American Journal of Educational Research, 2015, Vol. 3, No. 12, 1484-1488
Available online at http://pubs.sciepub.com/education/3/12/1

DOI:10.12691/education-3-12-1

Mathematical Problem Solving and Use of Intuition and Visualization by Engineering Students

Chih-Hsien Huang*

Ming Chi University of Technology
*Corresponding author: huangch@mail.mcut.edu.tw

Abstract Deciding on the truth value of mathematical statements is an essential aspect of mathematical practice in which students are rarely engaged. This study explored first year engineering students' approaches to mathematical statements with unknown truth values, taking a perspective that the construction examples is an activity of problem solving. Task-based interviews utilizing the think-aloud method revealed students problem solving processes in depth. The primary data sources were the protocols of 15 students to the questionnaire, three false statements involved basic concepts about derivative and definite integral. Through analysis of the data. The findings suggest that the factors the participants failed to solves problems include: mathematical intuition and prototype example hindered the constructing of counterexamples, there are two dangers in visualizing -figures can induce false conclusions and figures can mislead our reasoning.

Keywords: *engineering students, intuition, problem solving, visualization*

Cite This Article: Chih-Hsien Huang, "Mathematical Problem Solving and Use of Intuition and Visualization by Engineering Students." *American Journal of Educational Research*, vol. 3, no. 12 (2015): 1484-1488. doi: 10.12691/education-3-12-1.

1. Introduction

Determining the truth value of mathematical statements is an important component of the problem solving process. When dealing with uncertainty, mathematicians often try to decide on a statement's truth value with some degree of confidence before investing time in a proof or refutation attempt [5,15]. The proving processis complex and encompasses a multitude of reasoning activities including intuitive, informal, and formal reasoning. *Formal* reasoning is based on logic and deduction, and *informal* reasoning includes reasoning strategies such as visual, example-based, or pattern-based reasoning. This study hereby attempt to explore the use of intuitive and visual reasoning in advanced mathematical problem solving by engineering students in an interview setting. Through an analysis of engineering students' problem-solving protocols and responses, I examined the relationship between intuition and visualization in justifying the truth value of mathematical statement. This study explores (a) the ways intuition and visualization interact in the decision-making process, (b) the ways this decision-making process influences students' constructions of associated counterexamples for the statements.

2. Theoretical Framework

Intuition is especially important for deciding on the truth value of a mathematical statement because it can suggest what is plausible in the absence of a proof [4,12] and "provides a justification for, but is prior to, the search for convincing argument and, ultimately, proof" [3]. In the limited research on intuition in mathematics education, researchers have found a variety of types of intuitive reasoning used by students and mathematicians to evaluate mathematical conjectures. Inglis et al. [15] found that mathematicians' intuitive support for the truth or falsity of a mathematical statement was based on either suspected properties about mathematical objects or known relationships between mathematical concepts. Intuition constructs an automatic mental representation of a task, taking into consideration task cues, prior knowledge, and experience, and operates independently of working memory [9,11,14]. In this study, an intuition is "a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically meaningful, like a simple, given fact" [10]. Fischbein [11] offered two approaches for classifying intuitions, one based on roles or origins. In this classification system, intuitions can be affirmatory, conjectural, anticipatory, or conclusive. In the case of an affirmatory intuition, one affirms or makes a claim. A conjectural intuition is one in which an assumption about future events is expressed. Anticipatory and conclusive intuitions represent phases in the process of solving a problem. Anticipatory intuitions express a preliminary, global view that precedes an analytical solution to a problem. Conclusive intuitions summarize in a global, structured vision the solution to a problem that had previously been elaborated. Anticipatory intuitions are the cognition that implicitly emerges during an attempt at problem solving, immediately after a serious search for a problem-solving strategy. Anticipatory intuitions are holistic and associated with the feeling of conviction derived from comprehensive reasoning or proving.

Increasing attention has being paid to centrality of visualization in learning and doing mathematics, not just for illustrative purposes but also a key component of reasoning [1]. When considering the role of visual images in structuring intuitions, 'it is worth keeping in mind that visual representations are *not* by themselves intuitive knowledge' [11]. Visualization is a critical aspect of mathematical thinking, understanding, and reasoning. Researchers argue that visual thinking is an alternative and powerful resource for students to do mathematics [2,6], it is different from linguistic, logic-propositional thinking and manipulation of symbols. According to Duval [8], visualization can be produced in any register of representation as it refers to processes linked to the visual perception and then to vision. Zimmerman and Cunningham [21] contended that the use of the term "visualization" concerned a concept or problem involving visualizing. Nemirovsky and Noble defined visualization as a tool that penetrated or travelled back and forth between external representations and learners' mental perceptions [18]. Dreyfus contended that what students "see" in a representation would be linked to their conceptual structure, and further proposed that visualization should be regarded as a learning tool [7]. According to Arcavi, visualization has a powerful role in promoting understanding as both a support and an illustration of symbolic results and as a tool for solving conflicts between incorrect intuitions and correct solutions [1]. Visualization helps to grasp the hidden meaning of formal definitions. Arcavi emphasized that visualization is an operational mode: for him, the process of solving a problem is carried out through visualization [1].

3. Method

The participants in this study were 15 first-year undergraduate students at a university of technology in Taiwan, who had previously completed courses in derivatives and definite integrals. All students had successfully solved routine problems. They were given a questionnaire that included three false mathematical statements, and was designed to evaluate the students' abilities to generate examples (counterexamples) that are related to basic differentiation and integration concepts. These students had never previously tackled similar problems.

Statement1:If $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, then $f(x) \geq g(x)$. True or false? Justify your answer.

Statement 2: If f(x)andg(x)are both differentiable and $f(x) \geq g(x)$, $\forall x \in (a,b)$, then $f'(x) \geq g'(x)$, $\forall x \in (a,b)$.True or false? Justify your answer.

Statement 3:If f(x) and g(x) are both differentiable and $f'(x) \geq g'(x)$, $\forall x \in (a,b)$, then $f(x) \geq g(x)$, $\forall x \in (a,b)$. True or false? Justify your answer.

The students were asked to determine the accuracy of the mathematical statements and justify their answers. Data were gathered concerning the examples (counterexamples) that were produced by the participants. Each student worked individually on each problem. The investigation was carried out using clinical interviews. In case of difficulty, the interviewer also acted as a prompter. Interviews were audio-recorded and subjects' notes and figures were collected.

The data generated from (a) transcripts from the participants' task-based interviews using the think-aloud method, and (b) participants' written work on the tasks in the interviews were analyzed using the grounded theory approach [13]. The procedure of data analysis involved open, axial, and selective coding processes for qualitative data, following Strauss and Corbin, to produce descriptive categories [20]. The data were analyzed according to uses of intuitive and visual reasoning during the participants' processes of deciding whether mathematical statement was true or false and constructing counterexamples. Additionally, students' decision-making and construction processes were analyzed to determine students' decision-making and the connections between these processes. I will classify reasoning as intuitive if the student (a) stated that it was an intuition, instinct, gut feeling, or first thought; (b) used similarity to make an assessment of the task; or (c) was unable to justify the reasoning. Reasoning will be classified as visual if the student (a) introduce diagrams; (b) explicitly thinking of pictures or diagrams rather than algebraic representations.

4. Empirical Data and Analysis

4.1. Problem Solving in Statement 1

Five students, including S4, made logical errors in generating supportive examples.

S4: A greater integral correspond to a larger function. For example, given $f = x^2+1$ and $g = x^2$, f is larger than g. The integral of f from 0 to 1 is 4/3, which is also greater than the integral of g, 1/3, over the same interval.

I: The condition is that the integral of f is greater than the integral of g, but you said that the statement is true because the greater the integral is, the larger the function is.

S4: (Eight seconds of silence) I make a mistake. I should find out two different integrals, and then…(12sof silence) I have no idea. I forget how to do it.

Six students used graphical representations to generate examples to support their assertions. Although they connected integrals to areas, they did not understand the true relationship between the two. A typical response is as follows.

S6: The integral represents the area under the curve. A greater integral corresponds to a greater area [Figure 1]. Here, the greater integral is represented by the higher graph, so the function is greater.

I: You drew function graphs above the x-axis. If they were below the x-axis, or one were above the x-axis and the other were below the x-axis, would the results be the same?

S6: The Result would be the same, as long as the graphic of f above the graphic of g.

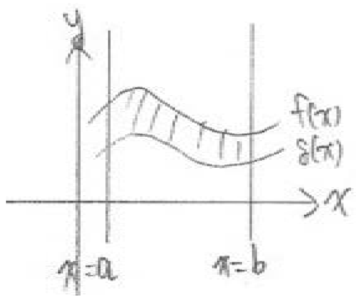

Figure 1. S6' supporting example to Statement 1

Six students asserted that the statement was false, but provided a counterexample that failed to refute the statement; they also did not understand the relationship between integrals and areas.

S9: An integral is an area, so a larger integral means a greater area. The area bounded by *f*(x), x = a, x = b and the x-axis is larger than that bounded by *f*(x), x = a, x = b and the x-axis [Figure2]. However, *f*(x) is smaller than *g*(x).

I: The integral value may be negative, but the area is positive.

S9: (Ten seconds of silence) If the area is above the x-axis, then the integral equals to the area. If the area below the x-axis, then the absolute value of the integral equals the area. (Ten seconds of silence) I think I made a mistake; the statement is correct.

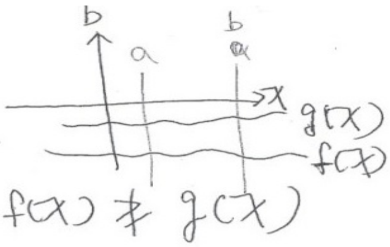

Figure 2. S9'supporting example to statement 1

Only one student accurately determined the truth value of Statement 1; he adopted a trial-and-error strategy to generate examples to evaluate the statement, until he constructed a counterexample:

S13: In the graph [Figure 3] that I drew, the area that that is bounded by *f*(x), x = a, x = b and the x-axis exceeds that bounded by *g*(x), x = a, x = b and the x-axis; therefore, the integral of *f* in [a, b] is greater than the integral of *g* in [a, b]. However, the function value of *f* in the interval [a, c] is smaller than the function value of *g* in the interval [a, c]. Therefore, the statement is incorrect.

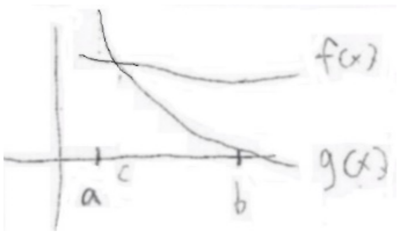

Figure 3. S13' counterexample to Statement 1

4.2. Problem Solving in Statement 2

Five students determined that Statement 2 was true; they did not define the range of the variable and so failed to generate supporting examples. S5 responded typically.

S5: If *f* (x) = x^2 +1 and *g*(x) = x, then *f* is greater than *g*, $f'(x)$ = 2x, $g'(x)$ = 1,and $f'(x)$is greater than $g'(x)$.

I: Why is $f'(x)$ not always greater than$g'(x)$.

S5: The statement is obviously true, for example, if x equals 1, since the value of $f'(x)$ is 2 is greater than the value of $g'(x)$ is 1.

Four students including S7, gave similar examples but refuted the statement.

S7: *f* = x^2 and *g* = x; *f* is greater than *g*, $f'(x)$ = 2x, and $g'(x)$ = 1. When x is less than 1/2, is greater than $f'(x)$.

I: When x is less than 1/2, is *f* always greater than *g*?

S7: No. *f* is not greater than *g*, x must be less than 1/2, and x^2 must be greater than x. Oh! I see. For this example, x must be less than zero.

These students focused on algebraic manipulation, but did not appreciate the importance of domain. Four students who refuted this statement noted the effects of constants in the differentiation process. S8, for example, also noticed the domain of the functions, "Since constant terms become zero after differentiation and the constant of integration affects the value of the function. For example, *f* = 2x+100, *g* = 3x+10, 0≦x≦10, *f* is greater than *g*, and $f'(x)$ = 2, and $g'(x)$ = 3 and $f'(x)$ is greater than $g'(x)$." Two students used graphical representations and slopes of tangents to argue that the mathematical statement was false (Figure 4). Interestingly, these students all marked the domain of the functions on the graphs. For example,

S12: I think a greater function does not imply a greater derivative. This statement is false. Therefore, I would like to find a counterexample to prove its falsity. The graph of *f* is above that of *g*. The graph of *g* is a straight line with a constant slope. However, the slope of *f* in the interval of *(a, b)* is not always greater than that of *g*. Therefore, I have proved that this statement is false.

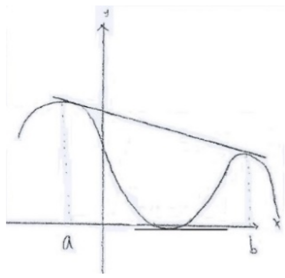

Figure 4. S12' counterexampleto Statement 2

4.3. Problem Solving in Statement 3

Statement 3 is the converse of Statement 2. However, the students performed differently in their responses to these two statements, revealing that Statement 3 was more difficult to evaluate than was Statement 2. However, most of the students neither noticed the interval nor generated examples or counterexamples to confirm or refute the statement. For example, S1 considered $f'(x)$ = 4x and $g'(x)$ = 2x, and claimed that $f'(x)$ is greater than $g'(x)$, and then provided *f* = $2x^2$ and *g* = x^2, and claimed that *f* is greater than *g*. This student did not realize that this example did not satisfy the conditions of the statement, ignoring the range of x and the constant of integration. UnlikeS1, S11 said that the statement was true,and used symbolic representation to generate a supporting example. However, she neglected the arbitrary constant c. She considered $f'(x)$ = 2x+1 and$g'(x)$ = 1, stating that$f'(x)$ is greater than $g'(x)$, for all x > 0; *f* is x^2+x+c; *g* is x+c, and *f* is greater than *g*, for all x > 0.

I: Are these two value of c the same?

S11: They are the same, because both are c.

I: Is c a fixed constant?

S11: No. c can be any constant.

I: So these two constants, c, can be any constant, right?

S11: Yes, but (10 seconds of silence)I made a big mistake; c can be any constant, so these two 'c's can be different. It didn't occur to me. (Seven seconds of silence) Now I claim that this statement is not true; the constant c of *f* is minus100, and the constant c of g is 100, and x should be between zero and one.

One student successfully generated counterexamples using algebraic representation. Unlike S11, S15 noted the effect of the constant of integration and the range of the variable; he considered "$f'(x) = 3$, $g'(x) = 2$, and stated that $f'(x)$ is greater than $g'(x)$, claiming that for f is3x+a, g is 2x+b, and f is greater than g, if $0 \leqq x \leqq 10$, and b>10-a." Two students used graphical representations to generate counterexamples. For example, S2 connected the derivatives to the slope of the tangent [Figure 5]: "the slopes of the tangents of f are greater than zero; the slopes of the tangents of g are less than zero, so $f'(x)$ is greater than $g'(x)$, but g is greater than f in the interval $[a,c_1]$." As in their responses to Statement 3, these two students both marked the domain of the functions on their graphs.

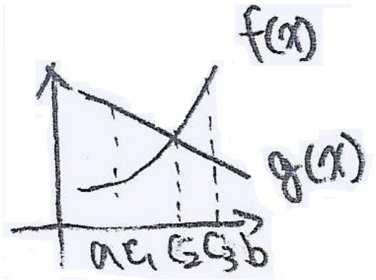

Figure 5. S2's counterexample to Statement 3

5. Discussion and Conclusion

5.1. Students' Use of Intuition and Systematic Intuitive Errors

From students' problem solving behavior of this study, and from judging the mathematical statements, the kind of intuition that can be used for students classified in this type of exercise is affirmatory intuition. In planning the solution, students used intuition to try to use symbolic and graphical representations. This intuition is classified as anticipatory intuition. In implementing the solution successfully, the intuition is used by way of trial and error, the kind of intuition that is used in this type of student learning is classified as anticipatory intuition.

Systematic intuitive errors are errors of intuitive reasoning that cause misrepresentations of situations and persist across situations and people. Many students made logical errors in deciding the truth value of mathematical statement (e.g., S4). According to Fischbein [11], we can affirm that the (false) equivalence between a statement and the converse is an intuition. Moreover, it is important to guide the students to the awareness of the structure of their argumentations, so that the knowledge of the non-equivalence between the statement and the converse becomes an intuitive knowledge. As Fischbein[11] wrote:

The training of logical capacities is a basic condition for success in mathematics and science education. We refer not only to a formal-algorithmic training. The main concern has to be the conversion of these mental schemas into intuitive efficient tools, that is to say in mechanisms organically incorporated in the mental behavioral abilities of the individual.

The participant of this study also hold the false intuition ' More A-More B ' [19] in the statement 1, for example, ' The greater the functionis, the larger the integralis ' (S4) And ' With a greater integral, the area is greater ' (S6). What is interesting is, students do not hold similar false intuition regarding the statement 2 and statement 3 about the concept of derivative, but about the concept of function. Such as S10, because 2 is greater than 1, so ' $2x$ is greater than x'; because 4 is greater than 2, so ' $4x$ is greater than $2x$ '. Differentiation is usually easier than integration, and the representation of area is easier than the representation of slope of tangent, according to the problem solving processes of the participants. This finding shows that the calculation complexity of the mathematical statement involved and the intensity of its connection with the graphical representation. These observations seem to relate to the inclination of students using intuitive laws. Many systematic intuitive errors can be classified as accessibility errors [14,16]. Accessibility is the ease with which certain knowledge is evoked or certain task features are perceived and is a crucial component of intuitive reasoning and decision-making [16]. These intuitive errors involve attribute substitution [16], when a more readily accessible attribute is substituted in a task for a less readily accessible attribute. For example, similarity is an attribute that is always accessible because it is processed intuitively [17]. Participants may intuitively notice similarities between a given concept (integral, $2x$, or $4x$) and familiar concept (area, 2, or 4) and substitute more accessible attributes for less accessible ones based on these similarities.

5.2. Students' Use of Visual Reasoning

One of the main heuristic strategies in many calculus tasks is to draw a graph of the function involved. However, most of the students had the strong inclination to use symbolic representation. What is interesting is, even if students use graphical representations to generate examples, only a few students could generate correct counterexamples. A possible reason is that there is no visual component in their concept image of the derivative and definite integral, this makes them difficulty to "see" the statements. This is particularly true for Statement 1. It is difficult to find an appropriate counterexample, if one can't expand one's limited concept image. The most significant expansion in the evoked concept image of function, in terms of being associated with learning events, is the use of visualization in the sense of Zimmermann and Cunningham: "Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding" [21]. Why is visualization important? The examples generated by students show that those using the symbolic representation were unable to meet the condition" $\forall x \in (a, b)$. On the contrary, the use of graphical representation allow students to control more assumed conditions at the same time while generating a counterexample. The global but not the local idea that the graph had could be associated with the statement, allowing the graph to act as a kind of generic example. In other words, visualization allowed students to control larger number of conditions simultaneously, while in the symbolic representation students may only control one requirement at a time. This finding provided some support to corroborate Fischbein's claim that visualization 'not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.' [11]. We suggest that visualization can be more than that: it can be the analytical process itself which concludes with a generic solution.

There are two dangers in visualizing. The first danger in visualizing is that *figures can induce false conclusions*. In fact, in this case (e.g., S8 and S9), it is not the figure that is incorrect and that brings us to the false conclusion. Rather, what is misleading is the reasoning 'behind' the figure. These incorrect—propositionally expressed—hypotheses activated an inaccurate figure, and this is what brings to a false conclusion. Nevertheless, this does not mean that the figure is incorrect as a figure. Rather it is the role of this figure as the activation of some incorrect hypotheses. Therefore, the error is in the informal reasoning which is behind the construction of these figures, and not in the figures themselves, or in the possibility of putting them to the test. This kind of error in using figures is pre-visual, since it depends on wrong hypotheses that are made before the figures are drawn. The second danger in visualizing is that f*igures can mislead one's reasoning*. This can happen when the reasoning is performed on the particular image that represents the mathematical statement without considering the consequences implied by it. Concerning problem solving performance of S2 and S6, though students know the mathematical concepts of derivative and definite integral, they are not capable of solving the mathematical statements. These kinds of errors in using figures are post-visual, since they depend on wrong hypotheses that are made on the drawn figure.

If we consider examples taken from mathematical problem solving, we see that the appeal to visualization is not direct, because it strongly depends on expertise. Moreover, discovery by visualization is mediated by the intuition of the generality of the conclusions obtained by means of it. Nevertheless, intuition and visualization are interconnected parts of a vast web of knowledge that results in the learning and in the application of a mathematical problem solving. It is the preservation of these interconnections that allows for the intuition of the generality of some conclusion and the consequent stabilization of certain beliefs.

Acknowledgement

This article is a part of a project that is supported by the National Science Council of Taiwan (NSC 102-2511-S-131-001-MY3).

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