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Applying the APOS Theory to Study the Student Understanding of Polar Coordinates

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Abstract A Genetic Decomposition (GD) is developed in this paper, in terms of the APOS instructional treatment of mathematics, for teaching and learning the polar coordinates. The design of the GD is based on the conclusions obtained by a written test and by oral interviews performed on a group of students of the Physics Department of the University of Neyshabur, Iran.

Keywords: *Polar and Cartesian Coordinates and Curves, APOS Theory, Genetic Decomposition (GD), Triad Mechanism of Schema Development*

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1. Introduction

There are several different coordinate systems used in mathematics, physics, engineering and the other applied sciences, all of them based on the same idea. Thus, a *coordinate system* can be defined as a rule for mapping pairs of numbers to points of the plane (or the space).

Descartes (1596-1650) introduced in 1637 the (x, y)-coordinates, known nowadays as the *Cartesian coordinates*, which opened the door to the development of the Analytic Geometry. In contrast to the classical Euclidean Geometry, this "new" geometry (followed later by the other today known non-Euclidean geometries) studies the properties of the classical geometric figures with algebraic methods. This led to the discovery of new important geometric properties of several figures, which otherwise could remain unknown. It was also the starting point of connecting two completely independent until that time branches of mathematics, the algebra with the geometry.

Another popular coordinate system with many applications to the differential and integral calculus, to complex numbers and to other branches of pure and applied mathematics, to physics, etc. is the system of *polar coordinates*. It is recalled that the polar coordinates of a point P of the plane are defined by a pair of numbers (r, θ), where r is the algebraic distance of P from a fixed point O of the plane, called the origin, and θ is the angle formed by the polar semi-axis OX and the straight line segment OP (see Figures 3 and 4 of Section 4). The numbers r and θ can be positive, negative or zero.

It becomes evident that, in contrast to its Cartesian coordinates, each point P of the plane has not a unique polar representation. Polar coordinates can be also defined for the points of the three dimensional space, but this is out of the purpose of this work, which is to study, in terms of the APOS theory for teaching and learning mathematics, the student difficulties for the understanding and proper use of polar coordinates in the plane.

For this, the rest of the paper is organized as follows: In Section 2 those elements of the APOS theory are briefly exposed, which are necessary for the understanding of our research. Also the triad mechanism of Piaget and Garcia is discussed for a mental schema development, which is going to be used later in the paper. In Sections 3 and 4 a written test and the oral interviews are described respectively, performed on a group of students of the Physics Department of the University of Neyshabur, Iran. The results of the quantitative and quantitative analysis made on the collected data are also presented in these two Sections, followed by the conclusions obtained. A preliminary generic decomposition proposed for the teaching and learning of the polar coordinates is designed in Section 5, which is based on the previous conclusions. Finally, Section 6 is devoted to a synopsis of our research and the statement of our plans for continuing it in the near future.

2. The APOS Theory and the Triad Mechanism of Schema Development

The APOS instructional treatment of mathematics was developed in the USA during the 1990's by Ed Dubinsky and his collaborators ([1,3] etc.). In earlier works the second author of this paper had the opportunity to apply the APOS approach for teaching the irrational numbers [7] and for assessing, with the help of fuzzy logic, the effectiveness of this application for improving the student performance [8]. Therefore, here only those elements of the APOS theory are recalled, which are absolutely necessary for the understanding of our current research. For more details the reader may look at [1,3,7,8], etc.

The APOS theory simply says that the teaching of mathematics should be based on helping students to use the mental structures that they already have and to build new, more powerful structures, for handling more and more advanced mathematics. The mental mechanisms to be used for this purpose are called *interiorization* and *encapsulation* and the related mental structures are *Actions*, *Processes*, *Objects* and *Schemas*. The initial letters of the above structures form the acronym APOS.

A mathematical concept is first formed as an action by applying transformations on certain entities to obtain other entities. As an individual repeats and reflects on an action, this action may be interiorized to a process enabling him/her to imagine performing the corresponding transformations without having to execute each step explicitly. When the individual becomes aware of a mental process as a totality and becomes able to construct transformations acting on this totality, then he/she has encapsulated the process to a cognitive object. A mathematical topic often involves many actions, processes and objects that need to be organized in a coherent mental framework, usually referred as a schema. The proper schema acquisition enables the individual to decide which processes to use for dealing with a mathematical situation.

The implementation of the APOS theory as a framework for teaching and learning mathematics involves a theoretical analysis of the concepts under study, called a *Genetic Decomposition (GD).* Dubinsky and his collaborators realized that, for each mental construction that comes out from a GD, one can find a computer task such that, if a student engages in this task, it is fairly likely to build the corresponding mental construction. This gave genesis to a pedagogical approach connected to the APOS theory and called the *ACE teaching circle* (for more details see [1,3,7], etc.).

A number of authors suggest the use of the *triad mechanism of Piaget and Garcia* [4] as means for supporting the schema development of their GD constructions (e.g. see [2], etc.). This general mechanism distinguishes three stages in the development of a concept. The *Intra stage* is characterized by the focus on a single mental construction in isolation from other actions, processes, or objects. The *Inter stage* is characterized by recognizing relationships between different actions, processes and objects, while the *Trans stage* is characterized by the construction of a coherent structure underlying some of the relationships discovered in the Inter stage. It is worth noting that it is only when the corresponding framework reaches the Trans stage of development that we can properly refer to it as a schema (the inter stage it is referred by some authors as a pre-schema).

3. The Written Test

The objective of the written test was to obtain a first idea about the student understanding and proper use of the polar coordinates and to find in which points they face difficulties. Nine questions had been designed for this test, which will be presented below together with the student responses. The questions involved converting points and equations from polar to Cartesian or from Cartesian to polar coordinates and sketching graphs of polar equations.

The subjects of our research were 26 students of the Physics Department of the University of Neyshabur, Iran. The test was performed during the academic year 2015-16. The students had completed one month ago the multivariable calculus course involving the concept of polar coordinates. The Stewart's [5] calculus book was used by the instructor and first author of this article as the main didactic source for this course. Note that all students participated voluntary in the test, at the end of which they received awards of appreciation. The score range of the written test was from 0 to 100 and the student scores are presented in Table 1. The student overall performance was assessed by calculating the *mean value* of their scores and the *Grade Point Average (GPA) index*.

We recall that the GPA index is a weighted average in which higher coefficients (weights) are assigned to the higher scores and therefore it is connected to the student quality performance. GPA is calculated by the formula $\text{GPA} = \frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n}$, where n is the total number of students and n_A, n_B, n_C, n_D and n_F are the numbers of students who demonstrated excellent (A), very good (B), good (C), fair (D) and unsatisfactory (F) performance respectively [6]. In case of the ideal performance ($n_A = n$) the above formula gives that GPA = 4, while in case of the worst performance ($n_A = n_F$) it gives that GPA = 0. Therefore, $0 \leq \text{GPA} \leq 4$ and consequently values of GPA greater than 2 indicate a more than satisfactory student performance.

Table 1. The student scores

Rank	**Score**	edarG
1	88.50	A
2	83.50	B
3	81.00	B
4	80.00	B
5	78.00	B
6	72.00	C
7	70.00	C
8	69.00	C
9	68.50	C
10	60.00	C
11	57.50	D
12	57.50	D
13	57.00	D
14	56.75	D
15	49.50	F
16	46.50	F
17	41.00	F
18	39.50	F
19	33.00	F
20	31.50	F
21	26.00	F
22	23.00	F
23	21.50	F
24	20.00	F
25	17.50	F
26	9.00	F

In order to perform our assessment we assigned to the student scores the above mentioned *linguistic labels (grades)* A, B, C, D and F as follows: A(100-85), B(84-75), C(60-74), D(50-59) and F(0-49). Note that, although the above assignment satisfies the common sense, it cannot be considered as being a standard one. For example, in a more strict assessment one can take A(100-90), B(89-80), C(79-70), D(69-60) and F(0-59), etc.

A straightforward calculation gives that the mean value of the student scores is 51.43, which demonstrates a fair (D) *mean student performance*. Also, from Table 1 one finds that $n_A = 1$, $n_B = 4$, $n_C = 5$, $n_D = 4$ and $n_F = 12$. Therefore, it is straightforward to check that GPA ≈ 1.15, which means that students demonstrated an unsatisfactory *quality performance.*

The first impression from the above *quantitative analysis* is that the overall student performance was problematic. Next, a detailed *qualitative analysis* of the student responses followed, in terms of which the main problems that the students faced during the written test were determined as follows:

Question 1: *Plot the point whose polar coordinates are: a)* $(2,\frac{5\pi}{6})$*, b)* $\left(1,-\frac{2\pi}{3}\right)$*, c)* $(1,-\frac{5\pi}{4})$*. Then find two other pairs of polar coordinates of the corresponding point, one with* $r<0$ *and one with* $r>0$.

The students succeeded in general to plot the polar point (r,θ). However some of them had problems when r and θ were negative. Some students found two other representations, but their points didn't lie in the correct quadrant. For example, the polar point $\left(1,-\frac{2\pi}{3}\right)$ is in the third quadrant, but some students found another representation for this point that wasn't in the third quadrant!

Question 2: *Plot the point whose polar coordinates are: a)* $\left(\sqrt{2},\frac{\pi}{4}\right)$*, b)* $(-2,\frac{3\pi}{4})$*, c)* $\left(-3,-\frac{\pi}{3}\right)$*. Then find the Cartesian coordinates of the point.*

In this question some students couldn't compute the values of $\cos\frac{3\pi}{4}$, or of $\sin\frac{3\pi}{4}$, or even some of them didn't know the values of $\sin\frac{3\pi}{4}$ or $\cos\frac{3\pi}{4}$. Also many students didn't pay attention to the fact that both the polar coordinates (r,θ), and the Cartesian coordinates (x,y) correspond to the same point of the plane. For example, they plotted correctly $(-2,\frac{3\pi}{4})$ in the fourth quadrant, but they converted this point to a Cartesian point (x,y) which wasn't in the fourth quadrant!

Question 3: *The Cartesian coordinates of a point are: a)* $\left(\sqrt{3},-1\right)$*, b)* $(-6,0)$

(i) *Find the polar coordinates* (r,θ) *of the point, where* $r>0$ *and* $0\le\theta<2\pi$.

(ii) *Find the polar coordinates* (r,θ) *of the point, where* $r<0$ *and* $0\le\theta\le2\pi$.

Most students faced problems in this question. The score assigned to each of the cases (i) and (ii) is 4 units and the class's means were 1.05 and 1.46 units respectively. This shows a bad student performance, especially in part (i), which was solved correctly by only one student. For converting the Cartesian coordinates (x,y) to the polar coordinates (r,θ) the students used the formulas $r^2=x^2+y^2$ and $\tan\theta=\frac{y}{x}$, but most of them couldn't compute the value of θ. For example, in part (i) of the question many students couldn't obtain the value of θ from formula $\tan\theta=\frac{y}{x}=\frac{-1}{\sqrt{3}}$. Also, similarly to the Question 2, many students didn't pay attention to the fact that the Cartesian coordinates (x,y) and the corresponding polar coordinates (r,θ) correspond to the same point.

Question 4: *Sketch the regions in the plane consisting of the points whose polar coordinates satisfy the conditions:*

a) $r\ge0,\ \frac{\pi}{4}\le\theta\le\frac{3\pi}{4}$

b) $2<r<3,\ \frac{5\pi}{3}\le\theta\le\frac{7\pi}{3}$.

Students demonstrated a good performance in general in this question. Some of them had problems on specifying in which quadrant are the angles $\theta=\frac{5\pi}{3}$ and $\theta=\frac{7\pi}{3}$.

Question 5: *Identify each of the below curves by finding their Cartesian equations*

a) $r^2=5$

b) $\theta=\frac{\pi}{6}$

c) $r=3\cos\theta$.

For solving this question the students used the formulas $r^2=x^2+y^2$, $\cos\theta=\frac{x}{r}$ and $\tan\theta=\frac{y}{x}$. However, the worst performance in the written test was on part (b) of this question, which was solved correctly by only one student.

Question 6: *Find the polar equation of the following curves*:

a) $x^2+y^2=3x$,

b) $y=3$,

c) $6y^2=x$.

For solving this question, the students used the formulas $x=r\cos\theta$ and $y=r\sin\theta$ for converting the Cartesian equations to polar equations.

Question 7: For each of the following curves decide if they are more easily expressed by a polar equation or a Cartesian equation and write down the corresponding equation.

a) A line through the origin forming angle $\theta=\frac{\pi}{3}$ with the positive x-axis

b) A vertical line through the point $(2,-2)$

c) A circle with radius and center $(1,2)$

d) A circle centered at the origin with radius 3.

The students' responses showed that they rather prefer to write a Cartesian equation than a polar one. For example, in part (d), most students had written $x^2+y^2=3^2$ instead of $r=3$. In case (a) also, where the polar equation $\theta=\frac{\pi}{3}$ is the logically expected response, most students tried to find a Cartesian equation. Also students demonstrated a bad performance in case (b). Its score is 3 and the class's mean was 0.807.

Question 8: *Sketch the curve of the polar equations given below*:

a) $r=3\sin\theta$

b) $r=\theta$

c) $r=1+\cos\theta$.

Question 9: *Figure 1 shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.*

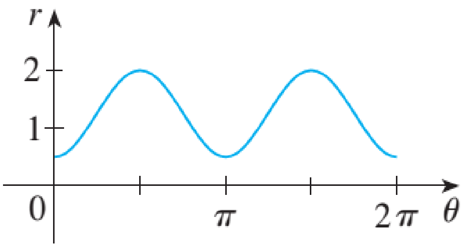

Figure 1. Gartesian Graph of Question 9

For solving questions 8 and 9 students used two different strategies:

1) They determined the values of r for some convenient values of θ and plotted the corresponding points (r,θ). Then joining these points they sketched the curve.

2) They first sketched the graph $r=f(\theta)$ in Cartesian coordinates. This enables to read at a glance the values of r that correspond to increasing values of θ. For instance, as θ increases from θ_1 to θ_2, r increases from *a* to *b*. In this way the students sketched the corresponding part of the polar curve. The rest of the curve can be drawn in a similar fashion.

Investigating the students' responses in the written test we determined the difficulties that they faced in performing the corresponding mental constructions. However, the need to obtain explanations for these difficulties, made us to decide to continue our research with the oral interviews.

4. The Oral Interviews

In an APOS based research different kinds of instruments are used to investigate students' understanding. Depending on the objectives of the particular study, these may include written questionnaires, interviews (audio-and/or videotaped), exams, etc. Interviews are the most important means by which data is gathered. The idea is to access data on different mathematical tasks in order to compare the thinking of students who had difficulty with the thinking of students who succeeded. In designing the interview questions, different sources may be used. The responses to a previously administered written exam or questionnaire might form the basis of such questions. In this case, students are asked to clarify their responses and/or to expand on them.

Nine out of the 26 students who participated in the written test were selected for our interviews; three top students (Maedeh who scored 88.50, Mohammad 83.50, Ahad 80), three medium students (Abdollahi 78, Talebi 60, Golshanian 57) and three bottom students (Ehsan 33, Atefeh 21.50, Gharchei 9). Based on student misunderstandings and mistakes in the written test we designed ten questions for the interviews.

The interviews gave us the opportunity to see and hear closer the problems that the students faced during the written test, and how they reacted. Each student was asked the proper questions according to the difficulties that he/she had faced in the written test. For students with false answers, the objective was to investigate the reasons which led to the mistakes and help them to understand the right way to proceed. On the contrary, for students with right answers the objective was to see whether they can generalize their thoughts correctly in more advanced questions than those of the written test.

Each interview lasted about one hour. The students, while explaining their solutions they also wrote their responses on paper. All nine interviews were recorded in video files. We used both the video recordings and the student papers for analyzing their responses. The most characteristic for each question student responses are presented below:

Question 1: *Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point. a)* $(-1,-\pi/4)$*, b)* $(-2,7\pi/3)$.

Atefeh: The angle $\frac{\pi}{4}$ is in the first quadrant … therefore the angle $-\frac{\pi}{4}$ is in the second quadrant.

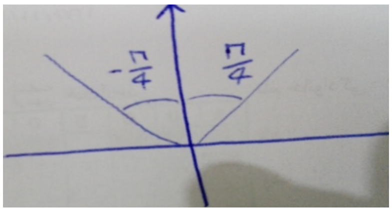

Photo 1. Atefeh's Figure

Golshanian: She divided the first quadrant into three equal angles claiming that each of these angles is equal to $\frac{\pi}{3}$ and she found that the angle 7π/3 is in the third quadrant!

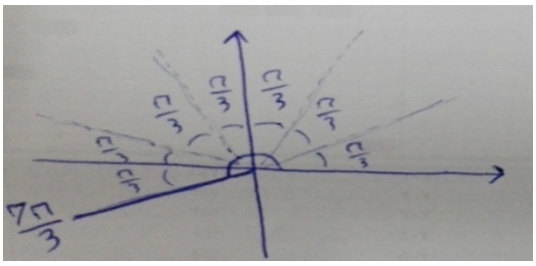

Photo 2. Golshanian's Figure

Question 2: *How many other polar representations can you write for the point with polar coordinates (r, θ)?*

In writing the polar representations of a fixed point in the plane most students didn't notice that each 360^o rotation of an angle will give other polar representations of that point. Two ways were used in general for finding many polar representations of each point in the plane. First the students found all paths which start from polar axis to reach the point clockwise, counterclockwise and negative r. Second, after finding a polar representation (r,θ) for the point, they used this representation to change the first and second component, r and θ, so that to comeback to the same point at the end.

Question 3: *Find the Cartesian coordinates of the point* $\left(-1,-\frac{\pi}{4}\right)$.

Question 4: *Compute the values of* $\sin\frac{5\pi}{3}$ *and* $\cos\frac{3\pi}{4}$.

In Questions 3 and 4 the students faced three problems. First some of them used the formulas $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$ instead of $x = r\cos\theta$ and $y = r\sin\theta$. Second some couldn't compute the values of $\cos\theta$ and $\sin\theta$ for some angles and third most students didn't notice that both the polar and the Cartesian representation of a point correspond to a fixed position in the plane.

Question 5: *Find the polar coordinates of the points (-1, -1) and (-1, 1).*

In converting the Cartesian to polar coordinates most students miscalculated the angle θ. They didn't notice that θ must be in the same quadrant with the point (x, y). The same thing had frequently happened in the written test too. When we asked the students to check the correctness of their response by plotting both the Cartesian and polar coordinates, they realized that the two points aren't in the same position and they corrected their responses by changing the angle from θ to $\theta+\pi$, or by changing the radius from r to $-r$. Talebi, for example, was one of the students having problems in this question.

Talebi: We have the Cartesian point $(-1,-1)$, therefore x is equal to -1 and y is equal to -1… (then she used formula $r=\sqrt{x^2+y^2}$ and … r is equal to $\sqrt{2}$ and $\tan\theta = \frac{-1}{-1} = 1$, therefore θ is equal to $\frac{\pi}{4}$. Therefore the point $(-1,-1)$ has polar coordinates $\left(\sqrt{2},\frac{\pi}{4}\right)$.

Instructor: Are you sure that the Cartesian point $(-1,-1)$ and the polar point $\left(\sqrt{2},\frac{\pi}{4}\right)$ correspond to the same position in the plane? Plot both points.

After plotting both points, $(-1,-1)$ Talebi corrected her response from $\frac{\pi}{4}$ to $\pi+\frac{\pi}{4}$.

Question 6: *Write a polar equation for each of the lines presented in Figure 2*:

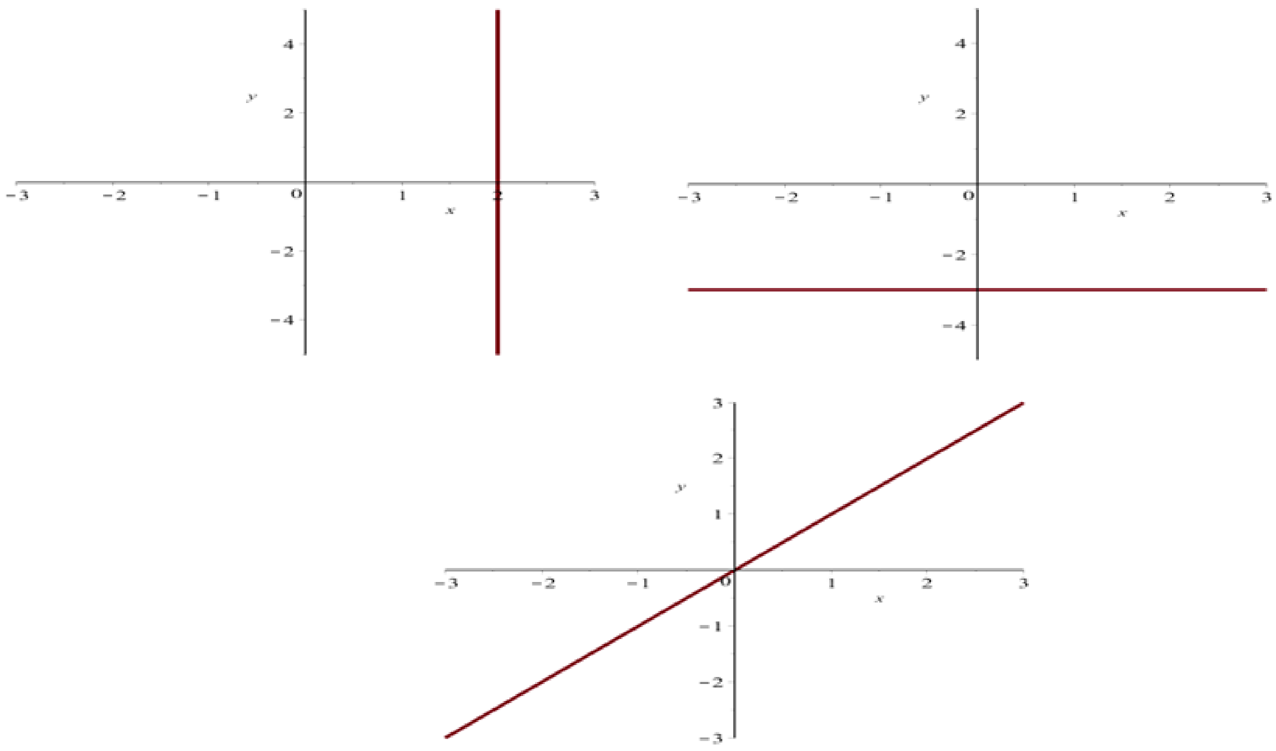

Figure 2. Lines of Question 6

First students wrote a Cartesian equation for each line and then they used the formulas $x = r\cos\theta$ and $y = r\sin\theta$ to convert it to a polar equation. They didn't attempt to do it straightforward. Some students couldn't write a Cartesian equation for a vertical line; for example $x = 2$ for the first line.

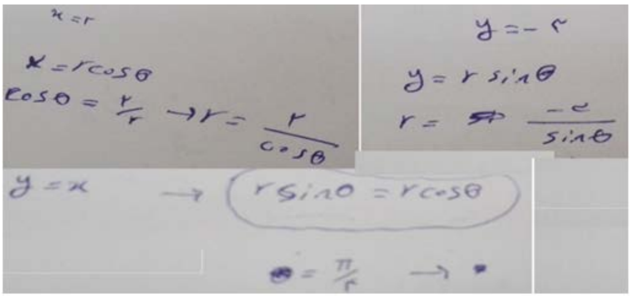

Photo 3. Mohammad's response

Question 7: *Sketch the polar curves a)* $\theta = \pi/4$, *b)* $r = 2$ *and find a Cartesian equation for each of them.*

Students demonstrated a good performance in general. Only Gharchei and Ehsan had problems.

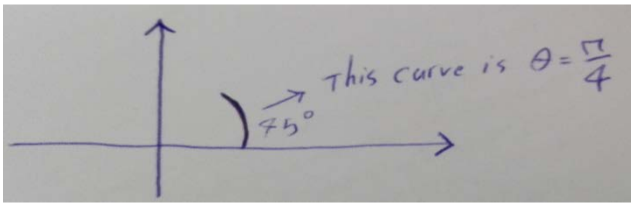

Photo 4. Gharchei's graph before instructor's help

Instructor: No, $\theta = \frac{\pi}{4}$ is the set of points of the plane with their radius forming angle $\frac{\pi}{4}$ with the x axis and having any length.

Gharchei: Oh,… It is a line that passes through the origin and forms angle $\frac{\pi}{4}$ with x axis.

Ehsan developed a completely wrong argument: When r is equal to 2 then $y = r\sin\theta$ … therefore I must sketch the graph $y = 2\sin x$.

Question 8: *Show that the polar equation* $r = a\sin\theta + b\cos\theta$, *with* $ab \neq 0$, *represents a circle, and find its center and radius.*

None of the students answered this question without the instructor's help. *Ahad* wanted to sketch the curve of the equation, but he was not able to do so. He said "I can sketch polar equations such as $r = a\sin\theta$ or $r = b\cos\theta$, but I can't sketch $r = a\sin\theta + b\cos\theta$ ". Two other students, *Mohammad* and *Maedeh*, wanted to square both sides of the equation to convert it to the form $r^2 = u(r,\theta)^2 + v(r,\theta)^2$ and they said that this is a circle equation! Even after my help, when they converted the polar equation to the Cartesian form $x^2 + y^2 = ay + bx$, they were not able to realize that this is a circle equation.

Question 9: *Identify the curve* $r = \tan\theta\sec\theta$ *by finding a Cartesian equation for it.*

Here one can use formulas $x = r\cos\theta$ and $y = r\sin\theta$ for converting the given equation to a Cartesian equation of the form $y = x^2$.

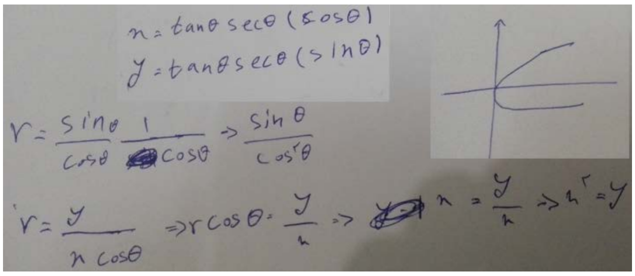

Photo 5. Ahad's response

Question 10: *Sketch the curve with polar equation* $r = \sin 2\theta$.

The students who used the plotting way (i.e. the first strategy applied in Questions 8 and 9 of the written test, see Section 3) were in trouble in selecting the convenient values of θ for finding the values of r. *Fatemeh Abdollahi* used the following table for sketching *r = sin2θ*.

θ	0	π/2	π	3π/2	2π
r	0	0	0	0	0

Photo 6. Abdollahi's Table

For all the above values of θ the value of r is equal to zero. Therefore she had only one point at the origin and couldn't continue her sketching.

Three students used the second strategy applied for Questions 8 and 9 of the written test by sketching first the graph of r as a function of θ in Cartesian coordinates. However they had problems for sketching this graph.

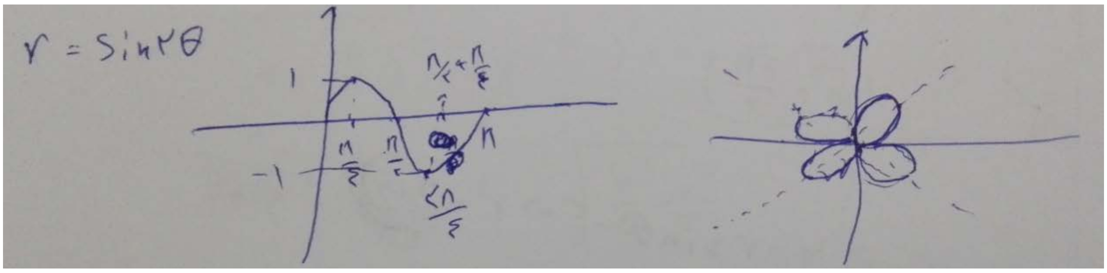

Photo 7. *Mohammad*'s response

Based on our analyses of students' responses in the written test and the oral interviews we concluded that the student difficulties are connected to the following factors:

- Incomplete previous knowledge of the trigonometric angles and functions
- Incomplete knowledge and use of the Cartesian coordinates and equations.

- Incomplete understanding of the polar coordinates and equations.
- Problems related to the transition from Cartesian to polar coordinates and vice versa.

The above conclusions helped us to design a preliminary GD proposed for teaching and learning the polar coordinates, which will be exposed in the next Section.

5. The Proposed Genetic Decomposition for Polar Coordinates

According to the APOS theory a GD comprises a description that includes mental constructions (actions, processes and objects) and the order in which it may be best for learners to experience them [3]. In what follows, we state mental constructions that a student might need to build in order to learn and understand properly the polar coordinates:

I. ***Prerequisite knowledge***

- A working knowledge of drawing an angle clockwise and counterclockwise in the plane; for example to understand the difference between $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.
- A working knowledge of sketching graphs of basic trigonometric functions like $y=\sin x$, $y=\cos 2x$, etc. Also the ability to compute the values of characteristic trigonometric functions like $\sin x, \cos\frac{7\pi}{2}, \sin\frac{4\pi}{3}, \cos\frac{17\pi}{6}$ etc.
- A working knowledge of Cartesian coordinates, for example plotting points (x,y), sketching line graphs and quadratic graphs in the plane, etc.

II. Polar coordinates

- The ability in the level of an *action* to identify a point P in the plane by an ordered pair (r,θ), where r and θ are positive numbers.
- Interiorization of the above action to a *process* by plotting a point (r,θ) in the plane, where r and θ can be positive or negative. It is recalled that, when $r < 0$, then the point P is determined in a distance $|r|$ on the opposite side of the straight line determined by the angle θ (see Figure 3).

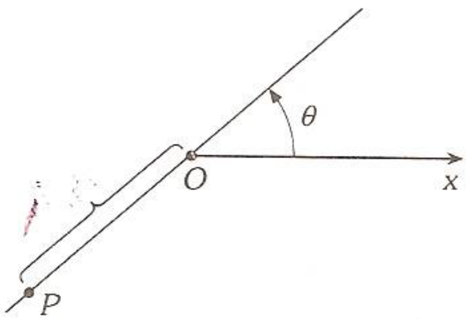

Figure 3. Polar coordinates with r<0

- An *object* conception occurs when students encapsulate all the above cases thus understanding that in a polar coordinate system each point has many representations.
- When the above actions, processes and objects are organized in a coherent *schema*, then students become able to sketch polar equations.

The above schema development can be described by the triad mechanism of Piaget and Garcia (see Section 2). In fact, **a** student being at the *intra* level of this mechanism can sketch a polar equation only by plotting some points (r,θ) and connecting them together (strategy 1 applied for Questions 8 and 9 of the written test). A student at the *inter* level can sketch a polar equation by first sketching the corresponding Cartesian curve (r as a function of θ) and then using it to sketch the polar equation (strategy 2 for Questions 8 and 9 of the written test), but he/she might face problems in certain quadrants or when r is negative. Finally, a student is at the *trans* level if he/she is able to sketch a polar equation in all quadrants correctly.

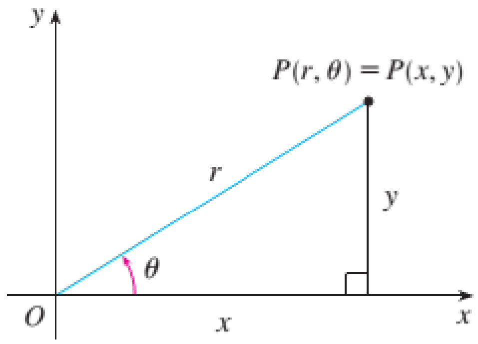

Figure 4. Conversion of polar to Cartesian coordinates and vice versa in the first quadrant

III. Conversion of polar to Cartesian coordinates and vice versa

- The ability in the level of an *action* to convert a polar point (r,θ) of the first quadrant to Cartesian point (x,y) and vice versa by using formulas $x=r\sin\theta$, $y = r \cos \theta$ and $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$ respectively (Figure 4).
- Interiorization of the above action to a *process* by understanding that a polar point (r,θ) and its corresponding Cartesian point (x,y) correspond to the same place in the plane and by using this conception to convert a point being at any quadrant from polar to Cartesian coordinates and vice versa.
- An *object* conception occurs when a student becomes able to convert a polar equation, $f(r,\theta)=0$, to a Cartesian equation, $g(x,y)=0$ or vice versa.
- The next step is the organization of the above actions, processes and objects to the student's *schema* of polar coordinates (see paragraph II).

The above exposition of the headlines of the proposed GD closes the first part of our research on the student understanding of polar coordinates. Next, instructional sequences based on the GD are planed to be developed and implemented with the help of computers (ACE teaching circle) and finally new data will be collected and analyzed in order to test and refine the GD and the pedagogical strategies that will be employed during the ACE circle.

6. Synopsis and Plans for Future Research

Our analysis of student responses in the written test and the oral interviews have shown that student difficulties for the understanding and proper use of the polar coordinates are connected to their previous incomplete knowledge of the trigonometric numbers and functions and of the Cartesian coordinates and equations and with problems related to polar coordinates and equations and to transitions from polar to Cartesian coordinates and vice versa.

The above conclusions led us to the design of a preliminary GD including actions, processes and objects that a student might need to build in order to learn and understand the polar coordinates. Our plans for continuing this research involve the development and implementation of instructional sequences (ACE circle) and the collection and analysis of new data from students in order to test and refine the GD and the ACE design.

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