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Designing an ACE Approach for Teaching the Polar Coordinates

Vahid Borji[1,*], Michael. Gr. Voskoglou[2]

[1]Department of Mathematics, Ferdowsi University of Mashhad, Iran
[2]Mathematical Sciences, School of Technological Applications, Graduate Technological Educational Institute of Western Greece, Patras, Greece
*Corresponding author: vahid.borji@stu.um.ac.ir, Vahid_math@yahoo.com

Abstract This paper continues a research started in an earlier work on teaching/learning the polar coordinates in the plane with the APOS/ACE instructional treatment of mathematics. Based on the Genetic Decomposition developed in our previous work, we design here an ACE circle and we apply it for teaching/learning the polar coordinates to a group of students of an Iranian university. For testing the effectiveness of this application the results of the experimental group are compared to those of a control group, for which the polar coordinates were taught at the same semester in the traditional, lecture-based way. The conclusions drawn enable us to make some minor revisions to the initial design of our Genetic Decomposition.

Keywords: *Cartesian and Polar Coordinates (PCO) and Curves, APOS Theory, Genetic Decomposition (GD), ACE Circle, Mathematics Teaching*

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1. Introduction

The *APOS theory*, developed by Ed Dubinsky and his collaborators in the USA during the 1990's [1,3], states that the teaching and learning of mathematics should be based on helping students to use the mental structures that they already have and to develop new, more powerful structures, for handling more and more advanced mathematics. These structures include ***A****ctions*, ***P****rocesses*, ***O****bjects* and ***S****chemas*, the acronym APOS being formed by the initial letters of the above four words. Two are the mental mechanisms involved in this process, the *interiorization* and the *encapsulation*. A mathematical concept is first formed as an action. As an individual repeats and reflects on an action, this action may be interiorized to a process. When the individual becomes aware of a process as a totality and becomes able to construct transformations on this totality, then the process has been encapsulated to an object. The actions, processes and objects involved in a mathematical topic need finally to be organized in an individual's coherent cognitive schema.

For example, if one can think of a function only through an explicit expression connecting the two variables involved, then he/she is having an action understanding of functions. On the contrary, a process understanding of a function enables the individual to think about it in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs. Further, an object understanding allows one to form sets of functions, to define operations on such sets, to equip them with a topology, etc. Finally, it is the schema structure that is used to see a function in a given mathematical or real world situation.

The implementation of the APOS as a framework for teaching and learning a mathematical topic requires a theoretical analysis of the concepts under study, called *Genetic Decomposition (GD).* Dubinsky and his collaborators realized that, for each mental construction that comes out from a GD, one can find a computer task such that, if a student engages in this task, it is fairly likely to build the corresponding mental construction. This gave genesis to a pedagogical approach connected to the APOS theory and called the *ACE teaching circle*. The ACE is a repeated circle of three components: ***A****ctivities* on the computer, ***C****lassroom* discussion and ***E****xercises* done outside the class [1,3,5].

In a recent work [2], based on the results of a written test and on oral interviews taken from students of the Physics Department of the University of Neyshabur, Iran, we have developed a GD for teaching and learning the *Polar Coordinates (PCO)* in the plane. It is recalled that the PCO of a point P of the plane are defined by a pair of numbers (r, θ), where r is the algebraic distance of P from a fixed point O of the plane, called the origin, and θ is the angle formed by the polar semi-axis OX and the straight line segment OP. The numbers r and θ can be positive, negative or zero (see Figure 1 of Section 2). It becomes evident that, in contrast to its Cartesian coordinates, each point P of the plane has not a unique PCO representation.

The present work continues the research performed in [2] by designing an ACE circle for teaching the PCO in the plane and by testing its effectiveness on an experimental group of university students. More explicitly, the rest of the paper is formulated as follows: In Section 2 the design of the ACE circle for teaching the PCO in the plane is developed. Section 3 describes our classroom experiment. There, the results of the experimental group are compared to the corresponding results of a control group, in which the PCO were taught in the traditional, lecture-based, way. Finally, Section 4 is devoted to our conclusions, which led us to make some minor changes to the initial design of our GD, and gave us some hints for further research.

2. Designing the ACE Circle for Teaching the PCO in the Plane

The implementation of the ACE circle and its effectiveness in helping students to create mental constructions and learn mathematics has been reported in several studies of the Dubinsky's research team; e.g. see [6,7], etc. For implementing the ACE circle the mathematical topic under consideration is divided to smaller subtopics each one of them corresponding to an iteration of the circle. The computer activities, which form the first step of the ACE approach, are deigned to foster the students' development of the appropriate mental structures. The students do all of their work in cooperative groups. In the classroom the instructor guides the students to reflect on the computer activities and their relation to the mathematical concepts being studied. They do this by performing mathematical skills without using the computer. They discuss their results and listen to explanations by fellow students or by the instructor of the mathematical meanings of what they are working on. The homework exercises are fairly standard problems related to the topic being studied. Students reinforce the knowledge obtained in the computer activities and classroom discussions by applying it in solving these problems.

In an earlier work [5] the second author of the present paper designed an ACE circle for teaching and learning the irrational numbers. Here we are going to design an analogous framework for the PCO in the plane by using Maple software for facilitating the student understanding. Our ACE design is based on the GD for the PCO developed in Section 5 of the article [2], which, for the needs of this work, can be divided in the following three parts:

1. Prerequisite knowledge (Cartesian coordinates, generalized angles, trigonometric functions).
2. Identifying and plotting points of the plane by their PCO and converting Cartesian to PCO and vice versa.
3. Sketching graphs of polar equations in all quadrants of the plane.

As we have observed in [2], one of the main reasons of student failure to understand and properly use the PCO is related to the problems that they face in dealing with generalized angles and trigonometric functions. Therefore the target of the ***first iteration*** of the ACE circle is to solve such kind of problems. As a first activity students draw different types of positive and negative angles like $\frac{\pi}{4}$, $-\frac{2\pi}{3}$, $\frac{27\pi}{4}$, $-\frac{15\pi}{7}$, etc. For drawing the angle $\frac{27\pi}{4}$ they write it as $\frac{27\pi}{4}=(6+\frac{3}{4})\pi=3(2\pi)+\frac{3\pi}{4}$, etc.

The rest of the first ACE iteration ιs allocated to computing trigonometric numbers of suitably chosen angles. The instructor recalls first how to compute the trigonometric numbers of the angles 0, $\pm\frac{\pi}{6}$, $\pm\frac{\pi}{4}$, $\pm\frac{\pi}{3}$, $\pm\frac{\pi}{2}$, $\pm\pi$ and 2π and then computations follow like

$$\cos\frac{31}{4}\pi=\cos\left(8\pi-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2},$$
$$\sin\frac{43}{6}\pi=\sin\left(7\pi+\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}=-\frac{1}{2},$$

etc.

In the ***second iteration*** of the ACE circle the instructor gives the necessary explanations for specifying a point of the plane in polar coordinates (Figure 1).

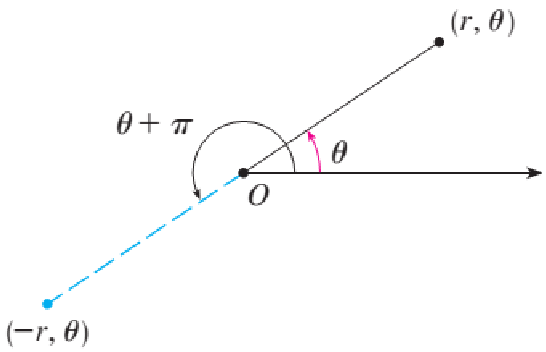

Figure 1. Polar coordinates of a point on the plane

Examples follow of converting the Cartesian coordinates (x, y) of a point P to PCO (r, θ) and vice versa. To find x and y when r and θ are known, one uses the equations:

$$x=r\cos\theta,\ y=r\sin\theta.$$

Conversely, to find r and θ when x and y are known, one uses the equations

$$r^2=x^2+y^2,\ \tan\theta=\frac{y}{x}.$$

The instructor emphasizes at this point that in converting from (x, y) to (r, θ) one must check first in which quadrant is the point (x, y) in order to compute correctly the angle θ. For example, for converting $\left(-\sqrt{3},-1\right)$ to PCO, one observes first that this point lies in the third quadrant. Therefore, since $\tan\theta=\frac{1}{\sqrt{3}}$, the angle θ is equal to $\pi+\frac{\pi}{6}$. In general, when converting from Cartesian to PCO or vice versa, both coordinates must correspond to the same point of the plane.

In the computer laboratory the students checked the correctness of their responses with commends in Maple. For example, converting (1, 1) to PCO we have:

$$> convert(1+I, polar)$$

$$polar\left(\sqrt{2}, \frac{1}{4}\pi\right).$$

Drawing the graphs of polar equations of the form $r = f(\theta)$ was another of the student difficulties observed in our earlier research [2]. Therefore, the ***third*** (and last) ***iteration*** of the ACE circle is focused on this matter. The students do the drawing of the graphs in two ways. The first way is to find the values of r for some suitable values of θ and to plot the corresponding points (r, θ). Then they join these points to sketch the curve. One of the problems that students faced in [2] using this method was the choice of the proper angles so that the polar curve could be sketched completely and this matter must be emphasized by the instructor during the ACE process. For example, the drawing of $r = 2\cos\theta$ is shown in Figure 2.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

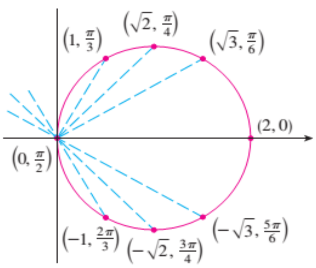

Figure 2. The graph of $r = 2\cos\theta$

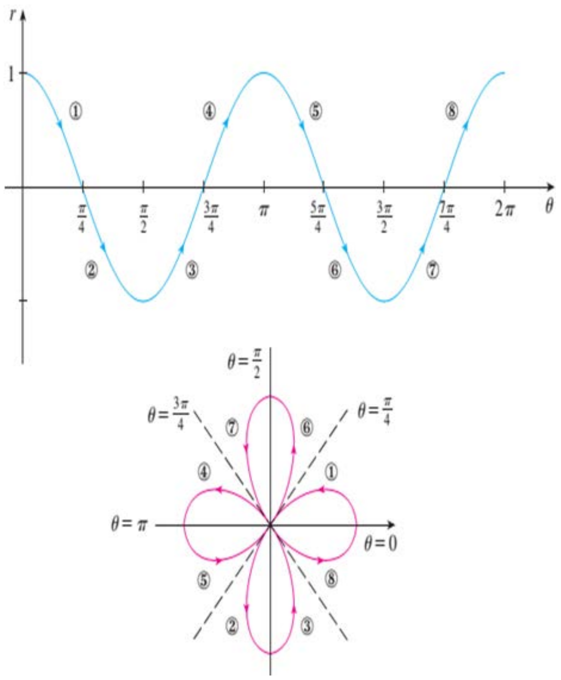

Figure 3. Drawing the polar graph of $r = f(\theta)$ (right) when its Cartesian graph (left) is given

In the second way $r = f(\theta)$ is first sketched in Cartesian coordinates. As θ increases from θ_1 to θ_2, in the Cartesian graph changes from r_1 to r_2. Consequently, the corresponding portion of the polar curve can be drawn (Figure 3). As θ changes, if r goes from 0 to negative values it means that the distance from the origin increases, but it lies on the opposite side of the pole (Figure 1).

The following commends and Figures 4-9 illustrate how one can use Maple for sketching the Cartesian and polar graphs of the equation $r = \sin\theta$:

$$\mathbf{plot}(\mathbf{sin}(\mathbf{x}),\ \mathbf{x} = \mathbf{0..2Pi});$$

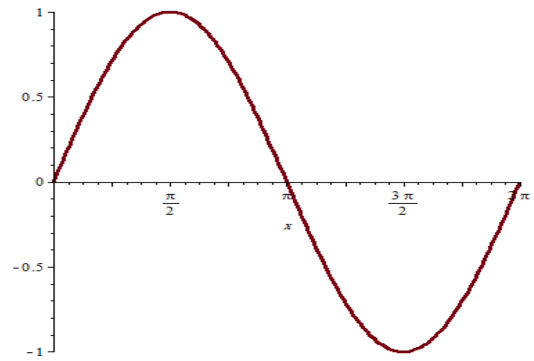

Figure 4. Cartesian graph of $r = \sin\theta$

$$\mathbf{polarplot}\left(\mathbf{sin}(\theta),\ \theta = \mathbf{0}..\frac{\mathbf{Pi}}{\mathbf{2}}\right);$$

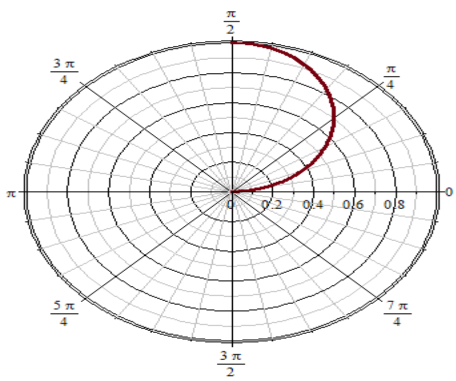

Figure 5. Polar graph of $r = \sin\theta$ in $[0, \frac{\pi}{2}]$

$$\mathbf{polarplot}\left(\mathbf{sin}(\theta),\ \theta = \frac{\mathbf{Pi}}{\mathbf{2}}..\mathbf{Pi}\right);$$

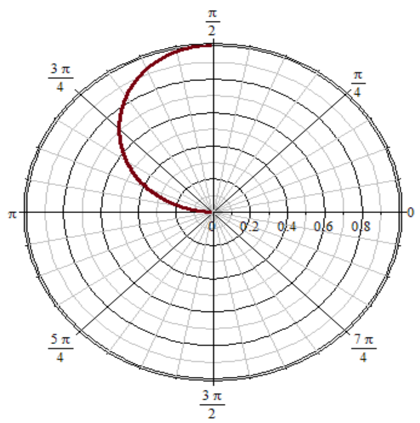

Figure 6. Polar graph of $r = \sin\theta$ in $[\frac{\pi}{2}, \pi]$

$$\mathbf{polarplot}\left(\sin(\theta),\ \theta = \mathbf{Pi}..\frac{\mathbf{3Pi}}{\mathbf{2}}\right);$$

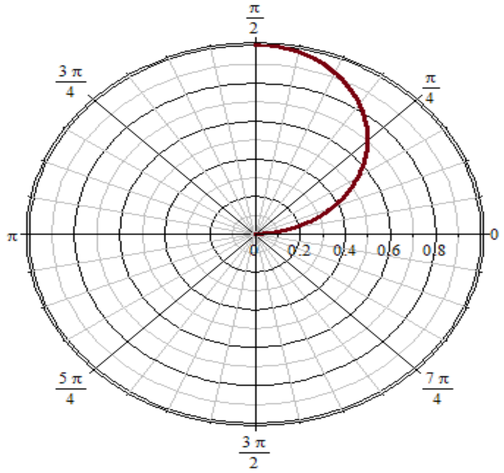

Figure 7. Polar graph of $r = \sin\theta$ in $[\pi, \frac{3\pi}{2}]$

$$\mathbf{polarplot}\left(\sin(\theta),\ \theta = \frac{\mathbf{3Pi}}{\mathbf{2}}..\mathbf{2Pi}\right);$$

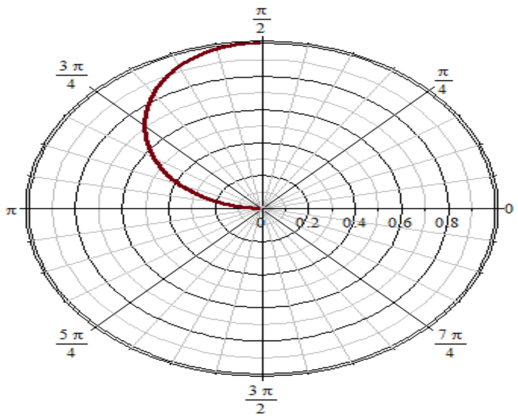

Figure 8. Polar graph of $r = \sin\theta$ in $[\frac{3\pi}{2}, 2\pi]$

$$\mathbf{polarplot}\left(\sin(\theta),\ \theta = \mathbf{0}..\frac{\mathbf{Pi}}{\mathbf{2}},\ \mathbf{thickness = 3}\right);$$

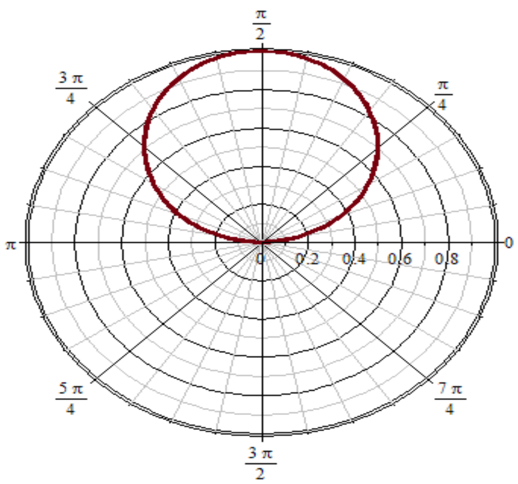

Figure 9. Polar graph of $r = \sin\theta$ in $[0, 2\pi]$

3. The Classroom Experiment

3.1. Methodology

Two groups of university students participated in this research. The first group (*control group*) was enrolled in a calculus course in the fall semester of 2016 at one of universities in Mashhad city, situated in Razavi Khorasan province of Iran. Eighteen students participated in the class and the subject was taught by a mathematics instructor in the traditional, lecture - based method.

The second group (*experimental group*) was enrolled in a calculus course in another university in Neyshabur city of Razavi Khorasan province of Iran at the same semester. The class was taught by the first of the authors of this article using the ACE cycle designed in Section 2 for teaching the PCO. Twenty students participated in this class. The textbook used in both classes was the Stewart's Calculus [4].

3.2. The Pre-Test

Before starting the teaching of PCO both groups completed a written pre-test. Since angles, trigonometric functions and Cartesian equations are fundamental prerequisites for learning polar coordinates and, as we found in our previous research [2], students have usually problems when dealing with them, the pre-test's questions, which are listed below, were based on those concepts.

1. *Sketch the angles* $\frac{\pi}{4}, -\frac{2\pi}{3}, \frac{27\pi}{4}, -\frac{15\pi}{7}$ *and find in which quadrant ends each one of them.*

2. *Compute the values:* $\cos\frac{7\pi}{3}$, $\sin(\frac{-\pi}{3})$, $\tan\frac{11\pi}{4}$

3. *Find all values of* θ *such that* $\tan\theta = -1$ *and* $0 \le \theta \le 2\pi$.

4. *Write the Cartesian equation of the below curves:*

a) A vertical line through the point (3, 2).

b) A circle with radius 4 and center (1, 2)

5. *Sketch the graphs of the functions y= sin2x and y=cos3x*

6. *Find the center and radius of the circle with equation* $x^2 + y^2 = 2x - 4y$.

The student answers were marked in a range from 0 to 100 and the scores obtained were the following:

Control group (G_1): 84, 70, 60, 59, 58, 40, 39, 38, 35, 32, 26, 25, 25, 15, 15, 10, 7, 5.

Experimental group (G_2): 83, 65, 65, 55, 50, 40, 40, 40, 38, 35, 35, 28, 25, 22, 22, 15, 15, 10, 7, 5.

Assigning to the student scores the *linguistic labels (grades)* A (100-85), B (84-75), C (74-60), D (59-50) and F (49-0) of excellent, very good, good, fair and unsatisfactory performance respectively, the results of the two groups are depicted in Table 1 as follows:

Table 1. Pre-Test Results

Grades	G_1	G_2
B	1	1
C	2	2
D	2	2
F	13	15
Total	18	20

The mean value of the student scores for G_1 is 35.72 and for G_2 is 34.75 demonstrating an unsatisfactory *mean performance* for both groups. Also, the *GPA index* [2: Section 3] is equal to $\frac{2+2*2+3*1}{18}$ = 0.5 for G_1 and $\frac{2+2*2+3*1}{18}$ = 0.45 for G_2 demonstrating a very poor *quality performance* for both groups. However, in both cases the control group demonstrated a slightly better overall performance than the experimental one.

3.3. The Post-Test

One week after the end of lectures on PCO a post-test was performed for both groups. For reasons of justice the questions of the post-test were designed by other mathematics faculties who are familiar with polar coordinates and none of the two instructors participated in their design. The questions of the post-test are the following:

1. *Plot the points whose polar coordinates are:* (2, -$\frac{5\pi}{8}$), (2, $\frac{91\pi}{4}$) and (-1, $\frac{3\pi}{4}$).

2. *Find the Cartesian coordinates of the point* (3, $\frac{9\pi}{4}$)

3. *Find the polar coordinates of the point* $(-2,-2)$

4. *Find the polar equation for the curve* $x^2 + y^2 = 3x$

5. *Identify the curve* $r = 4\csc\theta$ *by finding the Cartesian equation for it.*

6. *Sketch the curve with polar equation* $r = -\cos 2\theta$

7. *The figure below shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.*

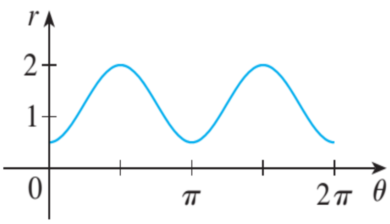

Figure 10. The Cartesian graph of $r = f(\theta)$

The student papers were marked together by both instructors for the control and the experimental group with scores ranging from 0–100. The student scores were the following:

Control group (G_1): 88, 75, 62, 55, 55, 45, 40, 36, 32, 30, 27, 27, 25, 20, 18, 15, 15, 10.

Experimental group (G_2): 100, 85, 80, 76, 76 65, 60, 60, 55, 48, 45, 42, 35, 30, 28, 25, 20, 20, 15, 15. The results of the two groups are depicted in Table 2 as follows:

Table 2. Post-Test Results

Grades	G_1	G_2
A	1	2
B	1	3
C	1	3
D	2	1
F	13	11
Total	18	20

For the control group, the mean value and GPA index are 37.5 and 0.61, while for the experimental group are 49 and 1.2 respectively. The above data show that, although both groups demonstrated again a less than satisfactory overall performance, the performance of the experimental group was clearly better than that of the control group. On comparing the performances of the two groups in the pre-test and the post-test one observes that both groups demonstrated a better performance in the post-test, although the questions of it were more difficult. However, the experimental group succeeded a much greater progress, which means that its students benefited better by the APOS/ACE instructional treatment for learning the PCO than the students of the control did by the traditional lecture-based approach.

3.4. Qualitative Analysis (Characteristic Student Answers)

In addition to the above quantitative analysis, a qualitative analysis of the student answers was performed which showed that the students of the experimental group had a better understanding of the PCO in contrast to the students of the control group that faced more problems. More explicitly, our conclusions for each question are the following:

Question 1: Most students of the control group had problems in specifying the angles $\frac{-5\pi}{8}$ and $\frac{91\pi}{4}$ and especially the negative one, while the students of the experimental group performed better.

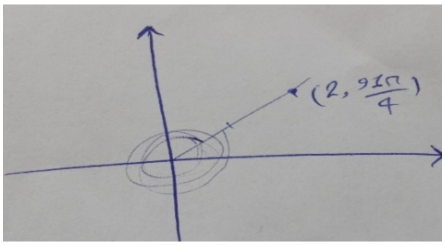

Photo 1. A wrong student answer to Question 1

Question2: Applying formulas $x = 3\cos\frac{9\pi}{4}$ and $y = 3\sin\frac{9\pi}{4}$ many students had problems for computing $\cos\frac{9\pi}{4}$ and $\sin\frac{9\pi}{4}$. Thirteen students of the experimental and six of the control group solved this question correctly.

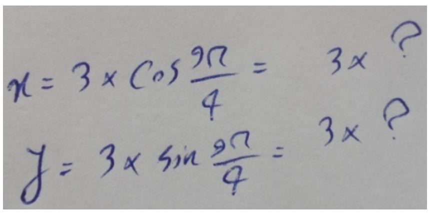

Photo 2. An incomplete student answer to Questio***n 2***

Question 3: The main problem here was the choice of the correct angle. Many students used the angle $\frac{\pi}{4}$, since they did not notice that the point $(-2,-2)$ is in the third quadrant.

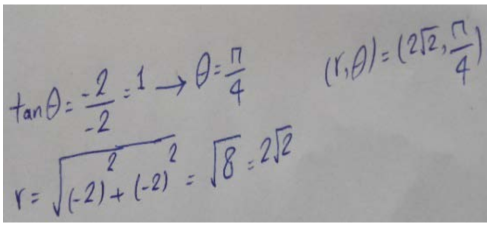

Photo 3. A wrong student answer to Question 3

Question 4: Most students of the control group gave the answer $r^2 = 3x$ without replacing x by *r cosθ*

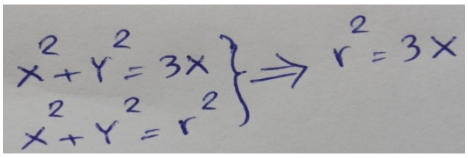

Photo 4. An incomplete student answer to Question 4

Question 5: Among the 28 students participated the post-test only three students of the experiment group could give a complete answer for this question.

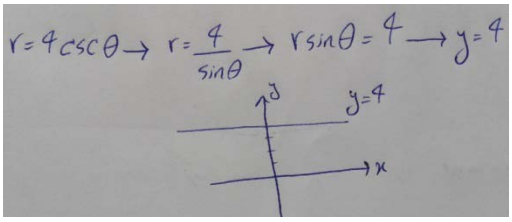

Photo 5. A complete student answer to Question 5

Question 6: The students of the control group used the first method described in the third iteration of the ACE circle of Section 2 (plotting points) for sketching the curve, while most students of the experimental group used the second method by sketching first the curve in Cartesian coordinates.

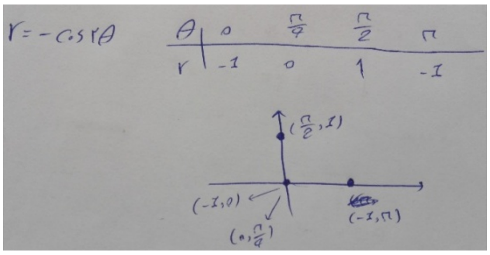

Photo 6. An incomplete answer to Question 6 from a student of the control group

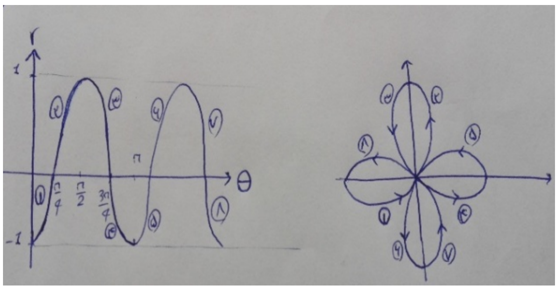

Photo 7. A correct answer to Question 6 from a student of the experimental group

Question 7: No student of the control group solved this question correctly in contrast to the 12 students of the experimental group that did it.

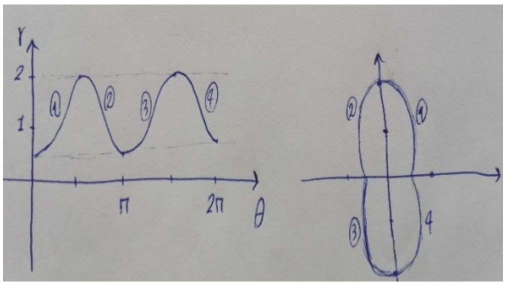

Photo 8. A correct student answer to Question 7

4. Discussion and Conclusions

From the material presented in this paper it becomes evident that the implementation of the ACE circle benefited much more the students of the experimental group for understating and using properly the PCO in the plane than the traditional lectures did for the students of the control group.

Note that the initial design a GD for teaching/learning a mathematical topic must be tested empirically and the test may lead to mixed results. Some of the initially suggested ideas may appear to be successful and others may need to be slightly or even radically changed. Also, certain things not included in the initial GD's design may be added. In other words, the GD may need a refinement in order to reflect what it has been found empirically. Revisions of the GD usually lead to changes in instruction, but they also give the impulsion for further empirical analysis.

In our case, after analyzing the results of the post-test, we decided to make some minor revisions only to the part of the initial GD (see Section 5 of [2]) related to the basic background needed for teaching/learning the PCO, i.e. Cartesian coordinates and graphs, angles and trigonometric functions. For example, we found that more emphasis should be given to the properties of the four quadrants of the Cartesian plane, to examples concerning the calculation of the trigonometric numbers of angles greater than 2π etc. On the other hand, taking into account the poor student performance in the pre-test, we decided that there is a need for further classroom experiments in order to study the effect of the APOS/ACE instruction on students with a higher mathematical background than those of the present experiment.

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