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Analytical Computation of the Caldirola-Kanai Oscillator Parameters by the Dynamic Invariant Method

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Abstract The method of the invariant dynamical linear operator is very simple and may be useful to solve the Schrödinger equation, in particular in the case of the problem of a harmonic oscillator with a time dependent mass and frequency. Indeed, we have successfully used this approach to the Caldirola-Kanai oscillator. In particular, we have obtained explicit expressions of the uncertainty product and the quantum correlation coefficient. The results obtained are in good agreement with those of the literature.

Keywords: *dynamical invariant method, harmonic quantum oscillator, quantum correlation coefficient, Heisenberg product of uncertainty*

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1. Introduction

The harmonic oscillator is a very importantproblem in quantum mechanics, because it is one of the few problems that can be solved and that has an exact analytical solution. Furthermore, the harmonic oscillator solution and algebraic formalism haveseveral applications in modern physics (e.g. quantum electrodynamics, quantum field theory, quantum optics, etc.) [1,2,3,4]. This is of great importance because it means we can produce analytical solutions to systems that are not quite as idealized as the simple linear harmonic oscillator. This fact gave rise to the development of many approximate methods to find the exact quantum states for time-dependent quantum systems [5-10,23,24]. One of the methods which was developed several decades ago by Lewis and Riesenfeld [11,12,22,25] introduces an invariant operator, known as either the Lewis-Riesenfeld invariant or the generalized invariant, for a time-dependent quantum harmonic oscillator and enables the determination of the exact quantum states of this system in terms of the eigenstates of the invariant operator.

In this work we apply the method of dynamic linear invariant for solving the problem of the harmonic oscillator with time-dependent mass and frequency. The computation of the product of the uncertainties in position and momentum as well as the correlation coefficient were carried out. Thus, the Heisenberg uncertainty principle is well verified all times [26-30]. This remains true even when time becomes infinite; the product of the uncertainties reaches a constant value. We also compared our results with those published in the literature [27,28,29].

2. The Method of Dynamic Linear Invariant

We consider a harmonic oscillator with time-dependent parameters described by the following Hamiltonian:

$$H(t)=P^2/2M(t)+\frac{M(t)\omega^2(t)}{2}q^2 + y(t)/2(pq-qp)-F(t)q \tag{1}$$

where the momentum p and the position q are canonical conjugates, M(t) is the time dependent mass, ω(t) is the frequency, y(t) is an arbitrary parameter of coupling between the momentum and the position and F(t) is an external force.

The time evolution of the system is described by the following Schrödinger equation:

$$\left[\frac{-\hbar^2}{2M(t)}\frac{\partial}{\partial q^2}+\frac{1}{2}M(t)q^2 - \frac{i\hbar y(t)}{2}-i\hbar y(t)q\frac{\partial}{\partial q}-F(t)q\right]\psi(q,t)=i\hbar\frac{\partial\psi(q,t)}{\partial t}. \tag{2}$$

In order to study the quantum motion of this type of system, we will solve the Schrödinger equation using the method of dynamic linear invariant.

2.1. Principle and Approach

The method of dynamic linear invariant was developed by Lewis and Riesenfeld [4]. It can be applied to solve the time-dependent Schrodinger equation. According to the

Lewis-Riesenfeld invariant theory [4], a Hermitian operator I(t) is called invariant if it satisfies the following invariant equation:

$$\frac{dI}{dt}=\frac{1}{i\hbar}[I,H]+\frac{\partial I}{\partial t} \tag{3}$$

then the general solution of the Schrodinger equation (2) can be written as:

$$\psi_\lambda(q,t)=e^{i\mu_\lambda(t)}\Phi_\lambda(q,t) \tag{4}$$

$\Phi_\lambda(q,t)$ is an eigenfunction of the linear invariant I(t)with the eigenvalue λ

$\mu_\lambda(t)$ is a phase function satisfying the following equation

$$\mathrm{i}\frac{\mathrm{d}\mu_\lambda(\mathrm{t})}{\mathrm{dt}}=\left\langle \Phi_\lambda \middle| \mathrm{i}\hbar\frac{\partial}{\partial \mathrm{t}}-\mathrm{H}(\mathrm{t}) \middle| \Phi_\lambda \right\rangle. \tag{5}$$

The invariant operator I(t)is expressed in the following form:

$$I(t)=\ \alpha(t)q+\beta(t)p+\gamma(t) \tag{6}$$

where $\alpha(t)$, $\beta(t) and \gamma(t)$ are time-dependent functions.

We replace I(t) in Equation 3, we obtain first-order differential equations expressed as follow:

$$\dot{\alpha}(t)=-\alpha(t)y(t)+\beta(t)M(t)\omega^2(t) \tag{7a}$$

$$\dot{\beta}(t)=\beta(t)y(t)-\frac{\alpha(t)}{M(t)} \tag{7b}$$

$$\dot{\gamma}(t)=-\beta(t)F(t). \tag{7c}$$

With the help of Eqs. 6 and 7, it is easy to derive the following equations :

$$\ddot{\beta}(t)+\frac{\dot{M}(t)}{M(t)}\dot{\beta}(t)+\Omega^2(t)\beta(t)=0 \tag{8}$$

where

$$\Omega^2(t)=\omega^2-y^2-\frac{\dot{M}(t)}{\dot{M}(t)}y-\dot{y} \tag{9}$$

$$I(t)=\beta(t)p-M(t)\left[\dot{\beta}(t)-\beta(t)\gamma(t)\right]+\gamma(t). \tag{10}$$

The next step is to find the eigenvectors of our system which give a set of solutions of the following equation:

$$I(t)|\Phi_\lambda\rangle=\lambda|\Phi_\lambda\rangle \tag{11}$$

with

$$\langle\Phi_\lambda/\Phi_{\lambda'}\rangle=\delta(\lambda-\lambda'). \tag{12}$$

By simple integration of equation (11), we obtain the function$\phi_\lambda(t)$ in the following form:

$$\Phi_\lambda(q,t)=\sqrt{\frac{1}{2\pi\hbar\beta(t)}}\exp\left\{\begin{matrix}\frac{iM(t)\left[\dot{\beta}(t)-\beta(t)y(t)\right]}{2\hbar\beta(t)}q^2\\ +\frac{i[\lambda-y(t)]}{\beta(t)}q\end{matrix}\right\}. \tag{13}$$

On the other hand, the calculation of the matrix element in equation 5, allows to obtain the following phase function:

$$\mu_\lambda(t)=\frac{1}{2}\int_0^t\frac{\left[\lambda-y(\tau)\right]^2}{M(\tau)\beta(\tau)}d\tau. \tag{14}$$

Therefore, the function$\psi_\lambda(q,t)$ is expressed as:

$$\psi_\lambda(q,t)=\sqrt{\frac{1}{2\pi\hbar\beta(t)}}\exp\left\{\begin{matrix}i\mu_\lambda(t)\\ +\frac{iM(t)\left[\dot{\beta}(t)-\beta(t)y(t)\right]}{2\hbar\beta(t)}q^2\\ +\frac{i[\lambda-y(t)]}{\hbar\beta(t)}q\end{matrix}\right\} \tag{15}$$

Notice that when β(t) vanishes, the phase functions $\mu_\lambda(t)$ diverge. In spite of this divergence, it can easily be shown that the wave function is always finite. Moreover, the general solution of the Schrödinger equation can be written as:

$$\psi(q,t)=\int_{-\infty}^{+\infty}g(\lambda)\psi_\lambda(q,t)d\lambda \tag{16}$$

$g(\lambda)$ is a Gaussian function which determines the state of the system, it is given by:

$$g(\lambda)=\frac{\sqrt{a}}{(2\pi)^{1/2}}\exp\left(-\frac{a^2}{4}\lambda^2\right) \tag{17}$$

with *a* is a real constant.

2.2. Gaussian Wave Packet: Fluctuation and Correlation

We express the wave function ψ_λ as follows:

$$\psi_\lambda=\sqrt{\frac{1}{2\pi\beta(t)}}\exp\left\{\begin{matrix}-\frac{if_1(t)}{2\hbar}\lambda^2+\frac{i}{\beta(t)\hbar}\left[q+\beta(t)f_2(t)\right]\\ +i\left[\begin{matrix}\frac{M(t)\left[\dot{\beta}(t)-\beta(t)y(t)\right]}{2\hbar\beta(t)}q^2\\ -\frac{y(t)}{\hbar\beta(t)}q+f_3(t)\end{matrix}\right]\end{matrix}\right\} \tag{18}$$

with

$$f_1(t)=\int_0^t\frac{d\tau}{M(\tau)\beta^2(\tau)} \tag{18a}$$

$$f_2(t)=\int_0^t\frac{\gamma(\tau)d\tau}{M(\tau)\beta^2(\tau)} \tag{18b}$$

$$f_3(t)=\int_0^t\frac{\gamma^2(\tau)d\tau}{M(\tau)\beta^2(\tau)} \tag{18c}$$

Thus, we obtain the general wave function of the gaussian wave packet as:

$$\psi(q,t)=\left(\frac{2B(t)}{\pi}\right)^{1/4}e^{i\Phi(t)}e^{i\Phi\left(C_1(t)q+C_2(t)\right)^2}\times e^{iB(t)\left(q+\beta(t)f_2\right)^2} \tag{19}$$

where $C_1(t), C_2(t), B(t)$ and $\Phi(t)$ are such as:

$$C_1(t)=\sqrt{\left[\frac{M(t)\left[\dot{\beta}(t)-\beta(t)y(t)\right]}{2\hbar\beta(t)}+\frac{2f_1}{\hbar^3\beta^2a^4\left(1+\frac{4f_1^2}{\hbar^2a^4}\right)}\right]} \quad (19a)$$

$$C_2(t)=\frac{2f_1f_2}{\hbar^3a^4\left(1+\frac{4f_1^2}{\hbar^2a^4}\right)}\frac{1}{C_1(t)}-\frac{\gamma}{2\hbar\beta C_1(t)} \quad (19b)$$

$$\Phi(t)=\frac{2f_1f_2}{\hbar^3a^4\left(1+\frac{4f_1^2}{\hbar^2a^4}\right)}-C_2^2(t)+f_3+\frac{1}{2}\tan^{-1}\frac{2f_1}{\hbar a^2} \quad (19c)$$

$$B(t)=\frac{1}{\beta^2\hbar^2a^2\left(1+\frac{4f_1^2}{\hbar^2a^4}\right)}. \quad (19d)$$

The computed average quadratic deviations in the state $\psi(q,t)$ have the following expressions:

$$\Delta q=\sqrt{\overline{q^2}-\overline{q}^2}=\frac{1}{2\sqrt{B}} \quad (20a)$$

$$\Delta p=\sqrt{\overline{p^2}-\overline{p}^2}=\hbar\sqrt{\frac{B^2+C_1^4}{B}} \quad (20b)$$

Using the definition of the following quantum correlation coefficients

$$C_{1,1}=\frac{1}{2}\left\langle\left\{(q-\overline{q}),(p-\overline{p})\right\}\right\rangle \quad (21)$$

where $\{\ \}$ is an anticommutator.

We succeeded in writing these coefficients in a new form

$$C_{1,1}=\hbar\frac{C_1^2}{2B} \quad (22)$$

The uncertainty relation of Heisenberg can be written as:

$$\Delta q\Delta p=\sqrt{\overline{q^2}-\overline{p}^2}=\frac{\hbar}{2}\sqrt{1+\frac{4C_{1,1}^2}{\hbar^2}}. \quad (23)$$

This last equation suggests that the product of uncertainties is minimal when the quantum correlation coefficient tends to zero. Therefore, to satisfy the Heisenberg uncertainty principle**,** this coefficient has to converge to a constant when time tends to infinity.

3. Application: Study of the Harmonic Oscillator with Time-dependent Mass and Frequency

In this section, we consider an example, which allows us to apply the theoretical approach developed in previous sections; The example we choose is the Caldirola-Kanai (CK) oscillator [4,13] with time-dependent mass and frequency. We apply the method of dynamic linear invariant for studying the (CK) oscillator. The aim is to test our method of resolution in the case of this type of oscillator.

The CK Hamiltonian has the following expression

$$H(t)=\frac{p^2}{2M(t)}+\frac{M(t)\omega^2(t)}{2}q^2. \quad (24)$$

3.1. Solution for the Differential Equation of β(t)

The second-order differential equation obtained in β(t)**(**Eq. 8)is given by

$$\ddot{\beta}(t)+\frac{1}{\omega}\left(\frac{M'}{M}+\frac{\omega'}{\omega}\right)\dot{\beta}(t)\omega^2(t)\beta(t)=0 \quad (25)$$

where M(t) is a time-dependent mass and $\omega(t)$ is a time-dependent frequency.

To solve this equation, we proceed to use the following variable change, which allows a simple computation:

$$Y=\int\omega(t)\,dt \quad (26)$$

with the initial condition

$$Y(0)=0. \quad (27)$$

Substitution of Y(t) of equation (26) into equation (2) yields:

$$\frac{d^2\beta(t)}{dY^2}+\frac{1}{\omega}\left(\frac{M'}{M}+\frac{\omega'}{\omega}\right)\frac{d\beta(t)}{dY}+\beta(t)=0 \quad (28)$$

In order to solve this differential equation and to obtain systematic solutions without making explicit choices of mass or frequency variation, we have considered the simplest situation in which we have considered that the Damping term $\frac{1}{\omega}\left(\frac{M'}{M}+\frac{\omega'}{\omega}\right)$ is a real constantσ such that:

$$\sigma=\frac{1}{2\omega}\left(\frac{M'}{M}+\frac{\omega'}{\omega}\right). \quad (29)$$

Integration of equation (29) gives:

$$M(t)=\frac{\alpha_0\exp(2\sigma\int\omega(t)dt}{\omega(t)}. \quad (30)$$

Thus equations (28) become

$$\beta(t)''+2\sigma\beta(t)'+\beta(t)=0. \quad (31)$$

It is a differential equation which admits three solutions under the three following conditions$|\sigma|<1$, $|\sigma|>1$ or $|\sigma|=1$

Time variation of the mass has been chosen to satisfy the expression M(t) such as:

$$M(t)=m_0e^{2\gamma t} \quad (32)$$

By a simple use of equation (30) we can obtain the frequency function as:

$$\omega(t)=\frac{\omega_0 e^{2\gamma t}}{\frac{\sigma\omega_0}{\gamma}\left(1-e^{2\gamma t}\right)+1}. \quad (33)$$

Now, we use the expressions (32) and (33)of M(t) and ω(t) to calculate the solutions of the second differential equation (31) corresponding to the three domains which the constant σ could respect.

a) $|\boldsymbol{\sigma}| = \mathbf{1}$

In this case, the solution of equation (31) can be written in the form:

$$\beta(t)=\frac{\beta_0}{\delta}e^{-\sigma Y}\left[\frac{\delta-i\sigma}{2}e^{i\delta Y}+\frac{\delta+i\sigma}{2}e^{-i\delta Y}\right] \quad (34)$$

where

$$\delta=\sqrt{1-\sigma^2} \quad (35)$$

Which leads to the function $f_1(t) = k$.

b) $|\boldsymbol{\sigma}| > 1$

The solution $\beta(t)$obtained in this case is the follows:

$$\beta(t)=\frac{\beta_0}{\delta}\left[\frac{\sigma+\delta}{2}e^{\delta Y}+\frac{\delta-\sigma}{2}e^{-\delta Y}\right]. \quad (36)$$

And the function $f_1(t)$ is given as:

$$f_1(t)=\frac{e^{2\gamma t}-1}{2m_0\beta_0^2 e^{2\gamma t}}. \quad (37)$$

c) $|\boldsymbol{\sigma}| < 1$

For the latter case, we determined the solution $\beta(t)$ in the form:

$$\beta(t)=\frac{\beta_0}{\delta}e^{-\sigma Y}\left[\frac{\delta-i\sigma}{2}e^{i\delta Y}+\frac{\delta+i\sigma}{2}e^{-i\delta Y}\right] \quad (38)$$

and the function $f_1(t)$ in this case can be written as:

$$f_1(t)=\frac{e^{-2\gamma t}-1}{2m_0\beta_0^2 e^{-2\gamma t}}. \quad (39)$$

3.2. Quantum Correlation Coefficient

From equations (19a) and (22) we can calculate the quantum correlations coefficients for the different domains of σ:

a) $|\boldsymbol{\sigma}| = \mathbf{1}$

In this case the quantum correlation coefficient obtained is:

$$C_{1,1}=\frac{4k^2}{\hbar^2 a^2}. \quad (40)$$

It is a non-time-dependent coefficient.

b) $|\boldsymbol{\sigma}| > 1$

The quantum correlation coefficient obtained is:

$$C_{1,1}=\frac{m_0 a^2\beta_0^2\omega_0}{2\sigma^3}\left[\frac{\left(1+n^2\right)e^{4\gamma t}-2n^2 e^{2\gamma t}+n^2}{\left[\frac{\sigma\omega_0}{\gamma}\left(1-e^{2\gamma t}\right)+1\right]^2}\right] \quad (41)$$
$$+n\frac{e^{2\gamma t}-1}{e^{2\gamma t}}$$

Notice that this coefficient is expressed in the form of a linear combination of real exponential time functions.

c) $|\boldsymbol{\sigma}| < 1$

For the last case, we obtain:

$$C_{1,1}=\frac{m_0 a^2\beta_0^2\omega_0}{2\sigma^3}\left[\frac{\left(1-n^2\right)e^{4i\gamma t}+2n^2 e^{2i\gamma t}-n^2}{\left[\frac{\sigma\omega_0}{\gamma}\left(1-e^{2i\gamma t}\right)+1\right]^2}\right] \quad (42)$$
$$+n\frac{e^{2i\gamma t}-1}{e^{2i\gamma t}}.$$

We notice that this coefficient is expressed in the form of a linear combination of complex exponential time functions.

3.3. Heisenberg Uncertainty Relation

In this part we will use the expression given in equation (23) to calculate the Heisenberg uncertainty product following the three domains of σ.

a) $|\boldsymbol{\sigma}| = \mathbf{1}$

In this case the Heisenberg product is given as:

$$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{4k^2}{\hbar^2 a^2}} \quad (43)$$

b) $|\boldsymbol{\sigma}| > 1$

We obtained the Heisenberg product as follows:

$$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{\left\{1+\frac{m_0 a^2\beta_0^2\omega_0}{2\sigma^3}\left[\frac{\left(1+n^2\right)e^{4\gamma t}-2n^2 e^{2\gamma t}+n^2}{\left[\frac{\sigma\omega_0}{\gamma}\left(1-e^{2\gamma t}\right)+1\right]^2}\right]+n\frac{e^{2\gamma t}-1}{e^{2\gamma t}}\right\}} \quad (44)$$

where n is a real constant as:

$$n=\frac{1}{\hbar a^2 m_0\beta_0^2} \quad (45)$$

c) $|\boldsymbol{\sigma}| < 1$

In this case we obtain:

$$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{\left\{1+\frac{m_0 a^2\beta_0^2\omega_0}{2\sigma^3}\left[\frac{\left[\left(1-n^2\right)e^{4i\gamma t}+2n^2 e^{2i\gamma t}-n^2\right]}{\left[\frac{\sigma\omega_0}{\gamma}\left(1-e^{2\gamma i t}\right)+1\right]^2}\right]+n\frac{e^{2i\gamma t}-1}{e^{2i\gamma t}}\right\}}. \quad (46)$$

Table 1. Comparative table of uncertainty products

	Qotni et al.	Achkar et al
|σ|>1	$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{m_0a^2\beta_0^2\omega_0}{2\sigma^3}sh\ (h(t))}$	$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{\sigma^2}{4\delta^2S^4}\left(1-sh(S(t)^{2\lambda})\right)^4}$
|σ|=1	$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{4k^2}{\hbar^2a^2}}$	$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{\sigma^2}{4\delta^2}}$
|σ|<1	$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{m_0a^2\beta_0^2\omega_0}{2\sigma^3}\sin\ (h(t))}$	$\Delta q\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{\sigma^2}{4\delta^2S^4}\left(1-\sin(S(t)^{2\lambda})\right)^4}$
with	$h(t)=\frac{\sigma\omega_0}{\gamma}(1-e^{2\gamma t})$	$h(t)=\frac{\sigma\omega_0}{\gamma}(1-e^{2\gamma t})+1$

4. Discussion

The different values of the uncertainty product obtained from the computation of the quantum correlation coefficient for Kanai-Caldirola harmonic oscillator, show that the Heisenberg uncertainty principle is well satisfied for any value of time t, and that this product depends mainly on the value of σ. The uncertainty relations obtained are closely related to the concept of coherent states, in the sense that the product Δq(t)Δp(t) turns out to be not minimum. This result has been interpreted by Pedrosa using the concept of squeezed states [15,17,18]. Moreover, the comparison of our results with those obtained by Achkar et al. [16] allows the usefulness and the efficiency of this method. The approach used by Achkar et al. is based on the Floquet theorem, which transforms the time-dependent Schrödinger equation to the time-independent one, the set of results is shown in Table 1.

The rewriting of our expressions of uncertainty products is intended to make easier the comparison of the two results. Thus, these expressions have the same form as that obtained by Achkar et al. for each domain of σ.

5. Conclusion

In this paper, we presented a new theoretical approach for a direct resolution of this problem of the quantum harmonic oscillator with time-dependent mass and frequency using a simple analytical approach based on the choice of a suitable variable change giving a symmetrical shape to the equation (Eq.28). Then we have introduced - as a first approximation - a constant, σ, combining the variation of M(t) and ω(t) (Eq.29), which allows us to have a differential equation with constant coefficients. Thus, the resolution becames systematic without any explicit choice of M(t) orω(t). The expression of β(t) has been determined for three domains of values of σ, we have shown that the solution β(t) is in the form of linear combinations of cos and sin functions, which tend towards the usual oscillating solution of the harmonic oscillator independent of time. Forms of solutions have been determined for various choices of explicit expressions of M(t) or ω (t) already used by other authors [1,2,4,5,11,12,14,16], and proved to be in agreement with the results of these authors deduced from different methods.

The results obtained show that the Heisenberg uncertainty principle is satisfied for any value of time. The comparison of our results with those obtained by other authors using other methods proves the efficiency of this method as a calculation tool.

Nomenclature

H(t)	Hamiltonien
P	momentum
M(t)	mass function
ω(t)	frequency
Q	position
ν	parameter
β(t)	solution of the motion equation
α(t)	time dependentparameter
γ(t)	time dependent coupling function
I(t)	linear invariant
$C_1(t)$	time dependentparameter
$C_2(t)$	time dependentparameter
B(t)	time dependentparameter
$f_1(t)$	time dependentparameter
$f_2(t)$	time dependentparamete
$\Phi(t)$	eigenfunction
$\psi(t)$	state of the system
g(t)	gausiun function
r_1(t)	time dependentparameter
k	arbitrary parameter
g_1(t)	time dependentparameter
ρ	probability density
μ_λ	wave function
λ	eigenvalue

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