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Magneto-Solid Mechanics and the Aleph Tensor

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Abstract In Reference [1], a new master system was formulated allowing to the analyze polarizable and/or magnetizable solids. The central distinction of the suggested approach consists in the systematic usage of the Cardinal Aleph tensor. By choosing different thermodynamic potentials, the suggested approach can be recommended for the analysis of a wide variety of static or dynamic engineering systems. Given the variety of possible applications, the system is relatively simple and can be analyzed not only computationally but also analytically. However, in order to make the analytical and computational results simpler and more transparent it makes sense to adjust the general system for different applications. Using our approach, we formulate the exact nonlinear system for dynamics of magnetizable fluid and solids. Then, we apply our exact nonlinear general system to ferrofluids and its linearized version to piezo-magnetic solids.

Keywords: *magnetizable substances, the Cardinal tensors, partial differential equations*

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1. Introduction

Many dynamic applications of piezo-electric and piezo-magnetic substances require modeling, which rigorously obey the basic principles of Newtonian mechanics and thermodynamics laws. This sort of modeling is required, for instance, when dealing with high-rate phenomena and, especially, shock-waves analysis. In the paper, we suggested such models for magneto-elastic substances. The most difficult in this analysis are not mathematical or computational difficulties, for which different models have been suggested over the 19th and 20th centuries. The most difficult are the obstacles implied by total misunderstanding of the fundamentals like energy, entropy, stresses. John von Neumann justly claimed that “Nobody knows what entropy really is” whereas Richard Feynman wrote “It is important to realize that in physics today, we have no knowledge what energy is.” Because of that we destined to build our theories on the more or less intuitive basic concepts. Of course, these concepts are not arbitrary, but they in no way can be treated as ultimate truth. More often, than not, we choose the fundamentals based on the limited applied targets, and deliberately sacrifice the universality for the sake of simplicity. In our opinion though, it is important to present the basic assumptions in the compact, observable, and clear form, preferably in mathematical form. For the studies of polarizable or magnetizable fluids or solids we suggested to use as the central concept the Cardinal tensor, described in [1]. In [1], we did so for the static problems for polarizable solids. In this paper, we generalize that approach for dynamic problems of magnetizable fluids and solids.

There are plenty of applications of magnetomechanics in general engineering. mechanics and physics, and in the defense related applications, in particular. Interested readers are referred to the world known manuals [2,3,6], as well as to the monographs [4,5,7,8,9,10], among many others. The classical medium level manual for theoretical physicists is [3]. One more important feature of [3] is the permanent emphasis of the crucial role of thermodynamics. The reader of this paper is referred to [3] if he or she is interested in the thermodynamics extensions.

In this paper, we touch thermodynamics of magnetization or polarization quite superficially. We are talking about the internal energy density ψ per unit mass of the substance, which is treated as a function of the polarization density $\vec{P}$ or magnetization density $\vec{M}$, and of the so called actual metrics X_{ab}. At the same time we ignore the thermal arguments like entropy density η or the absolute temperature T. Nonetheless, our master systems are, more or less, directly applicable to the adiabatic and isothermal cases. In the adiabatic case (in the absence of shock waves) the entropy density remains fixed pointwise, in the isothermal case the absolute temperature remains to be a fixed constant. Therefore, in the first case, the function ψ is just the internal energy density $e(X_{ab},P^a,\eta)$ or $e(X_{ab},M^a,\eta)$ at fixed η, whereas in the second case ψ is just the free energy density $\psi(X_{ab},P^a,T)$ or $\psi(X_{ab},M^a,T)$ at fixed T.

It is possible to treat simultaneously (with minor distinctions in notation) electric polarization and magnetization because we ignore any macroscopic currents, and in this situation, we can introduce the potential of the magnetic field.

2. The Simplest Master System for Magnetizable and Polarizable Substances

In the publication (Grinfeld M., Grinfeld P., 2019), we formulated a new master system applicable to the analysis of polarizable and/or magnetizable solids. The central distinction of the suggested approach consists in the systematic usage of the Cardinal Aleph tensor. By choosing different thermodynamic potentials, the suggested approach can be recommended for the analysis of a wide variety of static or dynamic engineering systems. Given the variety of possible applications, the system is relatively simple and can be analyzed not only computationally but also analytically. However, in order to make the analytical and computational results simpler and more transparent it makes sense to adjust the general system for different applications. It can be done in different ways.

Below, we present a slightly modified master system of Grinfeld and Grinfeld (2019). The modifications concern two aspects: a) we use the magnetization rather than the polarization terminology and b) we add the inertia terms targeting applications to dynamic problems.

When dealing with anisotropic polarizable substances, it is convenient to use the mixed Eulerian-Lagrangean description of continuum media.

Consider the immobile spatial coordinate system referred to the coordinates z^i (the reference indexes from the middle of the Latin alphabet i, j, k run the values 1, 2, 3) and assume that our space is Euclidean. In this space, we consider a material body B, referred to the material coordinates x^a (the material indexes from the beginning of the Latin alphabet a, b, c run the values 1, 2, 3 as well). We accept the standard concepts of the covariant and contravariant indexes and accept the standard agreement regarding summation over the repeat covariant and contravariant indexes of the same type (i.e., of the reference or material type).

In addition to two different coordinates, we distinguish between two different configurations - the initial and the current configurations of the body. Let the functions $z^i = z^i(x^a, t)$ be the Eulerian coordinates in the current configuration of the material point with the material coordinates x^a at the moment of time t. We use the notation $x^a = x^a(z^i, t)$ for the inverse of the function $z^i(x^a, t)$. Let us use the notation Z_{ij} for deformation-independent metrics of the reference spatial system, and the notation X_{ab} for the deformation-dependent metrics of the actual material configuration. These two metrics are connected by the relationships

$$X_{ab}(x,t) = Z_{ij} z^{i\cdot}_{\cdot a} z^{i\cdot}_{\cdot b}, \; Z_{ij} = X_{ab} x^{a\cdot}_{\cdot i} x^{b\cdot}_{\cdot j} \tag{1.1}$$

where the mixed shift-tensors $z^{i\cdot}_{\cdot a}$ and $x^{a\cdot}_{\cdot i}$ are defined as

$$z^{i\cdot}_{\cdot a} \equiv \frac{\partial z^i(x,t)}{\partial x^a}, \; x^{a\cdot}_{\cdot i} \equiv \frac{\partial x^a(z,t)}{\partial z^i} \tag{1.2}$$

The reference and the coordinate configurations are characterized by the current covariant bases $\boldsymbol{Z}_i(z)$ and contravariant bases and $\boldsymbol{X}_a(x,t)$, respectively.

We use the standard notation ∇_i and ∇_a for the reference and material contravariant differentiation in the metrics of the actual configuration.

Magnetization is a vector quantity. A distributed magnetization field is characterized by the density per unit mass $\boldsymbol{M}$ or per unit volume $\rho\boldsymbol{M}$, where ρ is the mass density. Vector $\boldsymbol{M}$ and can be decomposed with respect to the material basis $\boldsymbol{M} = M^a \boldsymbol{X}_a$ or the spatial basis. By definition, in vacuum, the magnetization vector $\boldsymbol{M}$ is equal to zero. Thus, it experiences a discontinuity jump across the body's boundary Σ.

The bulk energy density ψ per unit mass is given as a function of the actual material metrics X_{ab}, the Lagrangean components M^a of the magnetization vector per unit mass, and fixed material constants or tensors, which we do not mention explicitly:

$$\psi = \psi(X_{ab}, M^a) \tag{1.3}$$

The magnetoelastic Aleph tensor $\aleph^{ij}$ is defined as follows [1,11]

$$\aleph^{ij} \equiv 2\rho \frac{\partial \psi}{\partial X_{(cd)}} z^{i\cdot}_{\cdot c} z^{j\cdot}_{\cdot d} - \frac{1}{8\pi} H_k H^k Z^{ij} + \frac{1}{4\pi} H^i H^j \tag{1.4}$$

where H^i are the Eulerian component of the magnetic field.

The bulk dynamics equation reads

$$\rho\left(\frac{\partial V^i}{\partial t} + V^j \nabla_j V^i\right) = \nabla_j \aleph^{ij} \tag{1.5}$$

where V^i are the Eulerian components of the velocity field and ρ is the mass density.

The velocity field V^i is defined as

$$V^i(x,t) \equiv \frac{\partial z^i(x,t)}{\partial t} \tag{1.6}$$

We can also consider the velocity components as function of the Eulerian coordinates z^i: we will use the notation $V^i(z,t)$ for this function. The functions $V^i(x,t)$ and $V^i(z,t)$ are different function. This should not create any confusion even if we do not show the arguments explicitly - which of the two functions is meant should be clear from the context; for instance, in the equations (1.5) we mean $V^i = V^i(z,t)$.

The momentum condition at the boundary with vacuum reads

$$\left(\begin{array}{l} 2\rho \dfrac{\partial \psi}{\partial X_{(cd)}} z^{i.}_{.c} z^{j.}_{.d} + \\ \dfrac{1}{4\pi} H^i H^j - \dfrac{1}{8\pi} H_k H^k Z^{ij} \end{array}\right)_{sub} N_j = \tag{1.7}$$

$$\left(\frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij}\right)_{vac} N_j$$

The relationships (1.3) - (1.6) should be amended with the magnetostatics bulk equations and boundary conditions

$$H_i = -\frac{\partial \varphi(z,t)}{\partial z^i} \tag{1.8}$$

$$\nabla_i B^i = 0 \tag{1.9}$$

with the magnetic induction B^i defined as

$$B^i \equiv H^i + 4\pi\rho M^i \tag{1.10}$$

The equation (1.8) reflects the fact that in the absence of macroscopic electric current the magnetic field is irrotational. The equation (1.9) reflects the fact that the magnetic induction is always divergence-free.

At the interfaces, the fields $\varphi(z,t)$, $H_i(z,t)$, and $B^i(z,t)$ and/or their derivatives experience finite jumps. Those jumps ae not arbitrary but satisfy the boundary constraints of magnetostatics

$$[\varphi]_-^+ = 0 \tag{1.11}$$

and

$$\left[B^i\right]_-^+ N_i = 0 \tag{1.12}$$

The bulk equations (1.3) - (1.6), (1.8) - (1.10) should be amended with the following thermodynamics prompted relationship

$$H_a = \rho \frac{\partial \psi(X_{ab}, M^a)}{\partial M^a} \tag{1.13}$$

In order to get the mathematically closed master system, the relationships (1.1) - (1.13) should be amended with the initial conditions and conditions at infinity.

We notice, that the functions $\rho(z,t)$ and $V^i(z,t)$ satisfy the classical mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho V^i) = 0 \tag{1.14}$$

In hydrodynamics, the equation (1.14) is used explicitly. However, in mechanics of solids there is no need to use (1.14).

We proceed, inserting (1.10) in (1.9); then, we get

$$\nabla_i (H^i + 4\pi\rho M^i) = 0 \tag{1.15}$$

Also, using (1.8), we can rewrite (1.11) as follows

$$\nabla_i (-\nabla^i \varphi + 4\pi\rho M^i) = 0 \tag{1.16}$$

Combining the thermodynamic identity (1.13) with the magnetostatics relationship (1.8), we get

$$\rho \frac{\partial \psi(X_{ab}, M^a)}{\partial M^a} x^a_{.i} = -\nabla_i \varphi. \tag{1.17}$$

3. Model of Magnetizable Fluid

Fluid is a special case of solid deformable function. Often, fluids are described with the mathematical systems that are considerably simpler than the general master systems for solids. Sometimes, but not always, the simplification are possible for polarizable fluids. Let us consider a special model of the substance described by the following energy density ψ

$$\psi(X_{ab}, M^a) = \Psi(\rho) + \frac{\alpha}{2} X_{ab} M^a M^b, \tag{2.1}$$

where α is a positive constant.

In the energy density (2.1), the dependence upon the actual metrics X_{ab} enters not only explicitly through the term $X_{ab} M^a M^b$, but also implicitly through the density term $\Psi(\rho)$. Namely, we get

$$\frac{\partial \ln \rho^2}{\partial X_{ab}} = -X^{ab}. \tag{2.2}$$

Using (2.1), (2.2), we get ψ

$$\psi(X_{ab}, M^a) = \Psi(\rho(X_{ab})) + \frac{\alpha}{2} X_{ab} M^a M^b, \tag{2.3}$$

and we arrive at the following formula:

$$\frac{\partial \psi}{\partial X_{(cd)}} z^{i.}_{.c} z^{j.}_{.d} = -\frac{1}{2}\rho \Psi_\rho Z^{ij} + \frac{\alpha}{2} M^i M^j \tag{2.4}$$

as implied by the chain:

$$\frac{\partial \psi}{\partial X_{(cd)}} z^{i.}_{.c} z^{j.}_{.d} = \left(-\frac{1}{2}\rho \Psi_\rho X^{cd} + \frac{\alpha}{2} M^c M^d\right) z^{i.}_{.c} z^{j.}_{.d}$$
$$= -\frac{1}{2}\rho \Psi_\rho Z^{ij} + \frac{\alpha}{2} M^i M^j.$$

Inserting (2.3) in the definition (1.4) of the Aleph tensor, we get

$$\begin{array}{c} \aleph^{ij} \equiv -\rho^2 \Psi_\rho Z^{ij} + \alpha\rho M^i M^j - \\ \dfrac{1}{8\pi} H_k H^k Z^{ij} + \dfrac{1}{4\pi} H^i H^j \end{array} \tag{2.5}$$

Using (2.4), we can rewrite the bulk momentum equation (1.5) as follows

$$\begin{array}{l} \rho\left(\dfrac{\partial V^i}{\partial t} + V^j \nabla_j V^i\right) \\ = \nabla_j \left(\begin{array}{l} -\rho^2 \Psi_\rho Z^{ij} + \alpha\rho M^i M^j \\ -\dfrac{1}{8\pi} H_k H^k Z^{ij} + \dfrac{1}{4\pi} H^i H^j \end{array}\right). \end{array} \tag{2.6}$$

The bulk thermodynamics equation (1.13) for the model (2.3) implies

$$H_i = \alpha M_i. \tag{2.7}$$

Using (2.7), we get the following relationship for the magnetic induction

$$B_i = \frac{\alpha + 4\pi}{\alpha} H_i. \tag{2.8}$$

Combining the bulk magnetostatic equation (1.9) with (2.8), we get

$$\nabla_i H^i = 0. \tag{2.9}$$

Inserting (1.8) in (2.9), we arrive at the Laplace equation

$$\nabla_i \nabla^i \varphi = 0. \tag{2.10}$$

In the case of a magnetofluid media, the general momentum boundary condition (1.7) at the boundary with vacuum reads

$$\begin{gathered}\left(\begin{matrix} -\rho^2 \Psi_\rho Z^{ij} + \alpha\rho M^i M^j + \\ \frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij} \end{matrix} \right)_{fluid} N_j \\ = \left(\frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij} \right)_{vac} N_j. \end{gathered} \tag{2.11}$$

The magnetostatics boundary conditions imply

$$[\varphi]_-^+ = 0 \tag{2.12}$$

and

$$\frac{4\pi + \alpha}{\alpha} \nabla_i \varphi_{fluid} N^i = \nabla_i \varphi_{vac} N^i. \tag{2.13}$$

The system (2.6) - (2.13) should be amended with the mass conservation equation (2.14). Thus, in the case under study we eliminated the explicit use of the equations $x^a = x^a(z^i, t)$: this fact significantly simplifies the general master system form magnetizable solids.

4. The Linearized Master System for Deformable Magnetizable Solids

We choose an affine reference coordinate system with the time independent metrics Z_{ij}, Z^{ij}. Consider a uniform configuration with the uniform and time-independent shift tensors $x_{.i}^{a} = x_{.i}^{a\circ}$, $z_{.a}^{i.} = z_{.a}^{i.\circ}$, the uniform and time-independent metrics $X_{ab} = X_{ab}^{\circ}$, and $X^{ab} = X^{ab\circ}$, the vanishing fields $H^{\circ i} = M^{\circ a} = B^{\circ a} = V^{i\circ} = \varphi^{\circ} = 0$.

Let $\tilde{x}_{.i}^{a}$, $\tilde{z}_{.a}^{i.}$, $\tilde{X}_{ab}$, $\tilde{X}^{ab}$, $\tilde{H}^i$, $\tilde{M}^a$, $\tilde{B}^a$, $\tilde{V}^i$, $\tilde{\varphi}$ be the small time- and coordinate-dependent perturbations of the equilibrium fields. Let us establish the linearized master system for the perturbations.

To within the first order terms the relationship (1.4) implies

$$\tilde{\aleph}^{ij} \equiv 2\rho^{\circ} \left(\begin{matrix} \frac{\partial^2 \psi^{\circ}}{\partial X_{(ab)} \partial X_{(cd)}} \tilde{X}_{(ab)} + \\ \frac{\partial^2 \psi^{\circ}}{\partial X_{(cd)} \partial M^a} \tilde{M}^a \end{matrix} \right) z_{.c}^{i.\circ} z_{.d}^{j.\circ} \tag{3.1}$$

Differentiating (3.1), we get

$$\frac{1}{2\rho^{\circ}} \frac{\partial \tilde{\aleph}^{ij}}{\partial t} \equiv z_{.c}^{i.\circ} z_{.d}^{j.\circ} \left(\begin{matrix} \frac{\partial^2 \psi^{\circ}}{\partial X_{(ab)} \partial X_{(cd)}} (\nabla_a \tilde{V}_b + \nabla_b \tilde{V}_a) \\ + \frac{\partial^2 \psi^{\circ}}{\partial X_{(cd)} \partial M^a} \frac{\partial \tilde{M}^a}{\partial t} \end{matrix} \right) \tag{3.2}$$

where we used the relationship

$$\frac{\partial X_{ab}(x,t)}{\partial t} = \nabla_a V_b + \nabla_b V_a.$$

We can now rewrite (3.2) as follows

$$\frac{\partial \tilde{\aleph}^{ij}}{\partial t} \equiv \left(\begin{matrix} C^{abcd} z_{.a}^{k.\circ} z_{.b}^{l.\circ} z_{.c}^{i.\circ} z_{.d}^{j.\circ} \nabla_{(k} \tilde{V}_{l)} \\ + C_a^{cd} z_{.c}^{i.\circ} z_{.d}^{j.\circ} \frac{\partial \tilde{M}^a}{\partial t} \end{matrix} \right) \tag{3.3}$$

In (3.3) and below we use the following notation:

$$\begin{gathered} C^{abcd} \equiv 4\rho^{\circ} \left. \frac{\partial^2 \psi^{\circ}(X_{ab}, M^a)}{\partial X_{(ab)} \partial X_{(cd)}} \right|_{\substack{X_{cd} = X_{cd}^{\circ} \\ M^a = 0}}, \\ C_a^{cd} \equiv 2\rho^{\circ} \left. \frac{\partial^2 \psi^{\circ}(X_{ab}, M^a)}{\partial X_{(cd)} \partial M^a} \right|_{\substack{X_{cd} = X_{cd}^{\circ} \\ M^a = 0}}, \\ C_{ab} \equiv \rho^{\circ} \left. \frac{\partial^2 \psi(X_{ab}, M^a)}{\partial M^a \partial M^b} \right|_{\substack{X_{ab} = X_{ab}^{\circ} \\ M^a = 0}} \end{gathered} \tag{3.4}$$

The linearized bulk dynamics equation reads

$$\rho^{\circ} \frac{\partial \tilde{V}^i}{\partial t} = \nabla_j \tilde{\aleph}^{ij}. \tag{3.5}$$

Differentiating (3.5) with respect to t and using (3.3), we arrive at the linearized momentum bulk equation

$$\rho^{\circ} \frac{\partial^2 \tilde{V}^i}{\partial t^2} = \nabla_j \left(\begin{matrix} C^{abcd} z_{.a}^{k.\circ} z_{.b}^{l.\circ} z_{.c}^{i.\circ} z_{.d}^{j.\circ} \nabla_{(k} \tilde{V}_{l)} \\ + C_a^{cd} z_{.c}^{i.\circ} z_{.d}^{j.\circ} \frac{\partial \tilde{M}^a}{\partial t} \end{matrix} \right) \tag{3.6}$$

Linearizing the magnetostatics equation (1.16), we get

$$\nabla_i \nabla^i \tilde{\varphi} = 4\pi \nabla_i \left(\rho^{\circ} z_{.a}^{i.\circ} \tilde{M}^a \right). \tag{3.7}$$

Linearizing (1.17), we get eventually

$$-\nabla_i \frac{\partial \tilde{\varphi}}{\partial t} = C_{ab} x_{.i}^{a\circ} \frac{\partial \tilde{M}^b}{\partial t} + C_a^{bc} x_{.i}^{a\circ} z_{.b}^{k.\circ} z_{.c}^{j.\circ} \nabla_k \tilde{V}_j. \tag{3.8}$$

In order to establish (3.8) we first get, using equation (1.17)

$$-\frac{1}{\rho^\circ}\nabla_i\frac{\partial\tilde{\varphi}}{\partial t}=\begin{pmatrix}\frac{\partial^2\psi(X_{ab},M^a)}{\partial M^a\partial M^b}\frac{\partial M^b}{\partial t}\\+\frac{\partial^2\psi(X_{ab},M^a)}{\partial M^a\partial X_{(bc)}}(\nabla_b V_c+\nabla_c V_b)\end{pmatrix}x^{a\circ}_{.i} \quad (3.9)$$

and then, the symmetry, we rewrite (3.9) as

$$-\frac{1}{\rho^\circ}\nabla_i\frac{\partial\tilde{\varphi}}{\partial t}=\begin{pmatrix}\frac{\partial^2\psi(X_{ab},M^a)}{\partial M^a\partial M^b}\frac{\partial M^b}{\partial t}\\+2\frac{\partial^2\psi(X_{ab},M^a)}{\partial M^a\partial X_{(bc)}}\nabla_b\tilde{V}_c\end{pmatrix}x^{a\circ}_{.i} \quad (3.10)$$

At last, using the definitions (3.4), we rewrite (3.10) as (3.8).

Under the assumptions regarding the ground configuration, the linearized momentum boundary condition (1.7) implies

$$\begin{pmatrix}C^{abcd}z^{k.\circ}_{.a}z^{l.\circ}_{.b}z^{i.\circ}_{.c}z^{j.\circ}_{.d}\nabla_{(k}\tilde{V}_{l)}+\\C^{cd}_{a}z^{i.\circ}_{.c}z^{j.\circ}_{.d}\frac{\partial\tilde{M}^a}{\partial t}\end{pmatrix}_{sub}N_j=0 \quad (3.11)$$

whereas the linearized boundary conditions (1.11), (1.12) read

$$[\tilde{\varphi}]^+_-=0 \quad (3.12)$$

and

$$\left[\nabla^i\tilde{\varphi}-4\pi\rho^\circ z^{i.\circ}_{.a}\tilde{M}^a\right]^+_- N_i=0 \quad (3.13)$$

respectively.

5. Magnetoelastic Bulk Waves

Let us rewrite the system (3.6) - (3.8) as follows:

$$\rho^\circ\frac{\partial^2\tilde{V}^i}{\partial t^2}=\nabla_j\left(C^{ijkl}\nabla_{(k}\tilde{V}_{l)}+C^{ij}_{k}\frac{\partial\tilde{M}^k}{\partial t}\right) \quad (4.1)$$

$$\nabla_i\nabla^i\tilde{\varphi}=4\pi\nabla_i(\rho^\circ\tilde{M}^i) \quad (4.2)$$

$$C_{ki}\frac{\partial\tilde{M}^i}{\partial t}+C^{ij}_{k}\nabla_i\tilde{V}_j=-\nabla_k\frac{\partial\tilde{\varphi}}{\partial t} \quad (4.3)$$

where the magnetoelastic modules with the spatial indices are defined as follows:

$$\begin{aligned}C^{ijkl}&\equiv C^{abcd}z^{k.\circ}_{.a}z^{l.\circ}_{.b}z^{i.\circ}_{.c}z^{j.\circ}_{.d},\\C^{ij}_{k}&\equiv C^{cd}_{a}z^{i.\circ}_{.c}z^{j.\circ}_{.d}x^{a.\circ}_{.k},\\C_{ij}&\equiv C_{ab}x^{a\circ}_{.i}x^{b\circ}_{.j}\end{aligned} \quad (4.4)$$

Consider the following solutions of the bulk system (4.1) - (4.3):

$$\begin{bmatrix}\tilde{\varphi}\\\tilde{V}_m\\\tilde{M}^m\end{bmatrix}=\begin{bmatrix}\Phi\\W_m\\\mathrm{M}^m\end{bmatrix}e^{i(\omega t+k_i z^i)} \quad (4.5)$$

Inserting (4.5) in the equations (4.1) - (4.3), we arrive at the system of linear algebraic equations

$$(\rho^\circ\omega^2 z^{il}-C^{ijkl}k_jk_k)W_l=k_j\omega C^{ij}_{k}\mathrm{M}^k \quad (4.6)$$

$$-k_ik^i\Phi=4\pi ik_i\rho^\circ\mathrm{M}^i \quad (4.7)$$

$$i\omega C_{ki}\mathrm{M}^i+iC^{ij}_{k}k_iW_j=k_k\omega\Phi. \quad (4.8)$$

Excluding Φ between the equations (4.7), (4.8), we get

$$\omega\left(C_{ki}+4\pi\frac{k_ik_k}{|k|^2}\rho^\circ\right)\mathrm{M}^i+C^{ij}_{k}k_iW_j=0 \quad (4.8)$$

as implied by the chain:

$$\begin{aligned}&i\omega C_{ki}\mathrm{M}^i+iC^{ij}_{k}k_iW_j=k_k\omega\Phi,\\&\Phi=-4\pi i\frac{k_i}{|k|^2}\rho^\circ\mathrm{M}^i\rightarrow\\&\left(\omega C_{ki}+\omega 4\pi\frac{k_ik_k}{|k|^2}\rho^\circ\right)\mathrm{M}^i+C^{ij}_{k}k_iW_j=0.\end{aligned}$$

Let us introduce the following vectors

$$m_i\equiv\frac{1}{\omega}k_i,\quad \Delta_i\equiv\frac{k_i}{|k|}. \quad (4.9)$$

The vector m_i is not necessarily real. Obviously, the vector Δ_i is real and normalized:

$$|\Delta_i|=1. \quad (4.10)$$

Then, we can rewrite the equations (4.6), (4.8) as follows

$$(\rho^\circ z^{il}-C^{ijkl}m_jm_k)W_l=m_jC^{ij}_{k}\mathrm{M}^k \quad (4.11)$$

$$(C_{ki}+4\pi\rho^\circ\Delta_i\Delta_k)\mathrm{M}^i+m_iC^{ij}_{k}W_j=0. \quad (4.12)$$

Let us consider a special case, when the tensor C_{ki} has the form

$$C_{ki}=\gamma\rho^\circ z_{ki}. \quad (4.13)$$

Now, we can rewrite equation (4.12) as follows

$$(\gamma\rho^\circ z_{ki}+4\pi\rho^\circ\Delta_i\Delta_k)\mathrm{M}^i+m_iC^{ij}_{k}W_j=0. \quad (4.14)$$

Resolving (4.14) with respect to M^i, we get

$$\mathrm{M}^k=\frac{1}{\gamma\rho^\circ}\Lambda^{kl}m_iC^{ij}_{l}W_j \quad (4.15)$$

where Λ^{kl} is defined as

$$\Lambda^{kl}\equiv\frac{4\pi\Delta^k\Delta^l-(\gamma+4\pi)z^{kl}}{\gamma(\gamma+4\pi)}$$

Indeed, contracting (4.14), with Δ^k, we get

$$\Delta_i \mathrm{M}^i = -m_i \Delta^k \frac{C_k^{ij}}{(\gamma + 4\pi)\rho^\circ} W_j. \quad (4.16)$$

Now, combining (4.16) with (4.14), we get (4.15).

Eliminating M^k between the equations (4.11), (4.16), we get

$$\left(\rho^\circ z^{in} - C^{ijkn} m_j m_k\right) W_n = m_j C_k^{ij} \frac{1}{\gamma\rho^\circ} \Lambda^{kl} m_m C_l^{mn} W_n \quad (4.17)$$

or

$$\begin{pmatrix} \rho^\circ z^{in} - C^{ijmn} m_j m_m \\ -\frac{1}{\gamma\rho^\circ} \Lambda^{kl} C_k^{ij} C_l^{mn} m_j m_m \end{pmatrix} W_n = 0 \quad (4.18)$$

or else

$$\left\{ z^{in} - m_j m_m (c^{ijmn} + \Lambda^{kl} c_k^{ij} c_l^{mn}) \right\} W_n = 0 \quad (4.19)$$

where

$$c^{ijmn} \equiv \frac{1}{\rho^\circ} C^{ijmn}, \; c_k^{ij} \equiv \frac{1}{\rho^\circ} C_k^{ij}. \quad (4.20)$$

6. Conclusion

Many dynamic applications of piezo-electric and piezo-magnetic substances require modeling, that rigorously obey the basic principles of Newtonian mechanics and thermodynamics laws. This sort of modeling is required, for instance, when dealing with high-rate phenomena and, especially, shock-waves analysis. In the paper, we suggested such models for magneto-elastic substances.

In the section "The simplest master systems for magnetizable and polarizable substances" of this paper we postulated the closed master system that allows one to model magnetoelastic or electroelastic systems without any assumptions of smallness of deformations and electromagnetic fields. The central element of our model is the Cardinal Aleph tensor which has some common features with the stress tensor of the classical theory of elasticity and the Maxwell tensor of electromagnetic stresses. For the substances of any crystallographic symmetry the Cardinal tensors appear to be symmetric. In the section "Model of magnetizable fluid" we specify and simplify our general master system for the case of ferrofluid. In the section "The linearized master system for deformable magnetizable solids", we specify our general system for the case of small magnetic fields and deformation, thus, reducing the general nonlinear system to the linear one. At last, in the section "Magnetoelastic bulk waves" we provide a novel analysis of the linear piezomagnetic waves in solids.

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