the total energy is conserved. This means that the energy stored in the capacitor in the form of electric field and the energy stored in the inductor in the form of magnetic field oscillate back and forth. Therefore, an LC circuit is an electromagnetic oscillator [5].

2.2. A Series *RLC* Circuit

Section 2.1 described an LC circuit. However, when the spark gap is closed, some of the energy is lost due to the discharge through the air. Therefore, there is an effective resistance in the spark gap and the primary circuit is actually a series RLC circuit as shown in Figure 3.

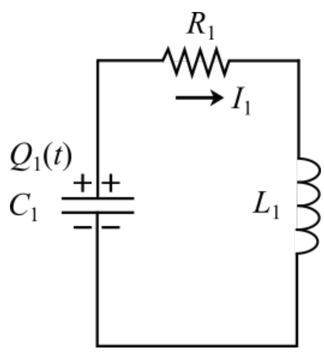

Figure 3. A series *RLC* circuit. While the capacitor is discharging, the current I_1 is flowing in the circuit. The capacitor C_1 is charged to $Q_1(t)$ at a time t

This section will review a series RLC circuit, while a capacitor is discharging. Since the voltage drop across the resistor is $-R_1I_1$, Kirchhoff's loop rule gives the following equation [6].

$$-L_1\frac{dI_1}{dt} - R_1 I_1 + \frac{1}{C_1}Q_1 = 0 \qquad (8)$$

Since the current is $I_1 = -dQ_1/dt$, equation (8) is written as follows,

$$L_1\frac{d^2Q_1}{dt^2} + R_1\frac{dQ_1}{dt} + \frac{1}{C_1}Q_1 = 0. \qquad (9)$$

If we divide the equation above by the inductance L_1, we get

$$\frac{d^2Q_1}{dt^2} + 2\beta_1\frac{dQ_1}{dt} + \omega_1^2 Q_1 = 0 \qquad (10)$$

Here, $\beta_1 = R_1/(2L_1)$ and $\omega_1 = 1/\sqrt{L_1C_1}$.

Without the second term, the charge $Q_1(t)$ in equation (10) satisfies the same differential equation in equation (2). The solution of equation (2) (equation (5)) shows that the charge $Q_1(t)$ oscillates at the angular frequency ω_1 with a constant amplitude. With the second term in equation (10), however, the system may undergo underdamped (damped oscillatory), critically damped, or overdamped motion, depending on the values of R_1, L_1, and C_1. The details of solving equation (10) are shown in Appendix B. In a Tesla coil, the values of R_1, L_1, and C_1 satisfies $\omega_1 > \beta_1$ (damped oscillation). For this case, Appendix B shows that the charge $Q_1(t)$ is expressed as follows,

$$Q_1(t) = e^{-\beta_1 t}(A\cos\omega_1' t + B\sin\omega_1' t) \qquad (11)$$

Here, $\omega_1' = \sqrt{\omega_1^2 - \beta_1^2}$ and A and B are constants that can be determined by the initial conditions. From the initial conditions in equation (4), we obtain $A = Q_0$ and $B = \beta_1 Q_0/\omega_1'$.

Equation (11) shows that the charge $Q_1(t)$ oscillates at the angular frequency ω_1', but its amplitude decreases exponentially. This is called damped oscillation (underdamped motion).

2.3. An Ideal Tesla Coil: Two Magnetically Coupled *LC* Circuits

An ideal Tesla coil is two magnetically coupled LC circuits as shown in Figure 4.

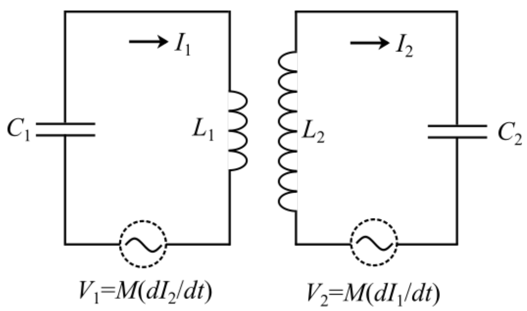

Figure 4. An ideal Tesla coil (two magnetically coupled *LC* circuits). The voltage source V_1 in the primary circuit is the induced voltage due to the oscillating current I_2 in the secondary coil. The voltage source V_2 in the secondary circuit is the induced voltage due to the oscillating current I_1 in the primary coil

When a spark gap is closed in the primary circuit, the current I_1 flowing through the primary coil oscillates with the angular frequency $\omega_1 = 1/\sqrt{L_1C_1}$ as described in section 2.1. This oscillating current in the primary coil provides an oscillating magnetic flux in the secondary coil. According to Faraday's law, this oscillating magnetic flux induces an oscillating voltage V_2 in the secondary coil. Therefore, the secondary circuit becomes an LC circuit with an oscillating voltage source $V_2 = M\frac{dI_1}{dt}$. Here, M is the mutual inductance between the primary coil (L_1) and the secondary coil (L_2). The current I_2 in the secondary LC circuit oscillates with the angular frequency $\omega_2 = 1/\sqrt{L_2C_2}$. Due to this changing current in the secondary circuit, the primary circuit also becomes an LC circuit with an oscillating voltage source $V_1 = M\frac{dI_2}{dt}$.

For the coupled LC circuits, Kirchhoff's loop rule for the primary circuit and the secondary circuit gives

$$
\begin{aligned}
&-L_1\frac{dI_1}{dt} + \frac{1}{C_1}Q_1 + M\frac{dI_2}{dt} = 0\\
&-L_2\frac{dI_2}{dt} - \frac{1}{C_2}Q_2 + M\frac{dI_1}{dt} = 0
\end{aligned} \tag{12}
$$

When the primary circuit is discharging, the secondary circuit is charging. Therefore, currents in the primary circuit and the secondary circuit are $I_1 = -dQ_1/dt$ and $I_2 = dQ_2/dt$, respectively. Then, equation (12) becomes

$$
\begin{aligned}
&\frac{d^2Q_1}{dt^2} + \omega_1^2 Q_1 + \frac{M}{L_1}\frac{d^2Q_2}{dt^2} = 0\\
&\frac{d^2Q_2}{dt^2} + \omega_2^2 Q_2 + \frac{M}{L_2}\frac{d^2Q_1}{dt^2} = 0
\end{aligned} \tag{13}
$$

Here, $\omega_1 = 1/\sqrt{L_1C_1}$ and $\omega_2 = 1/\sqrt{L_2C_2}$. The initial conditions are

$$
\begin{aligned}
&Q_1(0) = Q_0\\
&I_1(0) = -\frac{dQ_1(0)}{dt} = 0\\
&Q_2(0) = 0\\
&I_2(0) = \frac{dQ_2(0)}{dt} = 0
\end{aligned} \tag{14}
$$

Since solving equation (13) is generally beyond the level of mathematical skills for the students in an introductory physics course, it will not be discussed here. However, students who know the introductory level differential equation and linear algebra can solve equation (13). Moreover, physics major students will encounter the differential equations that are very similar to equations in equation (13) when they learn two coupled harmonic oscillators in their junior level mechanics course. Therefore, learning how to solve differential equations in equation (13) may be a great benefit to students. Appendix C shows the details of solving equation (13) when two LC circuits are oscillating at the same angular frequency, $\omega_1 = \omega_2 = \omega$.

When the coupling between the primary coil and the secondary coil is weak, Appendix C shows that the charges $Q_1(t)$ and $Q_2(t)$ are expressed as follows,

$$
\begin{aligned}
&Q_1(t) = (Q_0 \cos\tfrac{1}{2}k\omega t)\cos\omega t\\
&Q_2(t) = (\sqrt{\tfrac{L_1}{L_2}}Q_0 \sin\tfrac{1}{2}k\omega t)\sin\omega t
\end{aligned} \tag{15}
$$

Here, $k = M/\sqrt{L_1L_2}$ is a coupling coefficient. Equation (15) shows that both $Q_1(t)$ and $Q_2(t)$ are sinusoidally oscillating at the angular frequency ω. Their amplitudes (the quantities in the parenthesis in equation (15)) oscillate sinusoidally at the angular frequency $\frac{1}{2}k\omega$. The plots of $Q_1(t)$ and $Q_2(t)$ are shown in Figure 5.

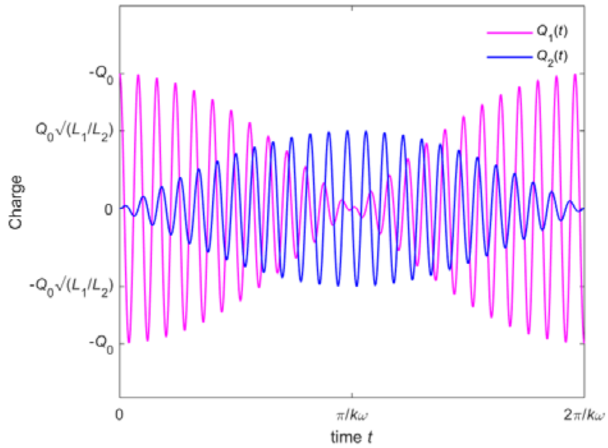

Figure 5. (Color online) Plots of $Q_1(t)$ and $Q_2(t)$ for two weakly coupled LC circuits. Here, the coupling coefficient $k = 0.04$ is used

Figure 5 shows that while the amplitude of $Q_1(t)$ gradually decreases from its maximum Q_0 at $t = 0$ to zero at $t = \pi/(k\omega)$, the amplitude of $Q_2(t)$ gradually increases from zero at $t = 0$ to its maximum $\sqrt{\frac{L_1}{L_2}}Q_0$ at $t = \pi/(k\omega)$. This means that some of the energy in the primary circuit is transferred to the secondary circuit after each cycle until all energy is transferred to the secondary circuit. Since the coupling coefficient is very small in the Tesla coil, the energy transfer rate is slow. Therefore, it takes several cycles before all energy is transferred to the secondary circuit. When all energy is in the secondary circuit, some of its energy is transferred back to the primary circuit at the same rate [1]. This process is repeated indefinitely in the ideal Tesla coil. Energy transfer is maximum when both primary and secondary circuits are tuned to oscillate at the same angular frequency, $\omega_1 = \omega_2 = \omega$.

We have considered the ideal Tesla coil where there is no resistance in the circuit. However, there is an effective resistance in the circuits, since a spark gap and coils have some resistance. We saw in section 2.2 that the charge in the capacitor is in damped oscillation when there is a resistance (see equation (11)). This means that the charge on the capacitor decreases after each cycle. Therefore, the voltage across the capacitor decreases. When the voltage is below the breakdown voltage of the spark gap, the spark gap opens and the capacitor in the primary circuit is then charged by the iron core transformer. This cycle of the Tesla coil repeats.

3. A Modern Tesla Coil with a Transistor

A spark gap used in the classical Tesla coil requires very high voltage to be closed. This causes a very loud noise. A spark gap also has a slower response time as a switch. To eliminate these drawbacks of a spark gap, modern Tesla coils use transistors. A transistor is a device that amplifies electronic signals. It can also be used to turn the electronic signals on and off very quickly. Since our modern Tesla coil uses a bipolar junction transistor, we will review it here.

A bipolar junction transistor (BJT) is made of two P-N junction diodes. A P-N junction diode is an electronic device that allows current to flow one direction. When the p-type region is shared at the junction, a BJT is called an NPN transistor. When the n-type region is shared at the junction, a BJT is called a PNP transistor. A BJT has three regions called base, collector, and emitter. An NPN transistor is shown in Figure 6 along with its symbol in the electric circuit. When a current is flowing into the base, an amplified current flows from the collector to the emitter. Then, the transistor is on (a switch is closed). When no current flows into the base, no current flows from the collector to the emitter. Then, the transistor is off (a switch is open).

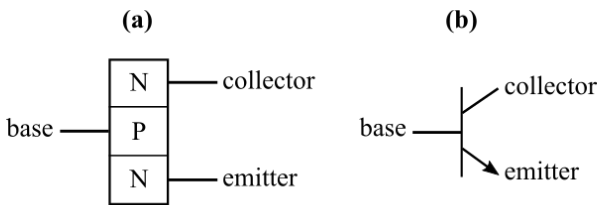

Figure 6. Bipolar junction transistor. **(a)** An NPN transistor **(b)** The symbol of an NPN transistor in the electric circuit

Since a transistor has a very fast response time, it is used as a very fast-acting switch in modern Tesla coils. There are several papers about modern Tesla coils with transistors [1,2,3,4]. However, their designs were complicated for students in an introductory physics course. To deliver the core concepts of Ampere's law, Faraday's law, an *RLC* circuit, and a transistor, we choose a modern Tesla coil with the simplest design. The circuit diagram of a modern Tesla coil used here is shown in Figure 7.

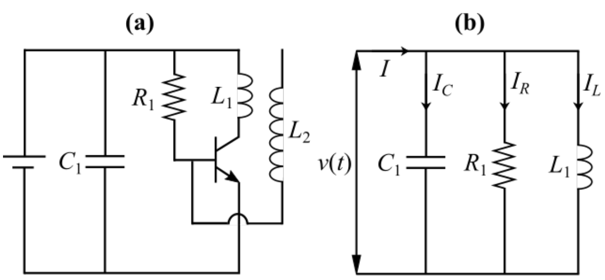

Figure 7. A modern Tesla coil with a transistor **(a)** The circuit diagram of the modern Tesla coil **(b)** The equivalent circuit diagram of the Tesla coil in **(a)** when the transistor is on

Now, we will describe how the modern Tesla coil in Figure 7 (a) works. Initially, the transistor is off. When the primary circuit is connected to DC voltage source, a current flows through the base of the transistor and the transistor is on. Therefore, a current flows through the primary coil. Then, the primary circuit becomes a parallel *RLC* circuit as shown in Figure 7 (b). The DC voltage source provides a constant current I to the parallel *RLC* circuit. To find the current flowing through the inductor, we apply Kirchhoff's junction rule, which states that the sum of currents entering a junction equals the sum of currents leaving the junction. Then, Kirchhoff's junction rule gives

$$I_C + I_R + I_L = I \tag{16}$$

Here, I_C, I_R, and I_L are currents flowing through a capacitor, a resistor, and an inductor. Current I is the current flowing into the junction and it is a constant. Since the voltage $v(t)$ across each element is the same in parallel connection, currents through a capacitor and a resistor are expressed as follows,

$$\begin{aligned} I_C &= \frac{dQ}{dt} = \frac{d(C_1 v)}{dt} = C_1 \frac{dv}{dt} \\ I_R &= \frac{v}{R_1} \end{aligned} \tag{17}$$

Using equation (17), equation (16) becomes

$$C_1 \frac{dv}{dt} + \frac{v}{R_1} + I_L = I \tag{18}$$

Since $v = L_1 \frac{dI_L}{dt}$, equation (18) becomes

$$L_1 C_1 \frac{d^2 I_L}{dt^2} + \frac{L_1}{R_1}\frac{dI_L}{dt} + I_L = I \tag{19}$$

If we divide the equation above by $L_1 C_1$, we get

$$\frac{d^2 I_L}{dt^2} + 2\gamma \frac{dI_L}{dt} + \omega^2 I_L = \omega^2 I \tag{20}$$

Here, $\gamma = 1/(2R_1C_1)$ and $\omega = 1/\sqrt{L_1 C_1}$. The general solution of equation (20) is [7],

$$I_L(t) = I_c(t) + I_p(t) \tag{21}$$

Here, $I_c(t)$ is a complementary function, which is the general solution of equation (20) when the right-hand side equals zero and $I_p(t)$ is a particular solution that satisfies equation (20). From Appendix B, the complementary function is given as follows,

$$I_c(t) = e^{-\gamma t}(A\cos\omega' t + B\sin\omega' t) \tag{22}$$

Here, $\omega' = \sqrt{\omega^2 - \gamma^2}$ and A and B are constants that can be determined by the initial conditions. Since the right-hand side is a constant in equation (20), a particular constant can satisfy equation (20). Therefore, we try a particular solution, $I_p(t) = D$. When $I_p(t) = D$ is plugged into equation (20), we find that $I_p(t) = D = I$.

Then, equation (21) is written as follows,

$$I_L(t) = e^{-\gamma t}(A\cos\omega' t + B\sin\omega' t) + I \qquad (23)$$

Since the initial current flowing through the inductor is zero and the initial voltage across the capacitor is zero, the initial conditions are

$$\begin{aligned} I_L(0) &= 0 \\ v(0) &= L_1 \frac{dI_L(0)}{dt} = 0 \end{aligned} \qquad (24)$$

From the initial conditions above, we obtain $A = -I$ and $B = -\gamma I / \omega'$. Therefore, equation (23) becomes

$$I_L(t) = -Ie^{-\gamma t}(\cos\omega' t + \frac{\gamma}{\omega'}\sin\omega' t) + I \qquad (25)$$

Equation (25) shows that the current I_L in the primary coil undergoes damped oscillation. This shows that the primary circuit in the modern Tesla coil in Figure 7 behaves the same as the primary circuit in a classical Tesla coil.

4. Conclusion

This article reviewed how a classical Tesla coil works. An ideal Tesla coil is two magnetically coupled *LC* circuits. First, we described an *LC* circuit as an electromagnetic oscillator which is one of the typical problems in an introductory physics course. By solving the differential equation for charge $Q(t)$, we showed that the current in the primary coil oscillates sinusoidally. The oscillating current in the primary coil provides an oscillating magnetic flux in the secondary coil. This oscillating magnetic flux induces an oscillating voltage in the secondary coil according to Faraday's law. Second, we described two magnetically coupled *LC* circuits by solving the second order linear differential equations appropriate for undergraduate students' level. When the coupling between the primary and the secondary coil is weak, we showed that the charges on the capacitors in the primary and the secondary circuit ($Q_1(t)$ and $Q_2(t)$) are sinusoidally oscillating with oscillating amplitudes. We also showed that the energy is transferred back and forth between the primary and the secondary circuits.

For the modern Tesla coil, we used the simplest design to deliver the core concepts of Ampere's law, Faraday's law, an *RLC* circuit, and a transistor. We showed that a parallel *RLC* of the primary circuit in the modern Tesla coil plays the same role as the series *RLC* of the primary circuit in the classical Tesla coil.

Using a transistor in a Tesla coil not only improves the efficiency of switching time, but it can also provide an educational benefit since a transistor is a key component in many electronic devices. Many students in electrical engineering will encounter transistors in the circuit design in their major courses. Therefore, a modern Tesla coil is a great multidisciplinary example in undergraduate STEM education since it combines the concepts in physics and electrical engineering.

Appendix

Appendix A

Appendix A will show a general method of solving a second order linear differential equation shown below.

$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0 \qquad (A1)$$

We notice that the sum of the second derivative of Q and a constant times Q itself becomes zero. The exponential function satisfies this. Therefore, we try the following as a solution [8].

$$\begin{aligned} Q(t) &= Ae^{st} \\ \frac{d^2Q}{dt^2} &= s^2Ae^{st} \end{aligned} \qquad (A2)$$

Here, A and s are constants. By plugging equation (A2) into equation (A1), we get

$$(s^2 + \omega^2)A = 0 \qquad (A3)$$

To have a solution other than $A = 0$, we get

$$s^2 + \omega^2 = 0 \rightarrow s = \pm i\omega \qquad (A4)$$

Since both $Ae^{i\omega t}$ and $Be^{-i\omega t}$ satisfy equation (A1), the general solution is a linear combination of $Ae^{i\omega t}$ and $Be^{-i\omega t}$.

$$Q(t) = Ae^{i\omega t} + Be^{-i\omega t} \qquad (A5)$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, equation (A5) is written as follows,

$$\begin{aligned} Q(t) &= A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t) \\ &= (A+B)\cos\omega t + i(A-B)\sin\omega t \\ &= D\cos\omega t + F\sin\omega t \end{aligned} \qquad (A6)$$

Here, $D = A + B$ and $F = i(A - B)$ are new constants.

Appendix B

Appendix B will show the details on solving a second order linear differential equation shown below.

$$\frac{d^2Q}{dt^2} + 2\beta\frac{dQ}{dt} + \omega^2 Q = 0 \qquad (B1)$$

We notice that the sum of the second derivative of Q, a constant times the first derivative of Q, and a constant times Q itself becomes zero. The exponential function satisfies this. Therefore, we try the following as a solution [8].

$$\begin{aligned} Q(t) &= Ae^{st} \\ \frac{dQ}{dt} &= sAe^{st} \\ \frac{d^2Q}{dt^2} &= s^2Ae^{st} \end{aligned} \qquad (B2)$$

Here, A and s are constants. By plugging equation (B2) into equation (B1), we get

$$(s^2 + 2\beta s + \omega^2)A = 0 \qquad \text{(B3)}$$

To have a solution other than $A = 0$, we get

$$s^2 + 2\beta s + \omega^2 = 0 \rightarrow s = -\beta \pm \sqrt{\beta^2 - \omega^2} \qquad \text{(B4)}$$

In a Tesla coil, the values of R, L, and C satisfy $\omega > \beta$ (damped oscillation). Therefore, the constant s in equation (B4) can be written as follows,

$$s = -\beta \pm i\sqrt{\omega^2 - \beta^2} = -\beta \pm i\omega'.$$

Here, $\omega' = \sqrt{\omega^2 - \beta^2}$ is the angular frequency of damped oscillation. Since both $Ae^{(-\beta + i\omega')t}$ and $Be^{(-\beta - i\omega')t}$ satisfy equation (B1), the general solution is a linear combination of $Ae^{(-\beta + i\omega')t}$ and $Be^{(-\beta - i\omega')t}$.

$$Q(t) = e^{-\beta t}(Ae^{i\omega' t} + Be^{-i\omega' t}) \qquad \text{(B6)}$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, equation (B6) is written as follows,

$$Q(t) = e^{-\beta t}(D\cos\omega' t + F\sin\omega' t) \qquad \text{(B7)}$$

Here, $D = A + B$ and $F = i(A - B)$ are new constants.

Appendix C

Appendix C will show the details on solving the differential equations in equation (13) when two LC circuits are oscillating at the same angular frequency, which is the condition for the maximum energy transfer between the primary circuit and the secondary circuit in a Tesla coil. When $\omega_1 = \omega_2 = \omega$, equation (13) becomes

$$\begin{aligned} \frac{d^2Q_1}{dt^2} + \omega^2 Q_1 + \frac{M}{L_1}\frac{d^2Q_2}{dt^2} = 0 \\ \frac{d^2Q_2}{dt^2} + \omega^2 Q_2 + \frac{M}{L_2}\frac{d^2Q_1}{dt^2} = 0 \end{aligned} \qquad \text{(C1)}$$

As we have seen in Appendix A, the exponential function satisfies the differential equations in equation (C1). Therefore, we try the following as a solution.

$$\begin{aligned} Q_1(t) &= A_1 e^{st} \\ Q_2(t) &= A_2 e^{st} \\ \frac{d^2Q_1}{dt^2} &= s^2 A_1 e^{st} \\ \frac{d^2Q_2}{dt^2} &= s^2 A_2 e^{st} \end{aligned} \qquad \text{(C2)}$$

Here, A_1, A_2, and s are constants. By plugging equation (C2) into equation (C1), we get

$$\begin{aligned} (s^2 + \omega^2)A_1 + \tfrac{M}{L_1}s^2 A_2 = 0 \\ \tfrac{M}{L_2}s^2 A_1 + (s^2 + \omega^2)A_2 = 0 \end{aligned} \qquad \text{(C3)}$$

To have a solution other than $A_1 = A_2 = 0$, the determinant of coefficients of A_1 and A_2 in equation (C3) should be zero.

$$\begin{vmatrix} s^2 + \omega^2 & \frac{M}{L_1}s^2 \\ \frac{M}{L_2}s^2 & s^2 + \omega^2 \end{vmatrix} = 0 \qquad \text{(C4)}$$

By calculating the determinant above, we get

$$\begin{aligned} &(s^2 + \omega^2)^2 - \tfrac{M^2}{L_1 L_2}s^4 = 0 \\ &(s^2 + \omega^2)^2 - k^2 s^4 = 0 \\ &s^2 + \omega^2 = \pm k s^2 \\ &s^2 = -\tfrac{\omega^2}{1 \mp k} \\ &s = \pm i\tfrac{\omega}{\sqrt{1-k}} = \pm ix, \quad \pm i\tfrac{\omega}{\sqrt{1+k}} = \pm iy \end{aligned} \qquad \text{(C5)}$$

Here, $k = M / \sqrt{L_1 L_2}$ (a coupling coefficient), $x = \omega / \sqrt{1-k}$, and $y = \omega / \sqrt{1+k}$. The general solution for $Q_1(t)$ and $Q_2(t)$ is a linear combination of Ae^{ixt}, Be^{-ixt}, Ce^{iyt}, and De^{-iyt}.

$$\begin{aligned} Q_1(t) = A_1 e^{ixt} + B_1 e^{-ixt} + C_1 e^{iyt} + D_1 e^{-iyt} \\ Q_2(t) = A_2 e^{ixt} + B_2 e^{-ixt} + C_2 e^{iyt} + D_2 e^{-iyt} \end{aligned} \qquad \text{(C6)}$$

The constants of $Q_2(t)$ in equation (C6) are not independent and they are related to constants of $Q_1(t)$ by equation (C3). By substituting $s = \pm ix$ and $s = \pm iy$ into equation (C3), we get

$$\begin{aligned} &\text{For } s = \pm ix \text{ case, } A_2 = -\sqrt{\tfrac{L_1}{L_2}}A_1 \\ &\text{For } s = \pm iy \text{ case, } A_2 = \sqrt{\tfrac{L_1}{L_2}}A_1 \end{aligned} \qquad \text{(C7)}$$

Then, equation (C6) becomes

$$\begin{aligned} Q_1(t) &= A_1 e^{ixt} + B_1 e^{-ixt} + C_1 e^{iyt} + D_1 e^{-iyt} \\ Q_2(t) &= -\sqrt{\tfrac{L_1}{L_2}}\left\{(A_1 e^{ixt} + B_1 e^{-ixt}) - (C_1 e^{iyt} + D_1 e^{-iyt})\right\} \end{aligned} \qquad \text{(C8)}$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the equations above are written as follows,

$$\begin{aligned} Q_1(t) &= F_1\cos xt + G_1\sin xt + H_1\cos yt + J_1\sin yt \\ Q_2(t) &= -\sqrt{\tfrac{L_1}{L_2}}\{(F_1\cos xt + G_1\sin xt) \\ &\qquad - (H_1\cos yt + J_1\sin yt)\} \end{aligned} \qquad \text{(C9)}$$

Here, $F_1 = A_1 + B_1$, $G_1 = i(A_1 - B_1)$, $H_1 = C_1 + D_1$, and $J_1 = i(C_1 - D_1)$ are new constants. By applying the initial conditions in equation (14) to equation (C9), we get

$$\begin{aligned} F_1 + H_1 = Q_0 \qquad (i) \\ xG_1 + yJ_1 = 0 \qquad (ii) \\ F_1 - H_1 = 0 \qquad (iii) \\ xG_1 - yJ_1 = 0 \qquad (iv) \end{aligned} \qquad \text{(C10)}$$

We get $F_1 = H_1 = Q_0 / 2$ from (i) and (iii) and $G_1 = J_1 = 0$ from (ii) and (iv) in equation (C10). Therefore, equation (C9) becomes

$$\begin{aligned} Q_1(t) &= \frac{Q_0}{2}(\cos xt + \cos yt) \\ &= Q_0 \cos(\tfrac{x+y}{2}t)\cos(\tfrac{x-y}{2}t) \\ Q_2(t) &= -\sqrt{\frac{L_1}{L_2}}\frac{Q_0}{2}(\cos xt - \cos yt) \\ &= \sqrt{\frac{L_1}{L_2}}Q_0 \sin(\tfrac{x+y}{2}t)\sin(\tfrac{x-y}{2}t) \end{aligned} \quad (C11)$$

Here, the following trigonometric identities are used.

$$\begin{cases} \cos\theta + \cos\varphi = 2\cos(\frac{\theta+\varphi}{2})\cos(\frac{\theta-\varphi}{2}) \\ \cos\theta - \cos\varphi = -2\sin(\frac{\theta+\varphi}{2})\sin(\frac{\theta-\varphi}{2}) \end{cases} \quad (C12)$$

Now, we are going to consider when the coupling is weak $(k << 1)$, which is the case for a Tesla coil. The x and y in equation (C5) are approximated as

$$\begin{aligned} x &= \frac{\omega}{\sqrt{1-k}} = \omega(1+\tfrac{1}{2}k) \\ y &= \frac{\omega}{\sqrt{1+k}} = \omega(1-\tfrac{1}{2}k) \end{aligned} \quad (C13)$$

Here, for $z << 1$, $(1+z)^n \approx 1+nz$ is used. From equation (C13), we get

$$\begin{aligned} \frac{x+y}{2} &= \omega \\ \frac{x-y}{2} &= \frac{1}{2}k\omega \end{aligned} \quad (C14)$$

If we substitute equation (C14) into equation (C11), we obtain

$$\begin{aligned} Q_1(t) &= (Q_0 \cos\tfrac{1}{2}k\omega t)\cos\omega t \\ Q_2(t) &= (\sqrt{\frac{L_1}{L_2}}Q_0 \sin\tfrac{1}{2}k\omega t)\sin\omega t \end{aligned} \quad (C15)$$

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