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Stability of Quadratic Functional Equation in Two Variables

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Abstract In this paper, we establish the general solution of a 2-variable quadratic functional equation $f(2x+y,2z+w)=f(x+y,z+w)-f(x-y,z-w)+4f(x,z)+f(y,w)$ and prove the generalized Hyers-Ulam stability of this functional equation.

Keywords: *solution, stability, quadratic functional equation*

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1. Introduction

One of the interesting questions in the theory of functional equations is the following (see [2]):

When is it true that a function which approximately satisfies a functional equation F must be close to an exact solution of F?

If there exists an affirmative answer we say that the equation F is stable. The stability problems of functional equations were raised by S. M. Ulam during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940 [15]:

Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exist a number $\delta>0$ such that if a function $f:G_1\to G_2$ satisfies the inequality $d(f(xy);f(x)f(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $T:G_1\to G_2$ such that $d(f(x);T(x))<\varepsilon$ for all $x\in G_1$?

If the answer is affirmative, we would say that the equation of homomorphism $T(xy)=T(x)T(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, his result was extended and generalized in several ways (see e.g. [13]). Th.M. Rassias [15] extended Hyers' theorem in the following form where Cauchy difference is allowed to be unbounded:

Let X and Y be real normed spaces with Y complete, $f:X\to Y$ be a mapping such that for each fixed $x\in X$ the mapping $t\to f(tx)$ is continuous on $\mathbb{R}$, and Assume that there exist constants $\varepsilon\ge 0$ and $p\in[0,\ 1)$ such that

$$\|f(x)-T(x)\|\le\frac{\varepsilon\|x\|^p}{1-2^{p-1}}$$

for all $x\in X$. In 1994, a generalization of Rassias' theorem was obtained by Gavruta P. Gavruta [10] in the spirit of Th. M. Rassias' approach.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, the reader is referred to [5,13,14] and references therein for further information on stability.

Let X and Y be vector spaces. For a mapping $f:X\times X\to Y$, consider the 2-variable quadratic functional equation:

$$\begin{aligned}f(2x+y,2z+w)&=f(x+y,z+w)\\&-f(x-y,z-w)+4f(x,z)+f(y,w)\end{aligned}\tag{1.1}$$

When $X=Y-\mathbb{R}$, we see the quadratic form given by $f(x,y)=ax^2+bxy+cy^2$ is a solution of (1.1). In fact, we can check that

$$\begin{aligned}&f(2x+y,\ 2z+w)\\&=a(2x+y)^2+b(2x+y)(2z+w)+c(2z+w)^2\\&=a(x+y)^2+b(x+y)(z+w)+c(z+w)^2\\&\quad-\left[a(x-y)^2+b(x-y)(z-w)+c(z-w)^2\right]\\&\quad+4ax^2+bxz+cz^2+4ay^2+byw+cw^2\\&=f(x+y,z+w)-f(x-y,z-w)\\&\quad+4f(x,z)+f(y,w).\end{aligned}$$

For a mapping $g:X\to Y$, Now, we consider the quadratic functional equation:

$$g(2x+y)=g(x+y)-g(x-y)+4g(x)+g(y) \quad (1.2)$$

In a one paper, by using the fixed point theorem method, C. Park [3] proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.2).

In this paper, we investigate the relation between (1.1) and (1.2). And we find out the general solution and the generalized Hyers-Ulam stability of (1.1).

2. The Relation between (1.1) and (1.2)

Theorem 2.1. Let $f: X\times X\to Y$ be a mapping satisfying (1.1) and let $g: X\to Y$ be the mapping given by

$$g(x):=f(x,x) \quad (2.1)$$

for all $x\in X$, then g satisfies (1.2).

Proof. By (1.1) and (2.1), we can show that

$$\begin{aligned}g(2x+y)&=f(2x+y,2x+y)\\&=f(x+y,x+y)+f(x-y,x-y)\\&\quad+4f(x,x)-f(y,y)\\&=g(x+y)-g(x-y)+4g(x)+g(y)\end{aligned}$$

for all $x,y\in X$.

Theorem 2.2. Let $a,b,c\in\mathbb{R}$ and $g: X\to Y$ be a mapping satisfying (1.2). If $f: X\times X\to Y$ is the mapping given by

$$f(x,y):=ag(x)+\frac{b}{4}\left[g(x+y)-g(x-y)\right]+cg(x) \quad (2.2)$$

for all $x,y\in X$, then f satisfies (1.1).

Proof. By (1.2) and (2.2), we can show that

$$\begin{aligned}&f(2x+y,2z+w)\\&=ag(2x+y)+\frac{b}{4}\begin{bmatrix}g(2x+y+2z+w)\\-g(2x+y-2z-w)\end{bmatrix}\\&=4ag(x)+ag(y)+ag(x+y)-ag(x-y)\\&\quad+\frac{b}{4}[4g(x+z)-g(y+w)\\&\qquad+g(x+z+y+w)-g(x+z-y-w)]\\&\quad-\frac{b}{4}[4g(x+z)-g(y+w)\\&\qquad+g(x-z+y-w)-g(x-z-y+w)]\\&\quad+4cg(z)+2cg(w)+cg(z+w)-cg(z-w)\\&=ag(x+y)+\frac{b}{4}[g(x+z+y+w)\\&\qquad-g(x+y-z-w)]+cg(z+w)\\&\quad-ag(x-y)+\frac{b}{4}\left[g(x-y+z-w)-g(x-y-z+w)\right]\\&\quad+cg(z-w)+4ag(x)+\frac{b}{4}\left[g(x+z)-g(x-z)\right]\\&\quad+cg(z)+ag(y)+\frac{b}{4}\left[g(x+z)-g(x-z)\right]+cg(w)\\&=f(x+y,z+w)+f(x-y,z-w)\\&\quad+4f(x,z)+f(y,w)\end{aligned}$$

for all $x,y,z,w\in X$. This completes the proof.

3. Solution and Stability Results

In the following theorem, we find out the general solution of the main functional equation (1.1).

Theorem 3.1. A mapping $f: X\times X\to Y$ satisfies (1.1) if and only if there exist two symmetric bi-additive mappings $S_1,S_2: X\times X\to Y$ and a bi-additive mapping $B: X\times X\to Y$ such that

$$f(x,y)=S_1(x,x)+B(x,y)+s_2(x,y)$$

for all $x,y\in X$.

Proof. We first assume that there exist two symmetric bi-additive mappings

$$S_1,S_2: X\times X\to Y$$

and a bi-additive mapping

$$B: X\times X\to Y$$

such that

$$f(x,y)=S_1(x,x)+B(x,y)+s_2(y,y)$$

for all $x,y\in X$. Then we have

$$\begin{aligned}&f(2x+y,2z+w)-f(x+y,z+w)+f(x-y,z-w)\\&=S_1(2x+y,2x+y)+B(2x+y,2z+w)\\&\quad+S_2(2z+w,2z+w)\\&\quad-[S_1(x+y,x+y)+B(x+y,z+w)+S_2(z+w,z+w)]\\&\quad+[S_1(x-y,x-y)+B(x-y,z-w)+S_2(z-w,z-w)]\\&=4S_1(x,x)+S_1(y,y)+4B(x,z)\\&\quad+B(y,w)+4S_2(z,z)+S_2(w,w)\\&=4[S_1(x,x)+B(x,z)+S_2(z,z)]\\&\quad+[S_1(y,y)+B(y,w)+S_2(w,w)]\\&=4f(x,z)+f(y,w)\end{aligned}$$

for all $x,y,z,w\in X$.

Conversely, we assume that f is a solution of (1.1). Define $f_1,f_2: X\to Y$ by $f_1(x):=f(x,0)$ and $f_2(x)=f(0,x)$ for all $x\in X$. One can easily verify that f_1,f_2 are quadratic. By [16], there exist two symmetric bi-additive mappings

$$S_1,S_2: X\times X\to Y$$

such that $f_1=S_1(x,x)$ and $f_2=S_2(x,x)$ for all $x\in X$. Define $B: X\times X\to Y$ by

$$B(x,y):=f(x,y)-\left[f(x,0)+f(0,y)\right]$$

for all $x,y\in X$. Then, it is easy to investigate that B is bi-additive. This completes the proof.

In the following theorem, let X be a vector space and Y be a Banach space. Given a function $f: X \times X \to Y$, we set

$$\begin{aligned} Df(x,y,z,w) &:= f(2x+y, 2z+w) \\ &- f(x+y, z+w) + f(x-y, z-w) \\ &- 4f(x, z) - f(y,w) \end{aligned}$$

for all $x, y, z, w \in X$.

Theorem 3.2. Let $f: X \times X \to Y$ be a mapping for which there exists a function $\varphi: X^4 \to [0,\infty)$, such that

$$\bar{\varphi}(x,y,z,w) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi\left(2^j x, 2^j y, 2^j z, 2^j w\right) < \infty \quad (3.1)$$

$$\|Df(x,y,z,w)\| \le \varphi(x,y,z,w) \quad (3.2)$$

for all $x, y, z, w \in X$. Then there exists a unique 2-variable quadratic mapping $A: X \times X \to Y$ such that

$$\|f(x,y) - A(x,y)\| \le \frac{1}{4}\bar{\varphi}(x,0,z,0) \quad (3.3)$$

for all $x, y \in X$. The mapping A is given by

$$A(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x, 2^n y\right)$$

for all $x, y \in X$.

Proof. Letting $y = 0$ and $w = 0$ in (3.2), we get

$$\left\|\frac{1}{4} f(2x, 2z) - f(x,z) + \frac{1}{2} f(0,0)\right\| \le \frac{1}{4}\varphi(x,0,z,0)$$

for all $x, z \in X$. Thus we obtain

$$\begin{aligned} &\left\|\frac{1}{4^{j+1}} f(2^{j+1}x, 2^{j+1}z) - \frac{1}{4^j} f(2^j x, 2^j z) + \frac{1}{4^{j+1}} f(0,0)\right\| \\ &\le \frac{1}{4^{j+1}} \varphi(2^j x, 0, 2^j z, 0) \end{aligned}$$

for all $x, z \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\begin{aligned} &\left\|\frac{1}{4^{j+1}} f(2^{j+1}x, 2^{j+1}y) - \frac{1}{4^j} f(2^j x, 2^j y) + \frac{1}{4^{j+1}} f(0,0)\right\| \\ &\le \frac{1}{4^{j+1}} \varphi(2^j x, 0, 2^j y, 0) \end{aligned}$$

for all $x, y \in X$ and all j. For given integers $l, m\ (0 \le l < m)$, we get

$$\begin{aligned} &\left\|\frac{1}{4^m} f(2^m x, 2^m y) - \frac{1}{4^l} f(2^l x, 2^l y) + \frac{1}{4^m} f(0,0)\right\| \\ &\le \frac{1}{4}\sum_{j=l}^{m-1} \frac{1}{4^j} \varphi(2^j x, 0, 2^j y, 0) \end{aligned} \quad (3.4)$$

for all $m > 1$ and all $x, y \in X$. It follows from (3.1) and (3.4) that the sequence $\left\{\frac{1}{4^n} f\left(2^n x, 2^n y\right)\right\}$ is Cauchy. Due to the completeness of Y, this sequence is convergent. So we can define the mapping $A: X \times X \to Y$ by

$$A(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x, 2^n y\right)$$

for all $x, y \in X$. By (3.2) and (3.1), we have

$$\begin{aligned} \|DA(x,y,z,w)\| &= \lim_{n\to\infty} \frac{1}{4^n} \left\|Df\left(2^n x, 2^n y, 2^n z, 2^n w\right)\right\| \\ &= \lim_{n\to\infty} \frac{1}{4^n} \varphi\left(2^n x, 2^n y, 2^n z, 2^n w\right) \end{aligned}$$

for all $x, y, z, w \in X$. So $DA(x,y,z,w) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.3).

Now let $A': X \times X \to Y$ be another 2-variable quadratic mapping satisfying (3.3). Then we have

$$\begin{aligned} &\|A(x,y) - A'(x,y)\| \\ &= \frac{1}{4^n}\left\|A\left(2^n x, 2^n y\right) - A'\left(2^n x, 2^n y\right)\right\| \\ &\le \frac{1}{4^n}\left\|A(2^n x, 2^n y) - f(2^n x, 2^n y)\right\| \\ &\quad + \frac{1}{4^n}\left\|A'(2^n x, 2^n y) - f(2^n x, 2^n y)\right\| \\ &\le \frac{2}{4^n}\bar{\varphi}\left(2^n x, 2^n y, 2^n x, 2^n y\right) \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in X$. So we can conclude that $A(x, y) = A'(x, y)$ for all $x, y \in X$. This proves the uniqueness of A. This completes the proof.

Remark 3.3. Let $f: X \times X \to Y$ be a mapping for which there exists a function $\varphi: X^4 \to [0,\infty)$ satisfying (3.2) such that

$$\bar{\varphi}(x,y,z,w) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}\right) < \infty$$

for all $x, y, z, w \in X$. By a similar method to the proof of Theorem 3.2, one can show that there exists a unique 2-variable quadratic mapping

$$A: X \times X \to Y$$

such that

$$\|f(x, y) - A(x, y)\| \le \frac{1}{4}\bar{\varphi}(x,0,y,0)$$

for all $x, y \in X$. The mapping A is given by

$$A(x, y) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all $x, y \in X$.

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