On the Procedural-Conceptual Based Taxonomy and Its Adaptation to the Multi-Dimensional Approach SPUR to Assess Students' Understanding Mathematics

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Abstract In this paper we propose a new cognitive taxonomy which is so-called the *PCK taxonomy* (based on Procedural and Conceptual Knowledge) and adapt this taxonomy to the multi-dimensional approach SPUR (Skills, Properties, Uses, Representations) in analyzing the written mathematics assessments given at some high schools located in Binh Dinh province, Vietnam. Base on the findings we discuss and propose methods to establish the written mathematics assessments in order to assess students' understanding mathematics more accurately.

Keywords: *mathematics assessment, understanding mathematics, SPUR, cognitive taxonomy, procedural knowledge, conceptual knowledge*

1. Introduction

One of important problems in teaching and learning mathematics is how to teach mathematics effectively and how can students understand good mathematical knowledge, apply the obtained knowledge to solve real life problems. In order to know how good a student's level of understanding mathematics is, firstly, it is necessary to consider the content of the curriculum, see if teaching knowledge for student is appropriate; secondly, it is necessary to implement students' knowledge assessment objectively, fairly and accurately.

In a mathematics class, students often have their own strengths in learning mathematics. They can attack problems by their own different ways. The learning method of each student is different, so the mathematics program needs to have a multi-dimensional perspective to show the appropriate balance with the students. If we innovate teaching methods expressing multi-dimensional approach to a mathematical knowledge, it should reflect this view as taught to have an appropriate and accurate assessment result.

Using a multi-dimensional approach as a tool in assessing students' understanding mathematics provides a foundation for building a teaching program and evaluating understanding mathematics in the most appropriate way. In this paper we mainly refer to the multi-dimensional approach of SPUR (Skills, Properties, Uses, Representations), which was discovered by Thompson & Senk (2008, [1]) (see also in Thompson & Kaur, 2011, [2]; Bleiler & Thompson, 2013, [3]; Wong & Kaur, 2015, [4]; Khaw & Kaur, 2017 [5]).

The *procedural* and *conceptual* aspects of knowledge was studied firstly by Hiebert and Lefevre (1986, [6]). After that there are some other researches on the relation between these types of knowledge (e.g. Baroody & Ginsburg, 1986, [7]; Rittle-Johnson and Siegler, 1998, [8]; Rittle-Johnson & Alibali, 1999, [9]; Barr, Doyle, et. al., 2003, [10]; Arslan, 2010, [11]; Rittle-Johnson and Schneider, 2015 [12]; Piper, et. al., 2016 [13]).

Smith et. al. (1996) proposed the *MATH taxonomy* (Mathematical Assessment Task Hierarchy) which is a modification of the Bloom taxonomy (1956, [14]). Based on this taxonomy, Wong and Kaur (2015, [4]) adapted the MATH taxonomy to the SPUR multi-dimensional approach to derive a set of assessment standards to examine the quality of mathematics written assessment from three secondary schools in Singapore.

The main aim of this paper is to propose a new cognitive taxonomy which is so-called the PCK *taxonomy* (based on the Procedural and Conceptual Knowledge), and to adapt this new taxonomy to the multi-dimensional approach SPUR in analyzing the written mathematics assessments given at some high schools located in Binh Dinh province, Vietnam. Based on the findings we discuss and propose methods to establish the written mathematics assessments in order to assess students' understanding mathematics more accurately.

2. Content

2.1. Understanding Mathematics and Assessment of Students' Mathematical Understanding

Skemp (1976, [15]) defined two types of understanding: *Relational* and *instrumental*. He wrote:

"Relational understanding in a mathematical situation consists of recognizing a task as one of a particular class for which one already knows a rule. To find the area of a rectangle, multiply the length by the breadth.

For a triangle, calculate half time the base time the perpendicular height. If the figure is a parallelogram, multiply the length of one pair of parallel sides by the perpendicular distance between them. The kind of learning which makes this possible, which I call instrumental learning, is the memorizing of such rules."

He described *relational understanding* as "knowing both what to do and why", and the process of learning relational mathematics as "building up a conceptual structure". *Instrumental understanding*, on the other hand, was simply described as "rules without reasons". He also wrote:

"I now believe that there are two effectively different subjects being taught under the same name, 'mathematics'."

Hiebert and Carpenter (1992, [16]) specifically defined *mathematical understanding* as involving the building up of the conceptual 'context' or 'structure' mentioned above. They also stated that:

"Understanding usually cannot be inferred from a single response on a single task; any individual task can be performed correctly without understanding. A variety of tasks, then, are needed to generate a profile of behavioral evidence."

Since understanding is a complex network, if we are to assess this understanding then we need to try and access the different connections that a student has. All too often, when we assess students' understanding in mathematics, we are gaining insight into only a small part of this network. Also, being able to carry out a mathematical task implies only that some understanding is there (this might only be links between the concept and a procedural model associated with that concept), not the extent of that understanding. Also, we can never have complete understanding; we can always develop understanding by developing more links, for example between apparently very different concepts that we have not associated together previously.

Barmby, Harries, Higgins and Suggate (2007, [17]) analyzed the following methods to assess students' mathematical understanding suggested by Hiebert and Carpenter (1992, [16]):

- Students' errors.
- Connections made between symbols and symbolic procedures and corresponding referents.
- Connections between symbolic procedures and informal problem-solving situations.
- Connections made between different symbol systems.

In this paper we consider the multi-dimensional approach *SPUR* to assess students' understanding mathematics.

2.2. SPUR - A Multi-dimensional Approach to Assess Students' Mathematical Understanding

One approach to a multi-dimensional perspective on mathematical understanding is that of SPUR, denoting for four dimensions Skills, Properties, Uses and Presentation, which was firstly proposed by Thompson and Senk (2008, [1]), and then developed by Thompson and Kaur (2011, [2]), Bleiler and Thompson (2013, [3]). Four dimensions of this approach is described as follows.

- *Skills* represents those procedures that students should master with fluency;

- *Properties* are the principles underlying the mathematics;

- *Uses* represents the applications of the concepts to the real world or to other concepts in mathematics;

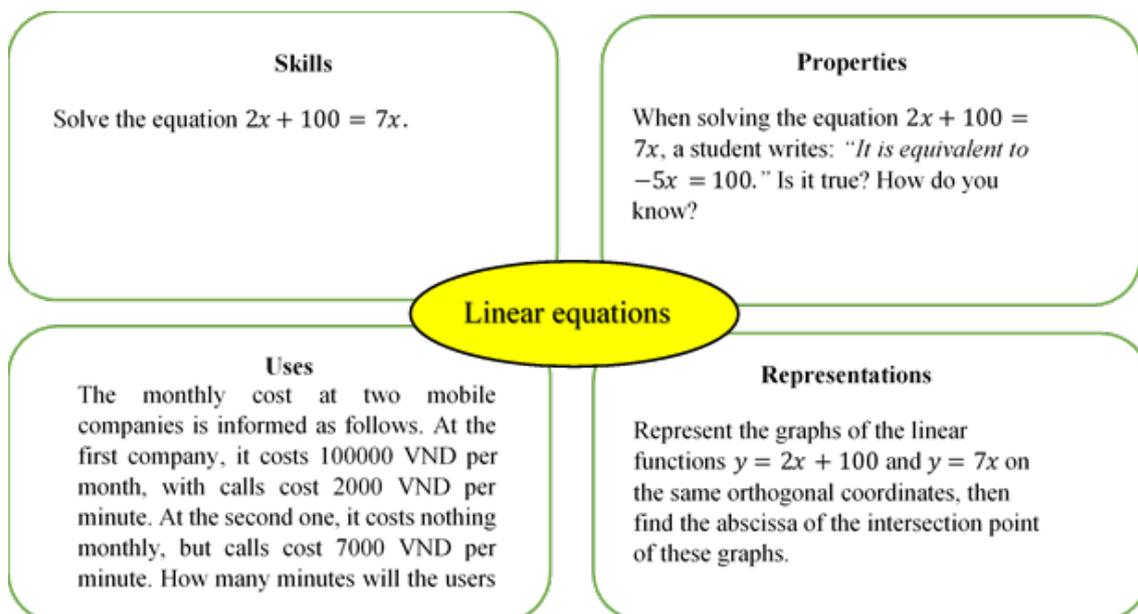- *Representations* are graphs, pictures, symbols, and other visual depictions of the concepts.

Skills

Solve the equation $2x + 100 = 7x$.

Properties

When solving the equation $2x + 100 = 7x$, a student writes: *"It is equivalent to $-5x = 100$."* Is it true? How do you know?

Uses

The monthly cost at two mobile companies is informed as follows. At the first company, it costs 100000 VND per month, with calls cost 2000 VND per minute. At the second one, it costs nothing monthly, but calls cost 7000 VND per minute. How many minutes will the users

Linear equations

Representations

Represent the graphs of the linear functions $y = 2x + 100$ and $y = 7x$ on the same orthogonal coordinates, then find the abscissa of the intersection point of these graphs.

Figure 1. Tasks illustrate for four dimensions of the SPUR approach

Although the SPUR is originally used for curriculum development, but then it is used as a powerful tool to assess students' mathematical understanding. This multi-dimensional assessment provides teachers with useful information about the depth of students' mathematical understanding. If the assessments are made only on one dimension, teachers may receive aberrant results on students' mathematical understanding. Whereas, assessments encompass all four dimensions will give teachers insights into strengths and weaknesses in their students' knowledge of the mathematical concepts. It allows teachers to orient the plan for next lessons.

Four dimensions of the SPUR approach is illustrated in Figure 1 with the considered knowledge is *linear equations*.

2.3. Integration of Conceptual and Procedural Knowledge

In many domains, students must learn both fundamental concepts and correct procedures for solving problems. Procedural and conceptual knowledge can be summarized in Table 1.

Table 1. Procedural and conceptual knowledge

Procedural knowledge	Conceptual knowledge
"Action sequences for solving problems."	"Explicit or implicit understanding of the principles that governs a domain and of the interrelations between pieces of knowledge in a domain." (Rittle-Johnson & Alibali, 1999, [9])
"Like a toolbox, it includes *facts*, *skills*, *procedures*, *algorithms* or *methods*."	"*Ideas*, *relationships*, *connections*, or having a *"sense"* of something." (Barr, Doyle, et. al., 2003, [10])
Learning that involves only memorizing operations with no understanding of underlying meanings."	"Learning that involves understanding and interpreting concepts and the relations between concepts." (Arslan, 2010, [11])
"To know *"how"* something happens in a particular way."	"To know *"why"* something happens in a particular way." (Hiebert & Lefevre, 1986, [6])

On the surface, conceptual and procedural knowledge appear to be clearly distinct. However, Hiebert and Lefevre (1986, [6]) made the following important points about these types of knowledge.

- Whereas conceptual knowledge, by definition, involves interconnected and, hence, meaningful knowledge, procedural knowledge may or may not be connected to other knowledge and, thus, may be meaningful or not.
- The distinction between these two types of knowledge can, in some cases, be fuzzy (Baroody & Ginsburg, 1986, [7]).
- Linking conceptual and procedural knowledge can greatly benefit the acquisition and application of the former, as well as the latter.

Rittle-Johnson and Siegler (1998, [8]) noted also that:

"An iterative relation between procedural and conceptual mastery, seems an especially plausible path of development".

The debate about whether skills or concepts is more important was replaced by a debate about their developmental order. Initially, two polar camps, namely the proponents of the *skills-first* view and those of the *concepts-first* view, dominated this debate. Baroody (2003, [18]) proposed the following four possible ways that skills and concepts can be related.

- Possibility 1: Skills first;
- Possibility 2: Concepts first;
- Possibility 3: Iterative development;
- Possibility 4: Simultaneous development.

He also offered in this paper three additional conclusions about the relations between conceptual and procedural knowledge (see more conclusions in the recent paper of Rittle-Johnson and Schneider, 2015, [6]).

1. Conceptual knowledge usually underlies procedural innovations.
2. Adaptive expertise involves the integration of conceptual and procedural knowledge.
3. Conceptual knowledge can play either a direct or an indirect role in the invention of procedures.

2.4. PCK Taxonomy

In this section we propose a definition for the cognitive taxonomy which is so-called the *PCK taxonomy* (**P** stands for **P**rocedural, **C** for **C**onceptual, and **K** for **K**nowledge). The PCK taxonomy consists of five levels:

- Level 1: (**PK1**) Familiar procedures.
- Level 2: (**PK2**) Complex procedures.
- Level 3: (**CK1**) Conceptual understanding.
- Level 4: (**CK2**) Conceptual applying.
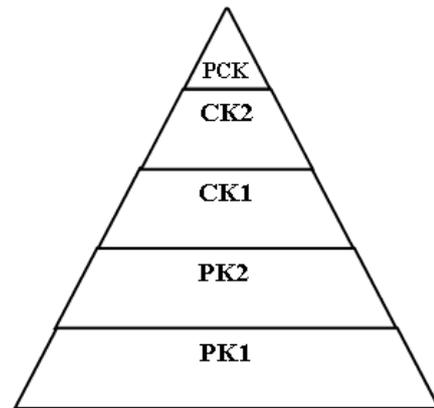- Level 5: (**PCK**) Procedural and conceptual knowledge integration.

Figure 2. Levels of the PCK taxonomy

The content of each level in the PCK taxonomy is described in Table 2, which is arranged from the lower to the higher level as shown in Figure 2.

Table 2. Content of levels in the PCK taxonomy

Categories	Students are able to …
(PK1) Familiar procedures	perform simple caculations; apply familiar formulas, procedures, algorithms.
(PK2) Complex procedures	perform complex calculations, high level reasons; link the different representations of the considered concepts in solving tasks.
(CK1) Conceptual understanding	recognize and explain the underlying concepts; recognize the relation of the considered concepts to the other concepts.
(CK2) Conceptual applying	apply the considered concept in solving problems; apply the considered concept to a real situation or to other concepts.
(PCK) Procedural and conceptual knowledge integration	integrate procedural knowledge to conceptual knowledge to solve problems; discover a new situation in which students need to integrate procedural and conceptual knowledge to solve.

We present in Table 3 some sample tasks to illustrate for five levels in the PCK taxonomy. The mathematical knowledge used here are linear function and equations which are important and familiar subjects for high school students.

Table 3. Examples to illustrate for levels in the PCK taxonomy

Categories	Examples
PK1	Solve the equation $3x = 50$.
PK2	Solve the equation $4x - 250 + \dfrac{2500}{x} = -5x + 50$.
CK1	Let be given the equation $mx = 50$, where m is a real parameter. Find the value of m such that the equation has a root $x = \dfrac{50}{3}$.
CK2	Find the value of m such that the graph of the functions $y = mx + 80$ and $y = 5x + 30$ intersect at the point whose abscissa equals to $\dfrac{50}{3}$.
PCK	A class would like to rent a bus for an excursion. The students ask prices at two tourist companies. The initial price at the company A is *800 K* VND*,* plused *20 K* VND for each kilometer, where *K=1,000*; The initial price at the company B is *300 K* VND, plused *50 K* VND for each kilometer. In your opinion, how many kilometers does this class need to spend in the excursion for which the cost paid for two companies are the same?

Here the mathematical knowledge *linear equations* is presented in different tasks with difficulty from the lower to the higher levels, corresponding to the levels in the PCK taxonomy, where in the fifth level (PCK) students need to integrate the conceptual knowledge on linear functions and equations to procedural knowledge on linear equations to mathematize the given realistic problem, reducing to solve a mathematical model.

2.5. Adapting the PCK to the SPUR to Assess Students' Understanding Mathematics

In this section we study the adaptation of the PCK taxonomy to the multi-dimensional approach SPUR to design the tasks assessing students' mathematical understanding.

Table 4. Two-dimensional matrix of the adaptation PCK to SPUR

SPUR's dimensions	Mathematics cognitive process by the PCK taxonomy				
	PK1	PK2	CK1	CK2	PCK
Skills	-	-	-	-	-
Properties	-	-	-	-	-
Uses	-	-	-	-	-
Representations	-	-	-	-	-

A description of each category of the PCK taxonomy across the four dimensions in this set of assessment standards, developed specifically for the purpose of this study, is given in Table 5.

Table 5. Description of each category in the adaptation of the PCK taxonomy and the SPUR

PCK's cognitive levels	Skills (S)	Properties (P)
PK1	Performing fluently simple calculations; applying proficiently familiar formulas, procedures, algorithms.	Recognizing which properties, mathematical principles being used in simple calculations, in applying familiar formulas, procedures, algorithms.
PK2	Performing exactly complex calculations, high level reasons; finding out the link between different representations of the considered concept.	Recognizing which properties, mathematical principles being used in complex calculations, in high level reasons; finding out the link between different representations of the considered concept by properties of each representation of this concept.
CK1	Recognizing quickly and exactly the underlying concepts used in the task, the relation of the considered concepts to the other concepts.	Recognizing the underlying concepts used in the task, the relation of the considered concepts to the other concepts based on their properties.
CK2	Applying proficiently the considered concept to solve problems, to a real situation, or to other concepts.	Applying properties of the considered concepts to solve problems; applying the concept to a real situation, or to other concepts with similar properties.
PCK	Integrating flexibly procedural knowledge and conceptual knowledge to solve problems; discovering new situations which need to integrate procedural and conceptual knowledge to solve.	Integrating properties of the concept to related principles, procedures, algorithms to solve problems; discovering new situations based on properties of the concept, integrating these properties and the corresponding procedures, algorithms to solve them.
PCK's cognitive levels	Uses (U)	Representations (R)
PK1	Performing familiar procedures or algorithms to solve mathematical models obtained by mathematization processes; applying familiar procedures or algorithms to solve other problems.	Performing familiar calculations in the standard representations of the considered concepts.
PK2	Performing complicated calculations, high level reasons to solve mathematical models obtained by mathematizing real situations; applying complex calculations, high level reasons to solve other problems.	Discovering new principles, algorithms to solve problems based on represented information of the considered concepts.
CK1	Determining the underlying concepts used in real situations or in process of solving problems.	Representing concepts in standard forms.
CK2	Applying the considered concepts to real situations or to other concepts.	Discovering new representations of the concepts.
PCK	Integrating procedural knowledge and conceptual knowledge to solve new (realistic) problems; discovering new (realistic) problems in which students need to integrate procedural and conceptual knowledge to solve.	Integrating procedures on represented information and conceptual knowledge to explain the represented information, deducing conclusion, solving problems.

2.6. Methods and Results

2.6.1. Research Questions

In this paper we will answer the following questions.

(1) What are the types of mathematical cognition by the PCK taxonomy represented in the mathematics written assessments at the considered schools' examination?

(2) What are the dimensions of mathematical understanding by the SPUR multi-dimensional approach represented in the mathematics written assessments at the considered schools' examination?

(3) What is the interpretation of the adaptation of the SPUR multi-dimensional approach to the PCK taxonomy in the mathematics written assessments at the considered schools' examination?

2.6.2. Research Subjects and Methodology

Our research subjects are the mathematics written assessments in Grade 10 at three high schools located in Quy Nhon city (Binh Dinh province, Vietnam), including Quoc Hoc high shool (QHH) – one of the best quality high schools in Quy Nhon city; Trung Vuong high school (TVH) - a little bit lower quality high school; and Tran Cao Van high school (TCVH) – a low quality high school in Quy Nhon city.

We focus our research on the assessments for the Chapter 2 in the mathematics curriculum as presented in the textbook *Algebra 10* (advanced) at three high schools mentioned above whose topic is "Functions", with the mathematical knowledge on linear and quadratic functions (resp. equations).

By analyzing the collected written assessments, we study the distributions of categories of cognitive demands in the PCK taxonomy, the dimensions of mathematical understanding by the SPUR multi-dimensional approach, and the interpretation of the adaptation of the SPUR to the PCK taxonomy represented in these mathematics written assessments, answering our research questions.

2.6.3. Data Analysis

We collected the written assessments at three considered schools for the Chapter 2 in the mathematics curriculum of the textbook *Algebra 10* (advanced). By the Document no. 8773/BGDĐT-GDTrH dated December 30, 2010 of the Vietnam Ministry of Education and Training (MOET), each assessment has two parts, one-part leaves for multiple-choice questions and the other for constructed-response questions. These schools are using the 10-scale grading system, in which the multiple-choice questions part occupies 60% (6 points) and the remaining one occupies 40% (4 points). Each assessment at QHH and TCVH consists of 19 questions or parts of questions, in which there are 15 multiple-choice questions and 04 constructed-response questions or part questions. At TVH, there are in total 16 questions or parts of questions in each assessment, in which there are 12 multiple-choice questions and 04 constructed-response questions or part questions. Each assessment is designed to meet *four levels* of the criteria for knowledge and skills applying now in Vietnam (including: *Recognizing*, *Understanding*, *Applying* (*low level*) and *Applying* (*high level*)), and

expressed in a *matrix of knowledge*. Students have to finish all questions in 45 minutes.

2.6.4. Results

In this section we answer three research questions mentioned above. We analyzed and assessed the collected written assessments at three schools QHH, TVH and TCVH, and then presented the number of questions or part questions corresponding to each cognitive level of the PCK taxonomy as in Table 6.

Table 6. Presentation of the mathematical cognitive levels of the PCK taxonomy

PCK's cognitive levels	Number of questions or parts of questions at three schools (percentage)
PK1	11 (20.37 %)
PK2	5 (9.26 %)
CK1	23 (42.59 %)
CK2	6 (11.11 %)
PCK	9 (16.67 %)

We have the following observations from Table 6.
- Many tasks in the assessments focus on the level CK1 – understanding concepts. Especially, more than half of questions at TCVH assess understanding concepts.
- The percentage of questions for the cognitive levels PK2 and CK2 is low. Especially, at TCVH, these kinds of tasks rarely appear.
- There are no realistic problems in all assessments at three schools, hence the integration of procedural and conceptual knowledge to solve realistic problems is missing.
- For some tasks, it is not clear to distinguish between the PK1, CK1 and PCK levels, because to solve these tasks, first of all students need to recognize which mathematical concepts are hidden, and then they solve the tasks by using familiar procedures. For example, let us consider the following multiple-choice question:

"*The domain of the function* $y = \dfrac{1}{x^2 + 2x - 3}$ *is*

(a) $\varnothing$ *(b)* $\{1; -3\}$

(c) $R \setminus \{1\}$ *(d)* $R \setminus \{1; -3\}$"

The mathematical concept hides behind this question is "quadratic equation". Whenever students recognize this underlying concept, the quadratic equation is easily solved by using a familiar procedure. For this question, we arrange it in the CK1 level. However, as we explain, this question can be considered as in the PK1 or PCK level.

On the other hand, if students use the property *"A fraction is well-defined if its denominator is non-zero"*, they can go directly to the procedure "Solving a quadratic equation". This is an adaptation between the **PK1** level in the PCK taxonomy to the dimension **Properties** in the SPUR multi-dimensional approach (denoted by **PK1-P**).

Now we are going to answer the second research question. The number of questions or part questions corresponding to each category of the SPUR multi-dimensional approach is presented in Table 7.

Table 7. Presentation of dimensions in the SPUR

SPUR's dimensions	Number of questions or parts of questions at three schools (percentage)
S	16 (29.63 %)
P	26 (48.15 %)
U	6 (11.11 %)
R	6 (11.11 %)

From this table, we have the following observations.

- The dimension **Properties** is assessed most frequently, especially with more than 48% at three schools, while the dimensions **Uses** and **Representations** are less appeared. One of the main reasons is that there are no realistic problems in the assessments, hence the category **Uses** appeared just as application of the considered concepts to other concepts.

- The same disadvantage appeared as in the PCK taxonomy, the distinction between dimensions **Skills** and **Properties** is not clear to recognize, because for some tasks, firstly students need to recognize which properties used to solve the problems, and then they use the known procedures to proceed solving them. By this process, we consider these tasks in the dimension **Properties**. For example, let us consider the following multiple-choice task.

"*The function $y = -4x^2 + 8x + 3$ is increasing monotone on which intervals*

(a) $(1; +\infty)$ *(b)* $(-1; +\infty)$

(c) $(-\infty; 1)$ *(d)* $(-\infty; -1)$?"

For this question, firstly students use the following property of the monotonicity of quadratic functions: "*For $a < 0$, the function $y = ax^2 + bx + c$ is increasing monotone on the interval $(-\infty; -\frac{b}{2a})$.*" Then they proceed by determining the coefficients a, b, c of the given function and replace into the formula. By this way, we consider this task in the dimension **Properties**. However, we may think that the monotonic property given above of quadratic functions is a known procedure to find the increasing intervals, then students can go directly to look for the coefficients a, b, c and replace them into the formula $(-\infty; -\frac{b}{2a})$. By this approach, we may consider this task as in the dimension **Skills**. This question can be considered as in the adaptation **PK1-P**.

On the other hand, students can draw the shape of the quadratic functions $y = ax^2 + bx + c$ when $a < 0$, based on the drawn shape, they can determine the interval for which this function is increasing monotone. This is actually an adaptation of the dimension **Representations** to the level **PK1**, which is denoted by **PK1-R**.

The disadvantages mentioned above show that we need to have an adaptation of the PCK taxonomy to the multi-dimensonal approach SPUR in the assessments of mathematical understanding. The number of questions or part questions corresponding to each category of the adaptation PCK to SPUR is presented in Table 8.

We have the following observation deduced from Table 8.

- For the adaptation *PCK-Skills*, the most frequently category assessed at three schools is **CK1-S,** with more than 42% questions or parts of questions assess fluent skill in recognizing the underlying concepts in solving problems.

- For the adaptation *PCK-Properties*, the most popular category assessed is **PK1-P**, with more than 48% questions or parts of questions focus on the recognition of which properties are used and applying familiar procedures and rules to solve the given problems.

- For the adaptation *PCK-Uses*, the category **CK2-U** appears more frequently, with more than 27% questions or parts of questions concentrate to assess the ability of students in applying the considered concept to other concepts. However, there are no questions or parts of questions at QHH assessing for the category **PK1-U**, and no questions at TVH and TCVH assessing for the category **PK2-U**. One of the main reasons is that there are no realistic problems in the assessments at three schools to use the familiar (resp. the complicated) procedures in solving the given tasks.

Table 8. Interpretation of the adaptation of PCK to SPUR

Categories of the adaption PCK-SPUR	Number of questions or parts of questions at three schools (percentage)
PK1-S	11 (20.37 %)
PK2-S	5 (9.26 %)
CK1-S	23 (42.59 %)
CK2-S	6 (11.11 %)
PCK-S	9 (16.67 %)
PK1-P	26 (48.15 %)
PK2-P	5 (9.26 %)
CK1-P	10 (18.52 %)
CK2-P	6 (11.11 %)
PCK-P	7 (12.96 %)
PK1-U	2 (3.70 %)
PK2-U	1 (1.85 %)
CK1-U	6 (11.11 %)
CK2-U	15 (27.78 %)
PCK-U	6 (11.11 %)
PK1-R	0 (0 %)
PK2-R	0 (0 %)
CK1-R	9 (16.67 %)
CK2-R	4 (7.41 %)
PCK-R	6 (11.11 %)

- For the adaptation *PCK-Representation*, in general there are not many questions assessing the adapted categories. Especially, there are no questions or parts of questions assessing the categories **PK1-R** and **PK2-R**. The most popular category appears in the assessments is **CK1-R**, with more than 16% questions or parts of questions focus on that.

3. Conclusion

Each written assessment is designed to meet four levels of the criteria for knowledge and skills applying now in

Vietnam (*Recognizing*, *Understanding*, *Applying* (*low level*) and *Applying* (*high level*)). However, these levels are quite general to establish written assessments. The SPUR multi-dimensional approach (resp. the PCK taxonomy) gives more specialized dimensions (resp. levels) for the assessment establishment. However, there are some disadvantage while considering which tasks belong to which dimensions or levels. Therefore, an adaptation of the SPUR multi-dimensional approach to the PCK taxonomy is necessary, solving obstacles appear.

Since there are no realistic problems in the written assessments given at three high schools, the dimension *Use* in the SPUR approach (and its adaptation to five levels of the PCK taxonomy) is in general missed. Hence, in establishing the written assessments, it is necessary to study the content of each adapted category in the adaptation PCK-SPUR to distribute the tasks. On the other hand, including more realistic problems in the written assessments will make all categories in the adaptation PCK-SPUR fulfilled. Moreover, the adaptation of the CK2 level to four dimensions of SPUR will increase its appearance whenever the tasks in the assessments are established with open-ended questions.

Acknowledgements

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