

The Influence of the Schools of Mathematical Thought to the Development of Mathematics Education

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Abstract In this article the effects of the ideas of the main schools of mathematical thought are studied on the development of mathematics education. Crucial problems that occupy nowadays the interest of those working in the field are also discussed, such as the future role of computers for the teaching and learning of mathematics. It is concluded that, although none of the existing schools of mathematical thought has succeeded in finding a solid framework for mathematics, most of the recent advances of this science were obtained through their disputations about the absolute mathematical truth. On the contrary, these disputations have created serious problems in the sensitive area of mathematics education, the most characteristic being the failure of the introduction of the “New Mathematics” to school education that distressed students and teachers for many years. The required thing for those working in this area of mathematics education is, without abandoning their persona; ideas, to search for a proper balance among the several philosophical aspects of mathematics. This will bring the required tranquillity in the area, in order to be developed smoothly for the benefit of the future generations.

Keywords: *philosophy of mathematics, platonism, formalism, intuitionism, logicism, mathematics education, problem-solving, mathematical modeling, computers in the teaching and learning of mathematics*

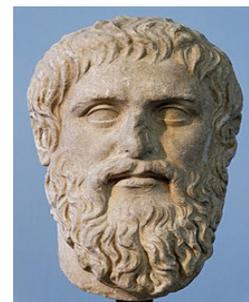
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1. Introduction

Philosophy emerged from the dialectics of the ancient Greeks, where commonplace beliefs and unanalysed concepts were interrogated and scrutinised. Philosophy may be defined as the systematic analysis and the critical examination of fundamental problems that involves exercise of the mind and intellect, including thought, enquiry, reasoning and its results: judgements, conclusions beliefs and knowledge [1].

In contrast to philosophy, *mathematics* used to be famous as being a clear, undisputable and without any further problems scientific branch. The Plato’s (Picture 1) view about the existence of an abstract, eternal and unchanged universe of mathematical forms dominated for centuries the scientific beliefs for the nature of mathematics¹. Consequently it was strongly believed that mathematics is not invented, but it is gradually discovered by humans (*Platonism*, e.g. see [2], Section 2). In a more general context, all those who believe that mathematics exists independently from the human mind belong to the school of *mathematical realism* and they are divided into

several categories with respect to their beliefs about the texture of the mathematical entities and the way in which we learn them ([2], Section 2).



Picture 1. Plato (424-377 BC)

However, the radical advances of Mathematics during the last two centuries, including the appearance of the non Euclidean Geometries, the axiomatic foundation of the Set Theory that enables one to consider four different forms of it (!), the proof of the Gödel’s Incompleteness Theorems, the eventual enrolment of informatics in the pure mathematical research, etc., as well as data collected from experimental researches of cognitive scientists and psychologists, have turned to a great deal currently the scientific views to the belief that mathematics is actually an *invention* of the human mind ([2], Sections 3 and 4). The *embodied mind theories* hold that mathematical thought is a natural outgrowth of the human cognitive

¹ In contrast to the Plato’s ideas the founder of the formal logic Aristotle (384-322 BC) believed that the mathematical forms are created by the human experience. Therefore, in terms of the modern terminology, whereas Plato was a *rationalist*, Aristotle supported ideas of *empiricism*.

apparatus, which finds itself in our physical universe. For example, the abstract concept of number springs from the experience of counting discrete objects. Thus humans construct, but do not discover, mathematics. There also exist intermediate theories stating that mathematics is a *mixture of human inventions (axioms, definitions) and discoveries (theorems)* [3].

In this dynamic environment of contravening ideas about the nature of mathematics the *philosophy of mathematics* was rapidly developed and the nowadays known *schools of mathematical thought* were gradually appeared in their typical form. It is recalled that the philosophy of mathematics is the branch of philosophy that studies the assumptions, foundations, and implications of mathematics, and aims to provide a viewpoint of the nature and methodology of mathematics, and to understand the place of mathematics in people's lives [4].

The target of the present study is to investigate the influence (positive and negative) of the several schools of mathematical thought to the development of mathematics education. The rest of the article is formulated as follows: The main schools of mathematical thought are briefly presented in Section 2, whereas Section 3 deals with the main focus of the article by examining how the beliefs of those schools have affected the sensitive area of mathematics education. A discussion follows in Section 4 on the future perspectives of mathematics education and the article closes with the general conclusions of this research that are stated in Section 5.

2. The Schools of Mathematical Thought

Two extreme philosophies have been tacitly appeared almost the orientation of mathematics almost from its origin as an autonomous science: *Formalism*, where emphasis is given to the content and *intuitionism*, where the attention is turned to problem-solving processes.

The axiomatic foundation of Geometry in Euclid's "*Elements*", the most famous in the world mathematical classic, is a characteristic example of the formalistic point of view. An analogous example for intuitionism is the less known to the West world Oriental counterpart "*Jiu Zhang Suan Shu*" (*Nine Chapters on Mathematics*) [5]. Although very different in form and structure from Euclid's "Elements", it has served as the foundation of traditional Oriental mathematics and it has been used as a mathematics text book for centuries in China and most of the other countries of Eastern Asia. Its title has been translated to English in various ways. Although "mathematics" seems to be a more accurate translation of "Suan Shu" than mathematical art, it seems that mathematics in the East was indeed more of an art as compared to mathematics in the West as a science.

Very many centuries later, during the 19th and the beginning of the 20th century, the well known *paradoxes* found in the Set Theory was the main reason of an intense dispute among the followers of the two philosophies, which however was extended much deeper into the mathematical thought.

Formalism claims that the mathematical statements may be thought of as statements about the consequences of certain string manipulation rules. For example, Euclidean

geometry is seen as consisting of some strings called "axioms", and some "rules of inference" to generate new strings (theorems) from the given ones. Apart from the *axiomatic foundation of mathematics*, the main beliefs of formalism include the *consistency* of the axioms and notions that does not permit the creation of absurd situations, the *law of the excluded middle* (something is either true or false) and the *possibility of the existence of a solution* (positive or negative) for each mathematical problem, even if it has not been found yet. For example, let A be the set of all sets. Then $A = A$ and also $A \neq A$, since A belongs to A and therefore A is a proper subset of A. But this is absurd, which means that the notion of the set A of all the sets is not consistent and therefore it does not exist

The main critique against formalism is that the genuine ideas and inspirations that occupy mathematicians are far removed from the manipulation games with the stings of axioms mentioned above. Formalism is thus silent on the question of which axiom systems ought to be studied, as none is more meaningful than another from a formalistic point of view.

The program of the leader of formalism David Hilbert (Picture 2) aimed to a complete and consistent axiomatic development of all branches of mathematic. However, the *Gödel's incompleteness theorems* put a definite end to his ambitious program. In fact, there is no system that can prove the consistency of another system, since it has to prove first its own consistency, which, according to the second of the above theorems it is impossible! Therefore, the best to hope is that the statement of a certain system's axioms, although it cannot be complete (first Gödel's theorem), it is consistent.



Picture 2. D. Hilbert (1862-1943)

On the other end, the main beliefs of *intuitionism* include the *primitive understanding of the natural numbers* (for the formalists proofs are needed for the consistency of the arithmetic operations among them) and the connection of the mathematical existence of an entity with the possibility of *constructing* it. For example, Zermelo proved that each non empty set can be well ordered (i.e. ordered in such way that each subset of it has a minimal element). However, for the intuitionists this theorem has not any value, since it does not suggest the way in which such an order could be constructed.

In addition, intuitionists do not accept neither the law of the excluded middle nor the possibility of the existence of a solution for each problem, although problems without a positive or negative solution have not been appeared in the history of mathematics until now. Further, the formalistic view that a notion's consistency guarantees its existence is

completely unacceptable for the intuitionists. L. Kronecker (1823- 1891), one of the pioneers of intuitionism, used to say that “God created the natural numbers, whereas all the other mathematical entities have been created by the humans”. The leaders of intuitionism were L. E. J. Brouwer (Picture 3) and H. Weyl (1885-1965).

In intuitionism, the term "construction" is not clearly defined, and that has led to criticisms. Attempts have been made to use the concepts of Turing machine or computable function to fill this gap, leading to the claim that only questions regarding the behaviour of finite algorithms are meaningful and should be investigated in mathematics. This has led to the study of the *computable numbers*, first introduced by Alan Turing [6] and associated with the theoretical computer science.



Picture 3. L. E. J. Brouwer (1881-1965)

The study of the history of mathematics reveals that there exists a continuous oscillation between formalism and intuitionism [7]. This oscillation is symbolically sketched in Figure 1, where the two straight lines represent the two philosophies, while the continuous broadening of space between the lines corresponds to the continuous increase of mathematical knowledge. According to Verstappen [8] the period of this oscillation is of about 50 years, which has been also crossed by Galbraith [9] by studying a diagram, due to Shirley, representing a parallel process between the alterations of the economical conditions and the changes appearing to the mathematical education systems of the developed west countries.

Examples of how the “mathematics pendulum” swung from one extreme to the other over the span of about a century, include the evolution from the mathematics of Bourbaki to the reawaking of experimental mathematics, from the complete banishment of the “eye” in the theoretical hard sciences to the computer graphics as an integral part of the process of thinking research and discovery and also the paradoxical evolution from the invention of “pathological monsters”, such as Peano’s curve or Cantor’s set – which Poincare said that should be cast away to a mathematical zoo never to be visited again – to the birth of Mandelbrot’s Fractal Geometry of Nature [10]. To Mandelbrot’s surprise and to everyone else’s, it turns out that these strange objects, coined *fractals*, are not mathematical anomalies but rather the very patterns of nature’s chaos.

Logicism is another important school of mathematical thought appeared in the beginning of the 20th Century. Logicians believe that mathematics is reducible to logic, and hence nothing but a part of logic. More explicitly, the concepts of mathematics can be derived from logical concepts through explicit definitions and the theorems from logical axioms through purely logical deduction.

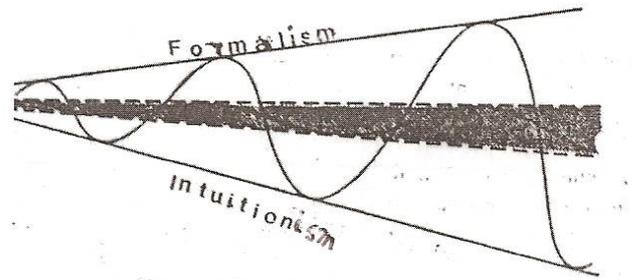


Figure 1. The oscillation between formalism and intuitionism

Gottlob Frege (Picture 4), the founder of logicism, built up arithmetic from logic by using a general principle of comprehension, which he called "**Basic Law V**". However Frege’s construction was flawed, when Russell proved that Basic Law V is inconsistent. Later, Bob Hale, Crispin Wright, and other logicians returned to a program close to Frege's by replacing the Basic Law V with the **Hume’s principle**². Although all the properties of numbers can be derived from the Hume's principle, this was not enough for Frege, who required Basic Law V to be able to give an explicit definition of the numbers. As he stated, “...this does not exclude the possibility that the number 3 is in fact Julius Caesar”! In addition, many of the weakened principles that they have had to adopt to replace Basic Law V no longer seem to be so obviously analytic, and therefore purely logical.



Picture 4. G. Frege (1848-1925)

In continuing with the schools of mathematical thought, **structuralism** is a position holding that mathematical theories describe structures, and that mathematical objects are exhaustively defined by their places in such structures; e.g. it maintains that all real numbers are completely defined by their places in the real line. Consequently, the kind of existence of each mathematical object depends on that of the structure in which it is embedded ([11], Chapter 10).

Although the above schools of mathematical thought and especially the first two of them are those which affected more the development of mathematics education, their catalogue does not end here. Thus, apart from mathematical realism, the embodied mind theories, etc., that have been mentioned in our Introduction, several other variations of mathematical thought have been appeared in the history of mathematics, each one having its own advantages and weaknesses [4,11].

² The Hume’s principle states that the number of objects falling under the concept F equals to the number of objects falling under the concept G , if and only if, the extension of F and the extension of G can be put into one-to-one correspondence.

3. The Effects of the Philosophy of Mathematics on the Development of Mathematics Education

The traditional components of the material of the school mathematics, i.e. Arithmetic, Euclidean Geometry, Trigonometry and Elementary Algebra, had remained stable for many years, almost from the time of Napoleon the Great!

However, as a consequence of the “mathematics pendulum” swing, dramatic changes also happened in the area of mathematical education during the last 50-60 years. First, the result of the post – war effort to bring mathematics as a teaching subject into harmony with mathematics as a science, as it has been developed since the last quarter of the 19th century, with an increasing gap between school mathematics and modern higher level mathematics, was the introduction, during the 60’s, of the “*New Mathematics*” in the curricula of studies. New chapters were added in the curricula, like Set Theory, elements from Linear and Abstract Algebra (matrices, determinants, algebraic structures, etc.) and Mathematical Logic, probability and Statistics and of course Mathematical Analysis up to the study of integrals in one variable and even simple forms of Differential Equations.

The way of presentation of the material was also changed, since the traditional inductive methods involving many examples and applications gave their place to a strict, axiomatic presentation that created many difficulties not only to the students, but also to the teachers,, who in addition were not adequately prepared to teach the new topics introduced in the curricula. Moreover, the volume of the total amount of material to be taught was enormously increased, since some space should be also found for the old, traditional school mathematics.

Therefore, it did not take many years to realize that the new curricula did not function satisfactorily all the way through, from primary school to university, even if the problems varied with the level [12]. Thus, and after the rather vague “wave” of the “*back to the basics*”, considerable emphasis has been placed during the 80’s on the use of the problem as a tool and motive to teach and understand better mathematics [13], with two main components: *Problem – Solving*, where emphasis was to the use of *heuristics* (solving strategies) for the solution of mathematical problems [14,15] and *Mathematical Modelling and Applications*, dealing with the formulation and solution of a special type of mathematical problems generated by corresponding problems of the real world and the everyday life [16,17]. The attention was turned also to *Problem - Posing*, i.e. to the process of extending existing or creating new problems [18].

The excessive emphasis given during the 80’s on the use of the heuristics for problem-solving has received several critiques [19,20] suggesting that the attention should be turned rather to the presentation of well prepared examples (solved problems) ant to the automation of rules. The argument was that these approaches facilitate better the *transfer of knowledge* and the acquisition by the students of the proper *schemas*, than the analytic methods of the problem-solving strategies that impose a heavy cognitive weight on students. On the contrary, mathematical

modelling has been evolved nowadays to a teaching method of mathematics, usually referred as *application-oriented teaching of mathematics* [21].

Another recent approach of mathematics education is the utilization of *informatics* as a tool for the teaching and learning of mathematics. In fact, the animation of figures and of mathematical representations, provided by suitable software mathematical packages, increases the students’ imagination and helps them to find easier the solutions of the corresponding problems. The role of mathematical theory after this is not to convince, but to explain.

Moreover, by thinking as a computer scientist does, students become aware of behaviours and reactions that can be captured in algorithms or can be analysed within an algorithmic framework. *Computational thinking*, the modern expression of algorithmic thinking [22], gives them nowadays a different framework for visualizing and analyzing, a whole new perspective of solving strategies. Figure 2 [23], represents how the two different modes of thinking, i.e. computational and *critical thinking*, are combined with the existing mathematical knowledge to solve a complicated problem. This approach is based on the fact that, when the already existing knowledge is adequate, the necessary for the problem’s solution new knowledge is obtained through critical thinking, while computational thinking is applied to design and to obtain the solution.

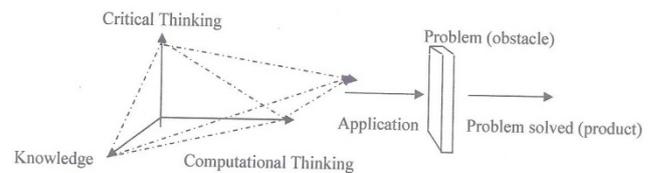


Figure 2. Computational thinking in Problem-Solving

In concluding, critical thinking is a prerequisite to knowledge acquisition and application to solve problems, but not a sufficient condition when we face complex problems of the real life (e.g. technological problems), which require also a pragmatic way of thinking such as the computational thinking is.

4. Discussion

In the name of introduction of modern mathematical topics in the school curricula, like Mathematical Analysis, Analytic Geometry, Probability Theory, etc., the teaching of the traditional *Euclidean Geometry* has drastically restricted and neglected. For example, nowadays we reach to the point that in the Greek Upper High School (Lyceum) the 3-dimensional Euclidean Geometry is not taught at all! This approach, according to the opinion of the majority of educators and researchers in the area of mathematics education, including the present author, is a big pedagogical mistake. In fact, although the traditional Geometry is nowadays out of the focus of the modern mathematical research, it remains an indispensable pedagogical tool for enhancing the student mathematical thinking and fantasy, since its objects are real, solid and within the student cognitive experiences. That is why many tertiary teachers of mathematics, taking into account

the weaknesses of their students in understanding the properties of space, they suggest that it is better for them to be taught in the Upper High School the geometry of space, than to learn the abstract properties of the integrals and other details of Mathematical Analysis, that could be taught more efficiently at the university level.

Another problem may be created by the mistaken view of a number of experts and educators that *mathematical modeling* could become a general, i.e. applicable in all cases, method for teaching mathematics. In fact, mathematical modeling has many advantages, because it connects mathematics to real world problems, thus revealing its usefulness to students and therefore increasing their interest for it. However, the attempt to teach everything through mathematical modeling hides the danger to neglect the mathematical content in favour of the applications.

Three years ago, I have presented in the ICTMA Newsletter [24] two mathematical modeling problems on the use of the derivative for calculating the extreme values of a function in one variable. One of those problems referred to the construction of a channel to run the maximum possible quantity of water, by folding the two edges of a metallic leaf so that to remain perpendicular to the surface of the rest of the leaf. I received a critique on it suggesting that it could be much more interesting, if I had left the choice of the angle of the edges of the leaf to my students. My answer [25] was that, if I had done so, it could be a good exercise on problem-posing, but my students, being busy by playing with the construction of the channel, would probably not learn anything about the derivatives!

A third and last thing that I want to mention here is connected to the use of the *computers* as a tool in the process of teaching and learning mathematics. Students today, using the convenient small calculators, can make quickly and accurately all kinds of numerical operations. Further, the existence of a variety of suitable mathematical software gives them the possibility to solve automatically all the standard forms of equations and systems of equations, to make any kind of algebraic operations, to calculate limits, derivatives, integrals, etc, and even more to obtain all the alternative proofs of the existing mathematical theorems.. Therefore, some experts of computer science have already reached to the conclusion that in the near future teachers will not be necessary for the process of learning mathematics, because everything will be done by the computers. "The use of horses became not necessary" they use to parallelize, "from the time that cars have been invented"!

However, this is actually an illusion. In fact, the acquisition of information is important for the learner, but the most important thing is to learn how to think logically and creatively. The latter is impossible, at least for the moment, to be achieved by the computers only, since computers have been created by the humans and they come into 'life' through programming, which was also done by a human being. Thus the old credo "garbage in, garbage out" is still valid. Therefore, although the computers dramatically exceed in speed, most probably they will never reach the quality of the human mind³. On

the other hand, the practice of students with numerical, algebraic and analytic calculations, through the solution of problems and the rediscovery of the proofs of the existing theorems, it is necessary to be continued for ever; otherwise students will gradually loose the sense of numbers and symbols, the sense of space and time, thus becoming unable to create new knowledge and technology.

Of course, there is no doubt that computation is nowadays an increasingly essential tool for doing scientific research. The information processing technologies aim at duplicating the capacity of the human mind adding the advantage of operating at higher speeds than the mind in computations (artificial intelligence). It is expected that future generations of scientists and engineers will need to engage and understand computing in order to work effectively with management systems, technologies and methodologies. However, all the above are related to the need of teaching the informatics and especially the computer programming to students and not to the teaching of mathematics. There, computers can play the role of a valuable tool that makes the learning process easier and more effective, but in no case they can replace the teacher of mathematics!

5. Conclusion

In the present work we studied the main schools of mathematical thought and their effects on the development of mathematics education. Crucial problems that occupy nowadays the interest of those working in this area were also discussed, like the future role of the computers for the teaching and learning of mathematics.

From the discussion performed it becomes evident that none of the existing schools of mathematical thought has succeeded in finding a solid framework for mathematics. However, most of the recent advances of this science were obtained through their disputations about the absolute mathematical truth. On the contrary, these disputations have created serious problems in the sensitive area of mathematics education, the most characteristic being the failure of the introduction of the "New Mathematics" to school education that distressed students and teachers for many years.

In Chinese philosophy *Yin* and *Yang* represent all the opposite principles [5]. It is important however to pay attention to the fact that these two aspects rather complement and supplement than opposing each other with the one containing some part of the other. This kind of philosophy seems to be a suitable one to for the field of mathematical education. In fact, although it is logical for each one of those working in the area to be closer to the ideas of a certain school of mathematical thought, what it is actually needed from them is to find a proper balance among the ideas of all those schools by accepting their advantages and pointing out their weaknesses.. In this way the area of mathematics education will find the required tranquillity to be developed smoothly for the benefit of the future generations.

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³ It is true that nowadays a new generation of computers has been created that are programmed to build new computers being better than themselves! However, this does not guarantee at all that eventually they will approach the quality of the human mind.

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