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On Instability of Dynamic Equilibrium States of Vlasov-Poisson Plasma

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Abstract The problem on linear stability of one–dimensional (1D) states of dynamic equilibrium boundless electrically neutral collisionless plasma in electrostatic approximation (the Vlasov–Poisson plasma) is studied. It is proved by the direct Lyapunov method that these equilibrium states are absolutely unstable with respect to small 1D perturbations in the case when the Vlasov–Poisson plasma contains electrons with stationary distribution function, which is constant over the physical space and variable in velocities, and one variety of ions whose distribution function is constant over the phase space as a whole. In addition, sufficient conditions for linear practical instability are obtained, the a priori exponential lower estimate is constructed, and initial data for perturbations, growing in time, are described. Finally, the illustrative analytical example of considered 1D states of dynamic equilibrium and superimposed small 1D perturbations, which grow on time in accordance with the obtained estimate, is constructed.

Keywords: *the Vlasov-Poisson Plasma, Dynamic Equilibrium States, the Direct Lyapunov Method, Instability*

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1. Introduction

The model of boundless electrically neutral collisionless plasma in electrostatic approximation (the Vlasov–Poisson plasma) continues to be one of basic mathematical models for modern plasma physics [1-6]. This is due to simplicity and clarity of this model as well as its obvious usefulness for solving the problem of controlled thermonuclear fusion (CTF).

As is well known, resolution of CTF problem is impossible without solving the stability problem of plasma equilibria [1]. This implies that development of mathematical stability theory occupies central position in studies of plasma and its properties.

A number of fundamental nature results was established early in the process of studying the stability of dynamic equilibrium states of the Vlasov–Poisson plasma [2,7,8,9].

Namely, the sufficient condition for linear stability of dynamic equilibrium states of the Vlasov–Poisson plasma was obtained in [7,8,9]. Moreover, it is shown in [7], [8] that this condition prohibits increasing in time small perturbations of dynamic equilibrium states of the Vlasov–Poisson plasma in the form of normal waves. Finally, the sufficient condition [7,8,9] for linear stability of dynamic equilibrium states of the Vlasov–Poisson plasma was generalized on finite perturbations in [2].

Here, in this paper, the sufficient condition [7,8,9] for linear stability of dynamic equilibrium states of the Vlasov–Poisson plasma will be converted. In other words, it will be demonstrated that the condition is both sufficient and necessary by its nature. In addition, it will be given rigorous description of the applicability area for this condition [10,11].

2. Formulation of the Problem

The simplified, but, nevertheless, quite substantial 1D version of the model boundless electrically neutral collisionless plasma in electrostatic approximation (the Vlasov–Poisson plasma) is considered in the following [3,12]:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{\partial U}{\partial x}\frac{\partial f}{\partial v}=0 \qquad (1)$$

$$\frac{\partial^2 U}{\partial x^2}=4\pi\left(1-\int_{-\infty}^{+\infty} f dv\right)$$

$$f=f(x,v,t)\ge 0;\quad f(x,v,0)=f_0(x,v)$$

where f is the distribution function of electrons; t is the time; x and v are coordinates and velocities of electrons; $U(x,t)$ is the potential of self–consistent electric field; π is the classic constant value (the ratio of a circle's circumference to its own diameter); f_0 is initial data for the function f. It is assumed that the distribution function f of electrons is periodic in the argument x or has appropriate asymptotic behavior at $|x|\to\infty$, and it vanishes in the argument v at $|v|\to\infty$.

From the physical point of view, the mixed problem (1) characterizes the plasma in the framework of such ideas about it as: 1) the plasma includes electrons and one kind of ions; 2) since the mass of electrons is extremely small compared with the mass of ions, it is believed that the latter are at rest and filling all the phase space with constant density equal to one; 3) the plasma temperature is supposed such that the speed of light is much larger than the mean thermal velocity of electrons, and the magnetic field cannot be considered; 4) collective interactions of electrons prevail over paired ones; therefore, one can neglect the collisions integral in the right–hand side of the Vlasov equation for the function f (see the first relation from system (1)).

The initial–boundary value problem (1) has exact stationary solutions in the form

$$f = f^0(v), \quad U = U^0: \quad \int_{-\infty}^{+\infty} f^0 dv = 1 \tag{2}$$

Here f^0 is arbitrary non–negative function of the independent variable v, and U^0 is a constant.

It is shown in [7,8,9] that the inequality

$$v\frac{df^0}{dv} \le 0 \tag{3}$$

is the sufficient condition for linear stability of dynamic equilibrium states (2) of the Vlasov–Poisson plasma. The meaning of condition (3) is that it highlights monotonically decreasing in velocities stationary distribution functions f^0 of electrons as stable ones.

It is proved in [7,8] that the inequality (3) ensures no growing on time small perturbations of dynamic equilibrium states (2) of the Vlasov–Poisson plasma in the form of normal waves.

However, in fairness, it should be noted that the condition (3) is derived using the total energy functional

$$E \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v^2 f dx dv + \frac{1}{8\pi}\int_{-\infty}^{+\infty}\left(\frac{\partial U}{\partial x}\right)^2 dx \tag{4}$$

and the integral of motion

$$C \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(f) dx dv \tag{5}$$

where Φ is arbitrary function of its argument, conserving in time along solutions to the mixed problem (1) due to the same evolutionary equation, namely the Vlasov equation for the distribution function f of electrons (see the first relation of system (1) again).

This circumstance is the cause for presence of some hidden relationship between the functional E (4) and the integral C (5). In turn, this relationship, as such, means that the inequality (3) is the sufficient condition for linear stability of dynamic equilibrium states (2) of the Vlasov–Poisson plasma with respect not to all of studied perturbations, but only to some of their subclass.

Further, in order to describe rigorously the applicability area for inequality (3) as the sufficient condition for linear stability of dynamic equilibrium states (2) of the Vlasov–Poisson plasma, non–singular change [3], [12], [13] of variables in the form [14]

$$f(x, v, t) = \rho(x, \lambda, t)\left(\frac{\partial u}{\partial \lambda}(x, \lambda, t)\right)^{-1}$$

$$v = u(x, \lambda, t); \quad \lambda \in (-\infty, +\infty)$$

$$\frac{d\lambda}{dt} \equiv 0; \quad \frac{\partial u}{\partial \lambda}(x, \lambda, t) \ne 0$$

is carried out.

3. Reformulation of the Problem

In the end, the initial–boundary value problem (1) can be rewritten as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial U}{\partial x}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} = 0 \tag{6}$$

$$\frac{\partial^2 U}{\partial x^2} = 4\pi\left(1 - \int_{-\infty}^{+\infty} \rho d\lambda\right)$$

$$u(x, \lambda, 0) = u_0(x, \lambda), \quad \rho(x, \lambda, 0) = \rho_0(x, \lambda)$$

Here λ is the Lagrangian coordinates of electrons; u is the velocity field; ρ is the density field of electrons; u_0 and ρ_0 are initial data for fields u and ρ, correspondingly. It is assumed that fields of the velocity u and the density ρ disappear in the argument λ at $|\lambda| \to \infty$, and they are periodical in the argument x or have desired asymptotic behavior at $|x| \to \infty$.

The mixed problem (6) has exact stationary solutions of the form

$$u = u^0(\lambda), \quad \rho = \rho^0(\lambda), \quad U = U^0 \tag{7}$$

$$\int_{-\infty}^{+\infty} \rho^0 d\lambda = 1$$

where u^0 is arbitrary increasing, ρ^0 is some non–negative functions of the independent variable λ; U^0 is certain constant value as before.

Subsequent consideration aims to find out whether exact stationary solutions (7) are stable with respect to small 1D perturbations $u'(x, \lambda, t)$, $\rho'(x, \lambda, t)$, and $U'(x, t)$.

To achieve this goal, it is realized linearization of the initial–boundary value problem (6) in the vicinity of its exact stationary solutions (7), allowing to obtain mixed problem in the form

$$\frac{\partial u'}{\partial t} + u^0\frac{\partial u'}{\partial x} = -\frac{\partial U'}{\partial x} \tag{8}$$

$$\frac{\partial \rho'}{\partial t} + u^0\frac{\partial \rho'}{\partial x} + \rho^0\frac{\partial u'}{\partial x} = 0, \quad \frac{\partial^2 U'}{\partial x^2} = -4\pi\int_{-\infty}^{+\infty} \rho' d\lambda$$

$$u'(x, \lambda, 0) = u'_0(x, \lambda), \quad \rho'(x, \lambda, 0) = \rho'_0(x, \lambda)$$

Here u_0' and ρ_0' are initial data for small 1D perturbations u' and ρ' of steady–state velocity u^0 and density ρ^0 fields of electrons.

Theorem 1. There is no the sufficient condition for linear stability of exact stationary solutions (7) to problem (6) with respect to 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8).

Proof. The functional

$$E_1 = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\begin{array}{l}\rho^0 u'^2 + 2u^0\rho' u' \\ +\dfrac{du^0}{d\lambda}\times\dfrac{d^2\Phi_1}{d\kappa^2}\left(\kappa^0\right)\kappa'^2\end{array}\right]d\lambda dx + \frac{1}{8\pi}\int_{-\infty}^{+\infty}\left(\frac{\partial U'}{\partial x}\right)^2 dx \quad (9)$$

$$\frac{d\Phi_1}{d\kappa}\left(\kappa^0\right) = -\frac{U^0 + u^{02}}{2}$$

$$\kappa = \kappa(x,\lambda,t) \equiv \rho\left(\frac{\partial u}{\partial\lambda}\right)^{-1} \ge 0$$

$$\kappa^0 = \kappa^0(\lambda) \equiv \rho^0\left(\frac{du^0}{d\lambda}\right)^{-1} \ge 0$$

$$\kappa' = \kappa'(x,\lambda,t) \equiv \rho'\left(\frac{du^0}{d\lambda}\right)^{-1} - \rho^0\frac{\partial u'}{\partial\lambda}\left(\frac{du^0}{d\lambda}\right)^{-2}$$

(linear analogue of the total energy integral) is conserved on solutions to the mixed problem (8).

Exact stationary solutions (7) to the initial–boundary value problem (6) are stable with respect to small 1D perturbations (8) if and only if the functional E_1 (9) is the definite one in sign. Unfortunately, by virtue of the Sylvester criterion [15], the integral E_1 does not possess distinctness in sign.

As a result, the sufficient condition for linear stability of exact stationary solutions (7) to the mixed problem (6) with respect to 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8) is really absent [10], [11]. This completes the **proof** of **Theorem 1**.

However, the sufficient condition for linear stability of exact stationary solutions (7) to the initial–boundary value problem (6) with respect to 1D perturbations (8) can be obtained yet if to subject the latter to additional demand. Specifically,

$$\int_{-\infty}^{+\infty}\left[\left(u^0\kappa^0 u'^2\right)\Big|_{\lambda\to+\infty} - \left(u^0\kappa^0 u'^2\right)\Big|_{\lambda\to-\infty}\right]dx \to 0 \quad (10)$$

In fact, the asymptotic form (10) imposes upper limit on allowable values of the kinetic energy for individual electrons.

Theorem 2. The inequality

$$u^0\frac{d\kappa^0}{d\lambda} \le 0 \quad (11)$$

is the sufficient condition for linear stability of exact stationary solutions (7) to problem (6) with respect to small 1D perturbations (8), (10).

Proof. In this case, the functional E_1 (9) will appear in such a way:

$$E_1 = -\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^0\frac{d\kappa^0}{d\lambda}u'^2 d\lambda dx + \frac{1}{8\pi}\int_{-\infty}^{+\infty}\left(\frac{\partial U'}{\partial x}\right)^2 dx \quad (12)$$

This implies that the integral E_1 (12) is the definite one in sign if the ratio (11) is fulfilled.

In the end, the relation (11) is indeed the sufficient condition for stability of exact stationary solutions (7) to the mixed problem (6) with respect to small 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8) that meet the additional requirement (10). This completes the **proof** of **Theorem 2**.

Incidentally, the ratio (11) is equivalent to the inequality (3) because this ratio can be written in the form similar to this inequality, namely

$$u^0\frac{d\kappa^0}{du^0} \le 0$$

Unfortunately, the asymptotic form (10) is not conserved in time along solutions to the initial–boundary value problem (8). In other words, small 1D perturbations (8), (10) are not complete closed partial class of solutions to the mixed problem (8).

This fact provides the reason to hypothesize about absolute instability of exact stationary solutions (7) to the initial–boundary value problem (6) with respect to small 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8). In this connection, the study will be aimed further at testing of this hypothesis truth.

4. The a Priori Exponential Lower Estimate of Growth on Time for Small 1D Perturbations

Next, linear instability of exact stationary solutions (7) to the mixed problem (6) with respect to 1D perturbations (8) will be set by the direct Lyapunov method [16]–[18] regardless on whether the inequality (11) holds or not. Concerning the relation (11), it will be shown that this relation is the necessary and sufficient condition for stability of exact stationary solutions (7) to the initial–boundary value problem (6) with respect to small 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8) of incomplete unclosed subclass (10). Moreover, sufficient conditions for linear practical instability [18]–[20] of exact stationary solutions (7) to the mixed problem (6) with respect to 1D perturbations (8) will be obtained. Finally, when these conditions for practical instability are correct, the a priori lower estimate, indicating that considered small perturbations grow over time, not slower than exponentially, will be constructed.

To show instability of any exact stationary solution (7) to the initial–boundary value problem (6) with respect to small 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8), one needs to be able to distinguish among them only one perturbation, but growing in time, at least, exponentially fast.

To that end, the partial class of solutions to the mixed problem (8), which is characterized by the property that its small 1D perturbations are deviations of electrons flight trajectories from current lines, corresponding to exact stationary solutions (7) to the initial–boundary value problem (6), is further investigated.

It is not complicated to describe these perturbations by means of the Lagrangian displacements field $\xi = \xi(x, \lambda, t)$ [21] which is determined by the equation

$$\frac{\partial \xi}{\partial t} = u' - u^0 \frac{\partial \xi}{\partial x} \tag{13}$$

With the help of relation (13), the mixed problem (8) can be reformulated as

$$\frac{\partial^2 \xi}{\partial t^2} + 2u^0 \frac{\partial^2 \xi}{\partial x \partial t} + u^{02} \frac{\partial^2 \xi}{\partial x^2} = -\frac{\partial U'}{\partial x} \tag{14}$$

$$\rho' = -\rho^0 \frac{\partial \xi}{\partial x}, \quad \frac{\partial^2 U'}{\partial x^2} = 4\pi \int_{-\infty}^{+\infty} \rho^0 \frac{\partial \xi}{\partial x} d\lambda$$

$$\xi(x, \lambda, 0) = \xi_0(x, \lambda), \quad \frac{\partial \xi}{\partial t}(x, \lambda, 0) = \left(\frac{\partial \xi}{\partial t}\right)_0 (x, \lambda)$$

The functional E_1 (9) will conserve on solutions to the initial–boundary value problem (14) too. Admittedly, it will take now the form

$$E_1 = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho^0 \left[\left(\frac{\partial \xi}{\partial t}\right)^2 - u^{02} \left(\frac{\partial \xi}{\partial x}\right)^2 \right] d\lambda dx + \frac{1}{8\pi} \int_{-\infty}^{+\infty} \left(\frac{\partial U'}{\partial x}\right)^2 dx \tag{15}$$

In accordance with the Sylvester criterion [15], it implies from the expression (15) that the integral E_1 is not the definite one in sign for small 1D perturbations $\xi(x, \lambda, t)$ (13), (14) in principle. This fact confirms only once more that choice the subclass (13) of small 1D perturbations (8) to demonstrate absolute instability of exact stationary solutions (7) to the mixed problem (6) is correct.

Theorem 3. Exact stationary solutions (7) to problem (6) are absolutely unstable with respect to small 1D perturbations (13), (14).

Proof. In the interests of subsequent statement, it is convenient to introduce into the study such additional functional:

$$M \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho^0 \xi^2 d\lambda dx \tag{16}$$

Using relations (13), (14), the original differential inequality [18,22,23,24,25]

$$\frac{d^2 M}{dt^2} - 2\nu \frac{dM}{dt} + 2\left(\nu^2 + 4\pi\right) M \geq 0 \tag{17}$$

can be constructed for the integral M (16), where ν is an arbitrary positive constant.

Since the procedure of relation (17) integration is described minutely in [18,22,23], only its results are reported further. Namely, if one supplement the inequality (17) by countable set of terms in the form

$$M\left(\frac{\pi n}{2\sqrt{\nu^2 + 8\pi}}\right) > 0 \tag{18}$$

$$\frac{dM}{dt}\left(\frac{\pi n}{2\sqrt{\nu^2 + 8\pi}}\right) \geq 2\left(\nu + \frac{4\pi}{\nu}\right) \times M\left(\frac{\pi n}{2\sqrt{\nu^2 + 8\pi}}\right);$$

$$M\left(\frac{\pi n}{2\sqrt{\nu^2 + 8\pi}}\right) \equiv M(0) \times \exp\frac{\pi n \nu}{2\sqrt{\nu^2 + 8\pi}},$$

$$\frac{dM}{dt}\left(\frac{\pi n}{2\sqrt{\nu^2 + 8\pi}}\right) \equiv \frac{dM}{dt}(0) \exp\frac{\pi n \nu}{2\sqrt{\nu^2 + 8\pi}}$$

(here $n = 0, 1, 2, \ldots$), it will provide just an opportunity to set the desired a priori exponential lower estimate

$$M(t) \geq C_1 \exp \nu t \tag{19}$$

for small 1D perturbations $\xi(x, \lambda, t)$ (13), (14) growing over time, where C_1 is the known positive constant value.

Before one continue the started above consideration, it is appropriate to highlight connection between the studied initial–boundary value problem (14) and countable set of relations (18) added to the original differential inequality (17).

Particularly, since the mixed problem (14) is linear, it is solvable thereby for small 1D perturbations in the form of normal waves [26]. Further, inasmuch as the functional E_1 (15) has no the property of definiteness in sign, the initial–boundary value problem (14) is solvable with respect to growing in time small 1D perturbations in the form of normal waves also. In addition, if the mixed problem (14) has, at least, one growing on time solution that meets small 1D perturbation in the form of normal wave, it will satisfy the inequality (17), terms (18), and the estimate (19) identically and automatically due to arbitrariness of a positive constant ν.

Thus, relations (18) do not preclude nowise from existence of growing over time solutions, which correspond small 1D perturbations in the form of normal waves, among solutions to the initial–boundary value problem (14) with additional demands

$$M(0) > 0, \quad \frac{dM}{dt}(0) \geq 2\left(\nu + \frac{4\pi}{\nu}\right) M(0) \tag{20}$$

to initial data $\xi_0(x, \lambda)$ and $\left(\partial \xi / \partial t\right)_0 (x, \lambda)$.

So, the subclass (20) of solutions to the mixed problem (14), increasing in time according to the a priori exponential lower estimate (19), is not empty. This conclusion will be supported further by the illustrative analytical example.

Hence, it is shown that the a priori exponential lower estimate (19) for growing over time small 1D perturbations (13), (14), (20) is obtained without any restrictions on exact stationary solutions (7) to the mixed problem (6). It follows from this that exact stationary solutions (7) to the initial–boundary value problem (6) are really absolutely unstable with respect to small 1D perturbations $\xi(x, \lambda, t)$ (13), (14), (20), and the inequality (11) represents actually the desired necessary and sufficient condition for linear stability of exact stationary solutions (7) to the mixed problem (6) with respect to 1D perturbations $u'(x, \lambda, t)$, $\rho'(x, \lambda, t)$, and $U'(x, t)$ (8) from the partial class (10).

Moreover, it is demonstrated that the first pair of ratios from the system of relations (18) is indeed desired sufficient conditions for practical instability of exact stationary solutions (7) to the initial–boundary value problem (6) with respect to small 1D perturbations (13), (14), (20). With regard to 1D perturbations $\xi(x, \lambda, t)$ (13), (14), (20) in the form of normal waves, inequalities of the relations system (18) are sufficient and necessary conditions for linear practical instability of exact stationary solutions (7) to the mixed problem (6).

Finally, in agreement with previously published monographs by other authors [19,20], if there is theoretical instability (on semi–infinite time intervals), practical instability may or may not be at the same time. However, it is established in this article that sufficient conditions (see inequalities from the system of relations (18)) for linear practical instability can be obtained when and only when there is no conditions for linear theoretical stability. By the way, found here sufficient conditions (see inequalities of the relations system (18) again) for linear practical instability are of constructive nature, since they can act as mechanism for testing and monitoring during implementation of physical experiments, execution of numerical calculations, realization of technological processes, etc.

As for the physical meaning of absolute linear theoretical instability of exact stationary solutions (7) to the initial–boundary value problem (6) with respect to 1D perturbations (13), (14), (20), established in the present paper, it consists in that the potential U^0 of steady–state self–consistent electric field is constant over the physical space so forces, which would be able both to protect stationary distribution function $u^0(\lambda)$, $\rho^0(\lambda)$ of electrons from ``smearing" over the phase space and to block development of growing in time small 1D perturbations $\xi(x, \lambda, t)$ (13), (14), (20), are absent in the studied Vlasov–Poisson plasma.

The physical meaning of obtained sufficient conditions (see inequalities from the system of relations (18)) for linear practical instability of exact stationary solutions (7) to the mixed problem (6) with respect to 1D perturbations (13), (14), (20) is that, gathered the information about temporal evolution of electron component of the considered Vlasov–Poisson plasma by recording equipment, with the help of these conditions for practical instability, one can answer the question whether small 1D perturbations $\xi(x, \lambda, t)$ (13), (14), (20) have tendency to unlimited, at least, exponential growth on time, and, therefore, to destructive effect on dynamic equilibrium states (7).

In conclusion, it is logical to mention the fact that the integral M (16) represents in the given article the Lyapunov functional increasing over time in accordance with equations of the initial–boundary value problem (14), (20). The distinctive feature of this growth is tremendous freedom which remains at positive constant value ν in the exponent from the right–hand side of inequality (19). Among other things, it allows us to interpret any solution to the mixed problem (14), (20), increasing with time according to the found a priori exponential lower estimate (19), as analogue of incorrectness example by Hadamard [27].

The illustrative analytical example of exact stationary solutions (7) to the initial–boundary value problem (6) and superimposed small 1D perturbations (13), (14), (20), which grow in time under sufficient conditions (see inequalities of the relations system (18) again) for practical instability, established in the present paper, in agreement with the obtained a priori exponential lower estimate (19), is constructed further.

5. The Example

Before proceeding to actual construction of the announced above illustrative analytical example, it is reasonable to show that results of articles [7,8], referring to prohibition on origin and development of growing over time perturbations in the form of normal waves by the sufficient condition (3) for linear theoretical stability of dynamic equilibrium states (2) of the Vlasov–Poisson plasma, cannot be used in relation to the necessary and sufficient condition (11) for linear theoretical stability of exact stationary solutions (7) to the mixed problem (6) as well as to 1D perturbations $\xi(x, \lambda, t)$ (13), (14), (20) in the form of normal waves.

In fact, let small 1D perturbations (13), (14) have the form of normal waves. Specifically,

$$\xi(x, \lambda, t) = \xi_1(\lambda)\exp\left(\alpha t + \beta x\right) \tag{21}$$

where ξ_1 is some function of its argument; $\alpha \equiv \alpha_1 + i\alpha_2$ is a certain complex, $\beta \equiv i\beta_1$ is an arbitrary purely imaginary constants; α_1 , α_2 , and β_1 are some real constant values; i is the imaginary unit.

Substituting the expression (21) in the first and the third equations from the system of relations (14), it is not hard to derive dispersion relation of the form

$$1 = -4\pi \int_{-\infty}^{+\infty} \frac{\rho^0 d\lambda}{\left(\alpha + u^0\beta\right)^2} \tag{22}$$

from there. Further, in the spirit of papers [7], [8], the real and the imaginary parts of equality (22) are separated from each other:

$$1 = -4\pi \int_{-\infty}^{+\infty} \frac{\rho^0[\alpha_1^2 - (\alpha_2 + u^0\beta_1)^2]d\lambda}{[\alpha_1^2 + (\alpha_2 + u^0\beta_1)^2]^2} \tag{23}$$

$$\int_{-\infty}^{+\infty} \frac{\rho^0(\alpha_2 + u^0\beta_1)d\lambda}{[\alpha_1^2 + (\alpha_2 + u^0\beta_1)^2]^2} = 0$$

Finally, using the second relation of system (23), the first equality from it can be reported much more clearly, namely:

$$1+4\pi(\alpha_1^2+\alpha_2^2)\int_{-\infty}^{+\infty}\frac{\rho^0 d\lambda}{[\alpha_1^2+(\alpha_2+u^0\beta_1)^2]^2} = 4\pi\beta_1^2\int_{-\infty}^{+\infty}\frac{\rho^0 u^{02} d\lambda}{[\alpha_1^2+(\alpha_2+u^0\beta_1)^2]^2} \quad (24)$$

The expression (24) demonstrates convincingly that it is internally consistent. In addition, there is no hint in it at the criterion (11) for theoretical stability of exact stationary solutions (7) to the initial–boundary value problem (6) with respect to small 1D perturbations $u'(x,\lambda,t)$, $\rho'(x,\lambda,t)$, and $U'(x,t)$ (8) from the subclass (10). Hence, the relation (24) is by no means the reason to deny origin and evolution of growing on time small 1D perturbations (13), (14), (20) of exact stationary solutions (7) to the mixed problem (6) in the form (21) of normal waves.

Now, after proof of not susceptibility for exact stationary solutions (7) to the initial–boundary value problem (6) and small 1D perturbations $\xi(x,\lambda,t)$ (13), (14), (20) in the form (21) of normal waves to influence of articles [7,8] results, there is every reason to pass on to direct designing of the conceived earlier illustrative analytical example.

Particularly, the representative of exact stationary solutions (7) to the mixed problem (6) is taken as

$$u = u^0(\lambda) = \lambda \quad (25)$$

$$\rho = \rho^0(\lambda) = \frac{2}{3\sqrt{\pi}}\frac{1+\lambda^2}{\exp\lambda^2}, \quad U = U^0$$

The choice of exact stationary solution (7) to the initial–boundary value problem (6) in the form (25) is explained by the fact that the field of velocity u^0 will become the independent variable v, and the field of density ρ^0 will turn into the distribution function f^0 of electrons in the course of fulfilment inverse non–singular change of variables $(x,\lambda,t)\to(x,v,t)$. Therefore, exact stationary solution (7) to the mixed problem (6) in the form (25) is simultaneously the dynamic equilibrium state (2) of the Vlasov–Poisson plasma.

It is not difficult to verify that the necessary and sufficient condition (11) for linear theoretical stability is true for exact stationary solution (7) to the initial–boundary value problem (6) in the form (25):

$$u^0\frac{d\kappa^0}{du^0} = \lambda\frac{d\rho^0}{d\lambda} = -\frac{4}{3\sqrt{\pi}}\frac{\lambda^4}{\exp\lambda^2} \le 0$$

Similarly, after inverse non–singular change of variables $(x,\lambda,t)\to(x,v,t)$, the criterion (3) for linear theoretical stability is fair for dynamic equilibrium state (2) of the Vlasov–Poisson plasma, corresponding with the solution (7), too.

However, contrary to these circumstances, increasing in time small 1D perturbations (13), (14), (20) in the normal waves form (21) of exact stationary solution (25) to the mixed problem (6) exist for all that.

Indeed, if to take $\alpha_1 \ne 0$, $\alpha_2 = 0$, and $\beta_1 = 1$ in the expression (21), then the dispersion relation (22) will appear for exact stationary solution (25) to the initial–boundary value problem (6) as

$$g(\alpha_1) \equiv 1+8\begin{Bmatrix}1-2\alpha_1^2 \\ +2\alpha_1^3\sqrt{\pi}\left(1-\operatorname{erf}\alpha_1\right)\times\exp\alpha_1^2\end{Bmatrix}\frac{\pi}{3} = 0 \quad (26)$$

Unfortunately, the author of this paper does not know what analytical methods roots of the equation (26) can be found. Therefore, it was solved him graphically. The results of this solving the dispersion relation (26) is shown in Figure 1.

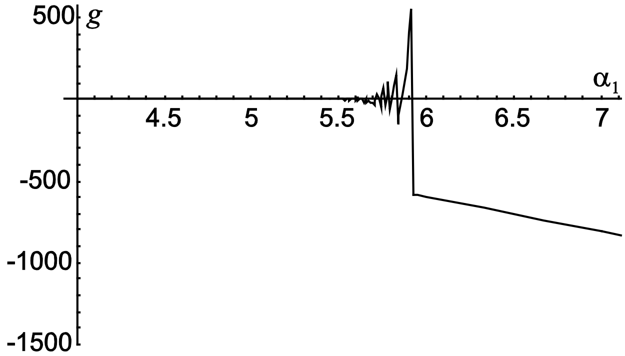

Figure 1. Graphical image solutions to the equation $g(\alpha_1)=0$ (26)

The graph of function g (26) demonstrates clearly that this relation has entire family of positive roots α_1. In turn, these roots of equation (26) just meet growing on time small 1D perturbations $\xi(x,\lambda,t)$ (13), (14), (20) in the normal waves form (21) of exact stationary solution (25) to the mixed problem (6). Moreover, taking as a constant ν, for example, the smallest positive root α_1 of the dispersion relation (26), it is not hard to see that the original differential inequality (17), countable set of terms (18), and the a priori exponential lower estimate (19) are performed for increasing over time small 1D perturbations (13), (14), (20) in the normal waves form (21) of exact stationary solution (25) to the initial–boundary value problem (6).

Thus, construction of the illustrative analytical example of exact stationary solutions (7) to the mixed problem (6) and superimposed small 1D perturbations $\xi(x,\lambda,t)$ (13), (14), (20), growing on time in accordance with the obtained a priori exponential lower estimate (19) in presence of discovered in this article sufficient conditions (see inequalities of the relations system (18)) for practical instability, is completed. Concurrently, this completes the **proof** of **Theorem 3** as well.

6. Conclusion

Finishing the presentation of this paper material, it is worth to dwell separately on how specifically sufficient conditions (see inequalities from the system of relations (18)) for linear practical instability can be applied in annex to resolution of CTF problem.

Let, for example, it is necessary to develop a device for plasma confinement, based on the use of dynamic equilibrium states (2), (7), as a unit of industrial fusion power plant.

In order that the device was reliable in operation, it is required to ensure practical stability for dynamic equilibrium plasma states (2), (7) with respect to all admissible perturbations. In particular, these dynamic equilibrium states must be stable in practical sense with regard to small 1D perturbations (13), (14), (20).

This can be achieved by constructing numerical and physical models, which are consistent with the linearized initial–boundary value problem (14), with control at reference time points $t_n \equiv \pi n \big/ 2\sqrt{\nu^2 + 8\pi}$ ($n = 0, 1, 2, ...$) for validity of inequalities in the relations system (18). In the process of these models constructing, one needs to focus main efforts on that inequalities from the system of relations (18) would not have been fair at the expense of those or other known external influences on increasing over time small 1D perturbations $\xi(x, \lambda, t)$ (13), (14), (20) (for example, due to violation of initial conditions (20)).

As a result, practical stability of dynamic equilibrium plasma states (2), (7) will guarantee, at least, with respect to small 1D perturbations (13), (14), (20) in the form (21) of normal waves, and so the desired device for plasma confinement, which is managed in real–time mode through a kind of feedback in the form of necessary and sufficient conditions (see inequalities of the relations system (18)) for linear practical instability, will be created.

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Nomenclature

f is the distribution function of electrons;
t is the time;
x and v are coordinates and velocities of electrons;
U is the potential of self–consistent electric field;
f_0 is initial data for the function f ;
f^0 is arbitrary non–negative function of the independent variable v ;
U^0 is a constant;
E is the total energy functional;
C is the integral of motion;
Φ is arbitrary function of its argument f ;
λ is the Lagrangian coordinates of electrons;
u is the velocity field;
ρ is the density field of electrons;
u_0 and ρ_0 are initial data for fields u and ρ ;
u^0 is arbitrary increasing, and ρ^0 is some non–negative functions of the independent variable λ ;
u' , ρ' , and U' are small 1D perturbations of u^0 , ρ^0 , and U^0 ;
u_0' and ρ_0' are initial data for small 1D perturbations u' and ρ' ;
E_1 is the linear analogue of total energy integral;
κ is the field of inverse vorticity;
Φ_1 is arbitrary function of its argument κ ;
κ^0 is some non–negative function of the independent variable λ ;
κ' is small 1D perturbation of κ^0 ;
ξ is the Lagrangian displacements field;
ξ_0 and $(\partial\xi/\partial t)_0$ are initial data for field ξ ;
M is the Lyapunov functional;
ν is an arbitrary positive constant;
n is non–negative integer;
C_1 is the known positive constant value;
ξ_1 is some function of its argument λ ;
$\alpha \equiv \alpha_1 + i\alpha_2$ is a certain complex, and $\beta \equiv i\beta_1$ is an arbitrary purely imaginary constants;
α_1 , α_2 , and β_1 are some real constant values;
g is short designation for the dispersive relation (26);
t_n are reference time points.

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