Journal of Automation and Control, 2013, Vol. 1, No. 1, 14-20
Available online at http://pubs.sciepub.com/automation/1/1/3
© Science and Education Publishing
DOI:10.12691/automation-1-1-3

Exponential Observer for a Class of Exothermal Axial Dispersion Reactors

N. Barje[1,*], M. E. Achhab[1], V. Wertz[2]

[1]Department of Mathematics, Faculty of Sciences, Université Chouaib Doukkali, El Jadida, Morocco
[2]CESAME, Université Cathoulique de Louvain, Louvain-La-Neuve, Belgium
*Corresponding author: nbarje@yahoo.fr

Received August 28, 2013; Revised September 21, 2013; Accepted September 25, 2013

Abstract In this work, an exponential observer is performed for an exothermal axial dispersion tubular reactor that involves one nonlinear sequential reaction. More precisely, the given state estimator is performed by using bounded observations and properties of the nonlinear set of partial differential equations. It is shown that the proposed observer admits a global unique solution and ensures asymptotic state estimator with exponentially decay error, when the temperature is available for measurement at the reactor outlet only. This result is confirmed by a numerical simulation.

Keywords: *distributed parameter systems, dissipativity, exponential observer, invariance, nonlinear systems, perturbed systems, tubular reactors, C_0-semigroup*

Cite This Article: N. Barje, M. E. Achhab, and V. Wertz, "Exponential Observer for a Class of Exothermal Axial Dispersion Reactors." *Journal of Automation and Control* 1, no. 1 (2013): 14-20. doi: 10.12691/automation-1-1-3.

1. Introduction

Over the last few years, extensive studies of the (bio)chemical processes monitoring and control have been elaborated and received considerable attentions, with a view to provide "good" controller and stabilizer designs for the (bio)process in consideration [1,2,3,4]. However, an important difficulty in the operation of a control (bio)process is related to the (poor) knowledge of the system's states, due to the fact that, the variables of the models used to describe their dynamics depend on spatial variable. This dependence, along with additional aspects such as the boundary conditions and presence of nonlinear terms in the model, increase the complexity to have access to the necessary measurements. Installing all the requisite sensors may not be physically possible. In such cases, the states can be estimated using state estimators (observers) [5,6,7]. For surveys on observer design methods for this class of systems, see, [8] and the references quoted therein. In [3], the (bio) chemical non-isothermal tubular reactors are regionally stabilized around a desired profile using asymptotic observers, if the temperature in the reactor is available for measurement. In this direction, this paper investigates the question of the conception of an exponential observer for a class of (bio) chemical non-isothermal tubular reactors, in the case where the measurements of the temperature may occur at the reactor outlet only. The kinetics of the process is characterized by first-order kinetics with respect to the reactant concentration C (mol/l) and by Arrhenius-type dependence with respect to the temperature T (K). The dynamics of the process are described by the following two energy and mass balance partial differential equations (PDEs) (see [4,9]):

$$\frac{\partial T}{\partial \tau} = D_a \frac{\partial^2 T}{\partial \xi^2} - \upsilon \frac{\partial T}{\partial \xi} - \frac{4h}{\rho C_p d}(T - T_C) - \frac{\Delta H}{\rho C_p} k_0 C e^{\frac{-E}{RT}} \quad (1)$$

$$\frac{\partial C}{\partial \tau} = D_a \frac{\partial^2 C}{\partial \xi^2} - \upsilon \frac{\partial C}{\partial \xi} - k_0 C e^{\frac{-E}{RT}} \quad (2)$$

with the boundary conditions are given, for $\tau \geq 0$, by

$$D_a \frac{\partial T}{\partial \xi}(0,\tau) = \upsilon(T(0,\tau) - T_{in}) \quad (3)$$

$$D_a \frac{\partial C}{\partial \xi}(0,\tau) = 0 = \upsilon(C(0,\tau) - C_{in}) \quad (4)$$

$$D_a \frac{\partial T}{\partial \zeta}(L,\tau) = 0 \quad (5)$$

$$D_a \frac{\partial C}{\partial \zeta}(L,\tau) = 0, \quad (6)$$

and the initial conditions are given, for $0 \leq \xi \leq L$, by

$$T(\xi,0) = T_0(\xi), \quad C(\xi,0) = C_0(\xi) \quad (7)$$

In the equations above, the following parameters D_a, υ, ΔH, ρ, C_ρ, k_0, E, R, h, d and T_C are the axial dispersion coefficient, the superficial fluid velocity, the heat of reaction, the density, the specific heat, the kinetic constant, the activation energy, the ideal gas constant, the wall heat transfer coefficient, the reactor diameter and the coolant temperature. T_{in} and C_{in} are

respectively the inlet temperature and the inlet reactant concentration. τ, ξ and L denote the time and space independent variables, and the length of the reactor, respectively. Finally T_0 and C_0 denote the initial temperature and reactant concentration profiles.

From a physical point of view it is expected that for all $(z,t) \in [0,1] \times [0,+\infty]$,

$0 \le T(z,t) \le T_{\max}$ and $0 \le C(z,t) \le C_{\max}$

where $T_{\max}$ could possibly be equal to $+\infty$.

Let consider the following dimensionless state transformation:

$$x_1 = \frac{T - T_{in}}{T_{in}}, \quad x_C = \frac{T_C - T_{in}}{T_{in}}, \quad x_2 = \frac{C_{in} - C}{C_{in}}$$

Let us consider also dimensionless time t and space z variables:

$$t = \frac{\tau \upsilon}{L}, \quad z = \frac{\xi}{L}$$

We shall assume in the rest of the paper that the coolant temperature is T_{in} (i.e., $x_C \equiv 0$), since x_C will be eliminated in the equation of the reconstruction error between the plan state and the observer state.

Then we obtain the following equivalent representation of the uncontrolled model (1)-(7):

$$\frac{\partial x_1}{\partial t} = \beta_a \frac{\partial^2 x_1}{\partial z^2} - \frac{\partial x_1}{\partial z} - \gamma x_1 + \alpha\delta(1 - x_2) e^{\frac{\mu x_1}{1+x_1}} \tag{8}$$

$$\frac{\partial x_2}{\partial t} = \beta_a \frac{\partial^2 x_2}{\partial z^2} - \frac{\partial x_2}{\partial z} + \alpha(1 - x_2) e^{\frac{\mu x_1}{1+x_1}} \tag{9}$$

with the boundary conditions

$$\beta_a \frac{\partial x_i}{\partial z}(z = 0, t) - x_i(z = 0, t) = 0, \quad i = 1, 2 \tag{10}$$

$$\beta_a \frac{\partial x_i}{\partial z}(z = 1, t) = 0, \qquad i = 1, 2 \tag{11}$$

and the initial conditions

$$x_1(z, t = 0) = x_1^0, \quad x_2(z, t = 0) = x_2^0 \tag{12}$$

The parameters α, γ, δ and μ are related to the original parameters as follows:

$$\mu = \frac{E}{RT_{in}}, \alpha = \frac{k_0 L}{\upsilon} e^{-\mu}, \gamma = \frac{4hL}{\rho C_p d\upsilon}, \delta = -\frac{\Delta H}{\rho C_p} \frac{C_{in}}{T_{in}}$$

The real constants α, β_a, γ and μ are strictly positive, whereas the constant δ is strictly positive ($\Delta H < 0$) for the exothermic reactions and strictly negative ($\Delta H > 0$) for the endothermic reactions. In this paper, we investigate the case of exothermic reaction (i.e., $T \ge T_{in}$), or equivalently the case when $0 \le x_1(z,t) \le x_{1,\max}$.

In the following sections, a conception of an exponential observer for the nonlinear system (8)-(12) is firstly given; then we state the existence of the global solution of the state estimator and prove that the estimation error converges exponentially to zero; at last, some conclusions and simulation results on the efficiency of the given conception can be obtained. The background of our approach can be found in [10,11,12].

2. Notations and Preliminaries

Let ($X, \|.\|$) be a real Banach space, A is the infinitesimal generator of $(T(t))_{t\ge0}$, N is a continuous nonlinear function from a closed subset D of X into X and I is the identity operator on X. Recall that

$$d(x; D) = \inf \|x - y\|, \qquad y \in D$$

For the following uncontrolled abstract Cauchy problem

$$\begin{cases} \dot{x} = Ax(t) + N(x(t)), \\ x(0) = x_0 \in D \end{cases} \tag{13}$$

we sate this important theorem that ensures the existence of the global unique solution .

Theorem 2.1 ([10], p. 355) If the following conditions are satisfied:

i) D is $(T(t))_{t\ge0}$-invariant, i.e., $T(t)D \subset D$ for all $t \ge 0$;

ii) For all $x \in D$,

$$\lim_{h \to 0^+} \frac{1}{h} d(x + hN(x); D) = 0$$

iii) N is continuous in D and there exists $l_N \in IR^+$ such that the operator $N - l_N I$ is dissipative on D (i.e., $\langle (N - l_N I)(x - y), x - y \rangle \le 0, \quad \forall x, y \in D$).

Then, (13) has a unique mild solution $x(t, x_0)$ on $[0, +\infty[$, for all $x_0 \in D$. Furthermore, if $(S(t))_{t\ge0}$ is defined on D by $S(t)x_0 = x(t, x_0)$, for all $t \ge 0$ and $x_0 \in D$, it is a nonlinear semigroup on D with $(A + N)$ as its generator.

We state also the following Theorem that will be needed to prove the exponential convergence of the estimation error.

Theorem 2.2 ([12], p. 109) Let A be the infinitesimal generator of a C_0-semigroup $(T_A(t))_{t\ge0}$ and D is linear bounded operator on H. The operator $A + D$ is the infinitesimal generator of a C_0 -semigroup $(T_{A+D}(t))_{t\ge0}$ which is the unique solution of the equation

$$T_{A+D}(t)x_0 = T(t)x_0 + \int_0^T T(t-s) D T_{A+D}(s) x_0 ds,$$

For all $x_0 \in H$. If in addition, $\|T_A(t)\| \le Me^{\omega t}$, then

$$\|T_{A+D}(t)\| \le Me^{(\omega + M\|D\|)t}$$

Throughout the sequel, we assume $H = L^2[0,1] \times L^2[0,1]$, the Hilbert space with the usual inner product $\langle z_1, z_2 \rangle = \langle x_1, x_2 \rangle_{L^2} + \langle y_1, y_2 \rangle_{L^2}$, for all $z_1 = (x_1, y_1)^T$ and $z_2 = (x_2, y_2)^T$ in H , and the induced

norm defined by $\left\|(x_1,x_2)^T\right\| = \sqrt{\|x_1\|_{L^2}^2 + \|x_2\|_{L^2}^2}$, for all $(x_1,x_2)^T \in H$.

Clearly the Hilbert space H is a real Banach Lattice (for more details, see [14]) where, $x \le y \Leftrightarrow$ ($x_1 \le y_1$, $x_2 \le y_2$), for almost all $z \in [0,1]$.

for all given $x=(x_1,x_2)^T \in H$, $y=(y_1,y_2)^T \in H$, for all given $x=(x_1,x_2)^T \in H$, $y=(y_1,y_2)^T \in H$, and $H^+ = \{x \in H: \ 0 \le x\}$ is a positive cone. Let Γ be a linear operator on D, then Γ is said to be positive linear operator if $0 \le \Gamma x$, for all $x \ge 0$, or equivalently $\Gamma H^+ \subset H^+$.

As a useful criterion for the invariance condition given by (i) of Theorem 2.1, we have the following proposition.

Proposition 2.3 [9] Let T(t) be a strongly continuous semigroup of bounded linear operators on a real Banach lattice X, generated by A, such that $\|T(t)\| \le Me^{\omega t}$ for all $t \ge 0$ for some $M \ge 1$ and $\omega \in IR$, then: $T(t)$ is positive if and only if the resolvent operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is a positive linear operator for all $\lambda > \omega$.

Where,

$$R(\lambda,A)x = \int_0^{+\infty} e^{-\lambda s} T(s)x ds, \qquad \forall x \in H.$$

3. Semi-linear State Estimator

The (PDEs) (8)-(12) describing the exothermic reactor dynamics can be written on its compact form as

$$\begin{cases} \dot{x} = Ax(t) + N(x(t)), \\ x(0) = x_0 \in D \end{cases} \tag{14}$$

where A is the linear operator defined by:

$$D(A) = \left\{ \begin{array}{l} x=(x_1,x_2)^T \in H : x, \dfrac{dx}{dz} \in H \text{ are absolutely} \\ \text{continuous}, \dfrac{d^2x}{dz^2} \in H, \beta_a \dfrac{dx_i}{dz}(0) - x_i(0) = 0, \\ \dfrac{dx_i}{dz}(1) = 0, \text{ for } \ i = 1,2 \end{array} \right\}$$

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \beta_a \dfrac{d^2.}{dz^2} - \dfrac{d.}{dz} - \gamma I & 0 \\ 0 & \beta_a \dfrac{d^2.}{dz^2} - \dfrac{d.}{dz} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{15}$$

It is shown in [13] that the linear operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on H, given by

$$T_A(t) = \begin{pmatrix} T_{A_1}(t) & 0 \\ 0 & T_{A_2}(t) \end{pmatrix}$$

Where $(T_{A_1}(t))_{t\ge 0}$ and $(T_{A_2}(t))_{t\ge 0}$ are the C_0-semigroups generated, respectively, by A_1 and A_2.

Remark 3.1 **i)** It is shown in [9] that $T_A(t)H^+ \subset H^+$ for all $t \ge 0$, (which is equivalent, by Proposition 2.3, to $R(\lambda,A)H^+ \subset H^+$, $\forall \lambda \in IR$), and $T_A(t)D \subset D$, for all $t \ge 0$.

ii) The C_0-semigroup $(T_A(t))_{t\ge 0}$ is exponentially stable (see [13]), i.e. there exist constants $M > 0$ such that

$$\|T_A(t)\| \le Me^{\frac{-1}{4\beta_a}t}, \qquad \forall t \ge 0$$

In particular, $\|T_A(t)\| \le Me^{\omega t}$, for all $t \ge 0$, with $\omega = 0$. Moreover, there exists a time t_M, such that

$$\|T_A(t)\| \le Me^{\frac{-1}{4\beta_a}t}, \qquad \forall t \ge t_M,$$

The nonlinear operator N is defined on

$$D := \left\{ \begin{array}{l} x=(x_1,x_2)^T \in H, 0 \le x_1(z) \text{ and } 0 \le x_2(t) \le 1, \\ \text{for almost all } z \in [0,1] \end{array} \right\}$$

for all $x=(x_1,x_2)^T$ in D,

$$N(x) = \begin{pmatrix} N_1(x) \\ N_2(x) \end{pmatrix} := \begin{pmatrix} \alpha\delta(1-x_2)e^{\frac{\mu\ x_1}{1+x_1}} \\ \alpha(1-x_2)e^{\frac{\mu\ x_1}{1+x_1}} \end{pmatrix} \tag{16}$$

3.1. State Estimator Conception

Hereafter we consider measurements of the state vector $x(t)$ are available at the reactor output only. In this case, the output function $y(.)$ is defined as follows: we consider a (very small) finite interval at the reactor output $[1-\omega,1]$ such that:

$$\begin{aligned} y(t) &= (Cx)(t) \\ &:= \int_0^1 \chi_{[1-\omega,1]}(a)x(a,t)da, \quad \forall t \in IR^+ \end{aligned} \tag{17}$$

Where, $\chi_{[1-\omega,1]}(a) = 1$ if $a \in [1-\omega,1]$ and $\chi_{[1-\omega,1]}(a) = 0$ elsewhere, with $0 < \omega < 1$ is a small number. The observer operator $C: H \to IR^2$ is linear bounded. For all $x, y \in H \times IR^2$,

$$\begin{aligned} \langle Cx, y\rangle_{IR^2} &= \langle \int_0^1 \chi_{[1-\omega,1]}(a)x(a,.)da, y\rangle_{IR^2} \\ &= \int_0^1 \langle x(a,.), \chi_{[1-\omega,1]}(a)y\rangle_{IR^2} da \end{aligned}$$

The adjoin operator C^* of C is defined for all $(z,t) \in [0,1] \times IR^+$ by:

$$(C^*y)(z) = \chi_{[1-\omega,1]}(z)y$$

For all $x \in H$,

$$\left\|C^*Cx\right\|^2 = \int_0^1 (\chi_{[1-\omega,1]}(z)(\int_0^1 \chi_{[1-\omega,1]}(a)x(a,.)da))^2 dz$$
$$\leq \omega \left\|\chi_{[1-\omega,1]}\right\|^2 \|x\|^2 = \omega^2 \|x\|^2$$

Then,

$$\left\|C^*C\right\| \leq \omega$$

An exponential observer for the system (8)-(12), when only the temperature is available for measurement, at the given reactor outlet, is given by

$$\frac{\partial \hat{x}_1}{\partial t} = \beta_a \frac{\partial^2 \hat{x}_1}{\partial z^2} - \frac{\partial \hat{x}_1}{\partial z} - \gamma \hat{x}_1 + \alpha\delta(1-\hat{x}_2)e^{\frac{\mu\ \hat{x}_1}{1+\hat{x}_1}} + gC_1^*(C_1x_1 - C_1\hat{x}_1) \quad (18)$$

$$\frac{\partial \hat{x}_2}{\partial t} = \beta_a \frac{\partial^2 \hat{x}_2}{\partial z^2} - \frac{\partial \hat{x}_2}{\partial z} + \alpha(1-\hat{x}_2)e^{\frac{\mu\ \hat{x}_1}{1+\ddot{x}_1}} + gC_2^*(C_2x_2 - C_2\hat{x}_2) \quad (19)$$

with the boundary conditions

$$\beta_a \frac{\partial \hat{x}_i}{\partial z}(z=0,t) - \hat{x}_1(z=0,t) = 0, \qquad i=1,2 \quad (20)$$

$$\beta_a \frac{\partial \hat{x}_i}{\partial z}(z=1,t) = 0, \qquad i=1,2 \quad (21)$$

and the initial conditions

$$\hat{x}_1(z,t=0) = \hat{x}_1^0, \quad \hat{x}_2(z,t=0) = \hat{x}_2^0 \quad (22)$$

The system (18)-(22) can be written on its compact form as

$$\begin{cases} \dot{\hat{x}}(t) = (A - GC^*C)\hat{x}(t) + N(\hat{x}(t)) + GC^*Cx(t), \\ \hat{x}(0) = \hat{x}_0 \in D \end{cases} \quad (23)$$

where, $x(t) = (x_1(.,t), x_2(.,t))^T$ is the state variable of (13) and $\hat{x}(t) = (\hat{x}_1(.,t), \hat{x}_2(.,t))^T$. The linear operator G is the observer gain, satisfying

$$G := \begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} = g\begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with I_1 is the identity operator of the Hilbert $L^2[0,1]$. The initial state $(x_1(0), x_2(0))^T$ of (12) is unknown while the initial state $(\hat{x}_1(0), \hat{x}_2(0))^T$ of the observer can be assigned arbitrarily. Thus, the estimation error is still an unknown quantity even if we know $(\hat{x}_1(0), \hat{x}_2(0))^T$.

3.2. Existence of the Global Solution

Let consider the following coupled system, given for all $(x(0), \hat{x}(0))^T \in D \times D$, by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ GC^*C & A - GC^*C \end{pmatrix}\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{pmatrix} + \begin{pmatrix} N(x(t)) \\ N(\hat{x}(t)) \end{pmatrix} \quad (24)$$

In aim to investigate the asymptotic behavior of the estimation error $x(.) - \hat{x}(.)$, we need to prove the existence of the solution of the augmented system (24), which remains in $D \times D$, on the whole interval $[0 + \infty)$, by applying Theorem 2.1. For this end we state the following lemmas concerning the nonlinearity involved in the dynamics (24). The proofs are similar to that given in [9].

Lemma 3.2.1 Consider the nonlinear operator (16). Then the operator $\begin{pmatrix} N - l_N I \\ N - l_N I \end{pmatrix}$ is dissipative on $D \times D$, where $l_N := \alpha e^{\mu}(1+\mu)(1+|\delta|)$.

Let define on $H \times H$, the distance

$$d_0((x,\hat{x})^T, D \times D)$$
$$= \inf\{\ d((x,\hat{x})^T, (y,\hat{y})^T)\ /\ (y,\hat{y})^T \in D \times D\ \}$$

Lemma 3.2.2 For all $(x,\hat{x})^T \in D \times D$,

$$\lim_{h \to 0^+} \frac{1}{h} d_0(\begin{pmatrix} x \\ \hat{x} \end{pmatrix} + h \begin{pmatrix} N(x) \\ N(\hat{x}) \end{pmatrix}, D \times D) = 0$$

From the Theorem 2.2, the operator $A - GC^*C$ is the infinitesimal generator of a C_0-semigroup

$$(T_{A-GC^*C}(t))_{t\geq 0} (:= \begin{pmatrix} T_{A_1 - G_1C_1^*C_1}(t) & 0 \\ 0 & T_{A_2}(t) \end{pmatrix}))_{t\geq 0}$$

where $(T_{A_1 - G_1C_1^*C_1}(t))_{t\geq 0}$ is the C_0-semigroup generated by the operator $A_1 - G_1C_1^*C_1$.

In the aim to study the invariance condition (i) of Theorem 2.1, it is useful to have the following Lemma:

Lemma 3.2.3 The C_0-semigroup $(T_{A-GC^*C}(t))_{t\geq 0}$, is $D_1 \times D_2$ invariant.

Proof From Remark 3.1, we have $\left\|T_{A_1}(t)\right\| \leq e^{\omega\ t}$, for all $t \geq 0$ with $\omega = 0$, it follows by Theorem 2.2 that,

$$\left\|T_{A_1 - G_1C_1^*C_1}(t)\right\| \leq e^{(\omega + \left\|G_1C_1^*C_1\right\|)t}, \qquad \forall t \geq 0$$

Let now prove that the semigroup $(T_{A_1 - G_1C_1^*C_1}(t))_{t\geq 0}$ is D_1 invariant, it is sufficient to demonstrate that the operator $A_1 - G_1C_1^*C_1$ is positive (i.e., $R(\lambda, A_1 - G_1C_1^*C_1)H^+ \subset H^+$ for all $\lambda > M\left\|G_1C_1^*C_1\right\|$, according to the proposition 2.3).

Consider that $\lambda > M\left\|G_1C_1^*C_1\right\|$ and $(x,y) \in H^+ \times D(A_1)$ such that:

$$R(\lambda, A_1 - G_1C_1^*C_1)x = y$$

Let prove that $y \in H^+$, we have

$$x = (\lambda I - A_1)y + G_1C_1^*C_1 y$$

(ie., $(\lambda I - A_1 + G_1C_1^*C_1)^{-1}x = y$).

Whence,

$$(\lambda I - A_1)^{-1}x = y + (\lambda I - A_1)^{-1}G_1C_1^*C_1y$$

Then, for almost all $z \in [0,1]$,

$$y(z) = ((\lambda I - A_1)^{-1}x)(z) - g(\int_{1-\omega}^{1} y(a)da)((\lambda I - A_1)^{-1}\mathrm{X}_{[1-\omega,1]}(.))(z)$$

It follows that,

$$y(z) = (R(\lambda, A_1)x)(z) - g(\int_{1-\omega}^{1} y(a)da)\int_0^{+\infty} e^{-\lambda t}T_{A_1}(t)\mathrm{X}_{[1-\omega,1]}(z)dt$$

From Remark 3.1, for all $\lambda \geq 0$, $R(\lambda, A_1)H^+ \subset H^+$

Then,

i) For almost all $z \in [0,1] \setminus [1-\omega,1]$, $y(z) \geq 0$.

ii) For almost all $z \in [1-\omega,1]$, there are three cases:

a) If $\int_{1-\omega}^{1} y(a)da = 0$, thus $y(z) \geq 0$.

b) If $\int_{1-\omega}^{1} y(a)da < 0$, thus $y(z) \geq 0$.

c) If $\int_{1-\omega}^{1} y(a)da > 0$, suppose that there exist a subset non negligible $V \subseteq [1-\omega,1]$, such that $y(z) < 0$ for almost all $z \in V$. We can suppose $[1-\omega,1] \setminus V$ negligible in the size, since ω is very small number, it follows,

$$\int_{1-\omega}^{1} y(a)da < 0$$

what is contradictory.

Thus,

$y(z) \geq 0$, for almost all $z \in [1-\omega,1]$.

It follows from (i) and (ii), that $y(z) \geq 0$ for almost all $z \in [0,1]$.

Consequently,

$$T_{A_1-G_1C_1^*C_1}(t)H^+ \subset H^+, \ \forall t \geq 0.$$

It follows that,

$$T_{A_1-G_1C_1^*C_1}(t)D_1 \subset D_1.$$

Besides, we have from Remark 3.1, $T_{A_2}(t)D_2 \subset D_2$.

Therefore,

$$T_{A-GC^*C}(t)D_1 \times D_2 \subset D_1 \times D_2, \qquad \forall t \geq 0$$

The proof is thus complete.

The following proposition demonstrates the existence of the unique mild solution on $[0,+\infty)$ of the coupled nonlinear system (24):

Proposition 3.2.4 For all $(x_0, \hat{x}_0)^T \in D \times D$, the dynamic system (24) has a unique mild solution $(x(t,x_0), \hat{x}(t,\hat{x}_0))^T$ in $D \times D$, for all $t \geq 0$.

Proof The linear bounded operator $\begin{pmatrix} A & 0 \\ GC^*C & A-GC^*C \end{pmatrix}$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ defined for all $t \geq 0$ by

$$T(t) = \begin{pmatrix} T_A(t) & 0 \\ S(t) & T_{A-GC^*C}(t) \end{pmatrix};$$

$$S(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \int_0^t T_{A_1-G_1C_1^*C_1}(t-s)G_1C_1^*C_1T_{A_1}(s)x_1ds & 0 \\ 0 & 0 \end{pmatrix}$$

(see [12], p., 30). In particular the C_0-semigroup $(T(t))_{t\geq 0}$ satisfies,

$$T(t)D \times D \subset D \times D, \qquad \forall t \geq 0$$

Indeed, it is proved in [9] that

$$T_A(t)(= \begin{pmatrix} T_{A_1}(.) & 0 \\ 0 & T_{A_2}(.) \end{pmatrix})D_1 \times D_2 \subseteq D_1 \times D_2$$

Besides, from Lemma 3.2.3, for all $x(=(x_1,x_2)^T)$, $\hat{x}(=(\hat{x}_1,\hat{x}_2)^T)$ in D, and for all $t \geq 0$ we have

$$T_{A-GC^*C}(t)\hat{x} = \begin{pmatrix} T_{A_1-G_1C_1^*C_1}(t) & 0 \\ 0 & T_{A_2}(t) \end{pmatrix}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \in D_1 \times D_2$$

Since

$$0 \leq T_{A_1-G_1C_1^*C_1}(.)G_1C_1^*C_1T_{A_1}(.)x_1, \ \forall x_1 \in D_1.$$

Then,

$$S(t)x \in D, \qquad \forall x \in D$$

Hence,

$$\begin{pmatrix} \int_0^t T_{A_1-G_1C_1^*C_1}(t-s)G_1C_1^*C_1T_{A_1}(s)x_1ds & 0 \\ 0 & T_{A_2}(t)\hat{x}_2 \end{pmatrix} + \begin{pmatrix} T_{A_1-G_1C_1^*C_1}(t)\hat{x}_1 & 0 \\ 0 & 0 \end{pmatrix} \in D_1 \times D_2$$

It follows that,

$$S(t)x + T_{A-GC^*C}(t)\hat{x} \in D, \qquad \forall t \geq 0$$

Therefore, for all $(x,\hat{x})^T \in D \times D$

$$T(t)\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \in D \times D, \qquad \forall t \geq 0$$

The condition (i) of Theorem 2.1 is thus satisfied. Conditions (ii) and (iii) of Theorem 2.1 follows respectively by Lemma 3.2.1 and Lemma 3.2.2 Finally by applying Theorem 2.1, the augmented system (24) admits a unique mild solution $(x,\hat{x})^T \in D \times D$ on the whole interval $[0,+\infty)$.

3.3. Convergence of the Estimation Error

The following main result deals with the proof of the exponential convergence of the estimation error.

Theorem 3.3.1: Given the exothermal axial-dispersion reactor model (8)-(12). Suppose that there exists a

bounded linear operator $G = g\begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$ with g is a positive number, such that $g < \frac{1-4\beta_a l_N}{4\beta_a \omega}$, then the dynamic system (18)-(22) is an exponential observer for the system (8)-(12).

Proof From Lemmas 3.2.1, 3.2.2 and 3.2.3, we prove by applying Theorem 2.1 that the evolution of the estimation error, given by

$$\dot{x}(t) - \dot{\hat{x}}(t) = (A - GC^*C)(x(t) - \hat{x}(t)) + N(x(t)) - N(\hat{x}(t))$$

admits for all $(x(0), \hat{x}(0))^T \in D \times D$ an unique mild solution on the whole interval $[0, +\infty)$, satisfying

$$x(t) - \hat{x}(t) = T_{A-GC^*C}(t)(x(0) - \hat{x}(0)) + \int_0^t T_{A-GC^*C}(t-s)(N(x(s)) - N(\hat{x}(t)))ds$$

such that, $(x(t), \hat{x}(t))^T \in D \times D$, for all $t \geq 0$.

It follows, for all $t \geq 0$, that

$$\|x(t) - \hat{x}(t)\| \leq \left\|T_{A-GC^*C}(t)\right\| \|x(0) - \hat{x}(0)\| + \int_0^t \left\|T_{A-GC^*C}(t-s)\right\| \|N(x(s)) - N(\hat{x}(s))\| ds$$

On the other hand, from Remark 3.1, there exists a time t_M such that,

$$\|T_A(t)\| \leq e^{\frac{-1}{4\beta_a}t}, \quad \forall t \geq t_M .$$

Consider $\alpha = \frac{1}{4\beta_a} - g\omega$. It follows from Theorem 2.2, that,

$$\left\|T_{A-GC^*C}(t)\right\| \leq M \exp(-\alpha t), \quad \forall t \geq t_M$$

Thus, for all $t \geq t_M$,

$$\left\|e^{\alpha\ t}(x(t) - \hat{x}(t))\right\| \leq \|x(0) - \hat{x}(0)\| + l_N \int_0^t e^{\alpha\ s} \|(x(s) - \hat{x}(s))\| ds$$

By applying Gronwall's Lemma ([11], p 639), for all $t \geq t_M$,

$$\|x(t) - \hat{x}(t)\| \leq M \|x(0) - \hat{x}(0)\| \exp((-\alpha + l_N)t)),$$

Therefore, the estimation error converges exponentially to zero if $g < \frac{1-4\beta_a l_N}{4\beta_a \omega}$.

4. Simulation Result

In order to test the performance of the proposed observers, the equations have been integrated by using a backward finite difference approximation for the first-order space derivative, with the following set of parameter values (see [3,4]):

Table 1. Process parameters for numerical simulations

Process parameters	Numerical values
β_a	0.167
υ	0.1 $m.s^{-1}$
L	1 m
δ	0.25
E	11.250 $cal.mol^{-1}$
k_0	10^6 s^{-1}
γ	0.2 s^{-1}
C_{in}	0.02 $mol.L^{-1}$
R	1.986 $cal.mol.L^{-1}$
T_{in}	340 K

The measurements are taken on the length interval $[3*L/4, L]$ i.e., $\omega = 3*L/4$, and the process model has been arbitrary initialized with the constant profiles $x_1(0,z) = 1, x_2(0,z) = 0, \hat{x}_1(0,z) = 0$, and $\hat{x}_2(0,z) = 1$. In order to response to the assumptions of the Theorem 3.3.1 we set $g = \frac{1-4\beta_a l_N}{2*4\beta_a \omega}$ for the observer design parameter.

Figure 1 and Figure 2 shows respectively the time evolution of the temperature and concentration errors $x_1 - \hat{x}_1$ and $x_1 - \hat{x}_2$ related to the exponential observer (18)-(22).

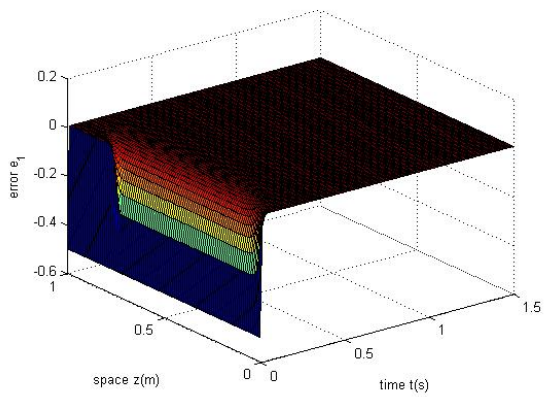

Figure 1. Error evolution of the temperature

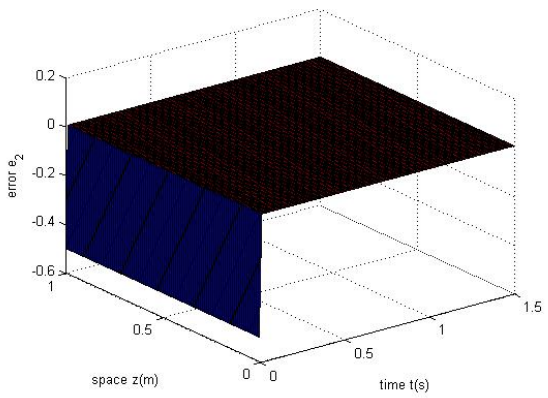

Figure 2. Error evolution of the product concentration

5. Conclusions and Prospects

In this work, we present an exponential observer to estimate the state variables initially unknown of a class of tubular reactor nonlinear models, namely exothermal axial dispersion tubular reactors involving sequential reactions for which the kinetics depends on temperature and reactant concentration. The given state estimator is based on measurements of the temperature at the reactor output only, and performed by a simulation study in which the parameters can be tuned by the user to satisfy specific needs in terms of convergence rate. It is shown in the simulations that the proposed observer is effective and satisfactory since it answers to difficulties of the reactant concentration measurements for a wide range of (bio)-chemical reactors.

An interesting topic for future research is to obtain a practicable feedback control (which is composed of original system and the observer) that stabilizes this class of non-isothermal tubular reactors around an optimal product concentration.

Acknowledgements

This paper presents research results of the Moroccan “Programme Thématique d’Appui à la Recherche Scientifique” PROTARS III, initiated by the Moroccan “Centre National de la Recherche Scientifique et Technique” (CNRST). The work is also supported by the Belgian Program on Interuniversity Poles of Attraction (PAI).

References

[1] D. Dochain, "Contribution to the analysis and control of distributed parameter systems with application to (bio)chemical processes and robotics". Ph. D. Thesis, Université Catholique de Louvain, Belgium 1994.

[2] C. Antoniades, P. D. Christofides, "Studies on nonlinear dynamics and control of tubular reactor with recycle", Nonlinear Anal 47, 5933-5944, 2001.

[3] I.Y. Smets, D. Dochain, J.F. Van Impe, "Optimal Temperature Control of a Steady-State Exothermic Plug-Flow Reactor", AIChE Journal, Vol. 48, No. 2, 279-286, 2002.

[4] Y. Orlov, D. Dochain, "Discontinuous Feedback Stabilization of Minimum-Phase Semi linear Infinite-Dimensional Systems With Application to Chemical Tubular Reactor Models", IEEE Trans. Aut. Contr, vol. 47, 1293-1304, 2002.

[5] D., Dochain, M. Perrier and B.E. Ydstie, "Asymptotic observers for stirred tank reactors". Chem. Eng. Sci., 47, 4167-4178, 1992.

[6] M. Perrier, S. Feyo de Azevedo, E.C. Ferreira and D. Dochain, "Tuning of observer-based estimators: theory and application to the on-line estimation of kinetic parameters", Control Eng. Practice, 8, 377-388, 2000.

[7] R. Miranda, I. Chairez, and J. Moreno, “Observer design for a class of parabolic PDE via sliding modes and backstepping”, in Proc. Int. Workshop Var. Struc. Syst., Mexico City, Mexico, pp. 215-220, Jun. 2010.

[8] Z. Hidayat, R. Babuska, B. De Schutter, and A. Nunez, "Observers for linear distributed-parameter systems: A survey", Proceedings of the 2011 IEEE International Symposium on Robotic and Sensors Environments (ROSE 2011), Montreal, Canada, pp. 166-171, Sept. 2011.

[9] M. Laabissi, M.E. Achhab, J. Winkin, D. Dochain, "Trajectory analysis of non-isothermal tubular reactor nonlinear models", Systems & Control Letters, 42, pp. 169-184, 2001.

[10] R.H, Martin, "Nonlinear operators and differential equations in Banach spaces", John Wiley and Sons, 1976.

[11] Pazy, A., "Semigroups of linear operators and applications to partial differential equations", Springer-Verlag, New York, 1983.

[12] Curtain, R.F., J. Zwart, "An Introduction to Infinite Dimensional Linear Systems Theory", Springer. New York. 1995.

[13] J. Winkin, D. Dochain, P. Ligarius, "Dynamical Analysis of Distributed Parameter Tubular Reactors", Automatica 36, 349-361, 2000.

[14] R., NAGEL, "One-Parameter Semigroups of Positive Operators, Lecture Notes in Mathematics", vol. 1184, Springer, New York, 1986.