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Spherical Harmonic on a Four Sphere

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Abstract In this paper, we will analyse the scalar harmonics on a four sphere using a associated Legendre function. Then, we will use these modes to construct two types of vector harmonics on a four sphere. Finally, we will also construct three types of tensor harmonics on a four sphere. As there is a relation between de Sitter spacetime and four sphere, these modes are related to the modes on de Sitter spacetime.

Keywords: *de Sitter spacetime, four sphere, degrees of freedom, Einstein tensor*

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1. Introduction

The de Sitter universe is a maximally symmetric solution to Einstein's field equations. It models a universe where the dynamics are governed by the cosmological constant. The universe in the inationary era is thought to be approximated by de Sitter spacetime [1,2,3,4]. Furthermore, the universe is expected to be approaching de Sitter spacetime asymptotically. So, quantum field theory on de Sitter spacetime is very important. The de Sitter spacetime can be obtained from an analytical continuation of a four sphere [5]. A new scenario in which it is possible to have a ghost-free massive gravity has been analysed and this work has direct cosmological applications [6]. Cosmological redshift grows with the expansion of the universe. It is possible to analyse this redshift using the Friedmann-Robertson-Walker metrics with constant spacetime curvature [7]. Furthermore, FRW cosmology with con-positively defined Higgs potentials has also been analysed. In this analysis the classical aspects of dynamics of scalar models with negative Higgs potentials in the FRW cosmology were studied. These models were found to appear as effective local models in non-local models. After a suitable field redefinition these models have the form of local Higgs models with a negative extra cosmological term. However, the total Higgs potential is non-positively defined and has rather small coupling constant. It may be noted that bulk viscous cosmological models in $f(R,T)$ theory of gravity have been studied [9]. These models have provided a way to reconstruction anisotropic cosmological models of accelerating universe [10].

Numerical study of spherically symmetric solutions on a cosmological dynamical background using the BSSN Formalism have been performed [11] Quan-tum non-hermal radiation of non-stationary rotating de Sitter cosmological model have also been recently analysed [12]. In these model Hamilton-Jacobi method was used to study of quantum non-thermal radiation of non-stationary rotating de Sitter cosmological model. It is shown that there exist seas of positive and negative energy states in the vicinity of the cosmological event horizon. Apart from that there also exists a forbidden energy gap between the two seas. The forbidden energy gap vanishes on the surface of the cosmological event horizon. Thus, the positive and negative energy levels overlap. Recently, the atness problem in cosmology has been analysed [13]. A new quantitative argument has shows that it does not exist in cosmological models which collapse in the future. New fermions have been introduced into particles physics, such that they are singlet under the gauge group of the Standard Model [14]. This allows using the seesaw type-I mechanism to obtain small neutrino masses and explaining the phenomenon of neutrino oscillations. This mass is a free parameter. these neutrinos can inuence evolution of the universe. They can also be responsible for the baryon asymmetry of the universe. Hence, de Sitter spacetime is an important spacetime and it is also important to understand quantum field theory on this spacetime.

Quantum field theory is one of the most important scientific developments of the last centenary. In quantum field theory, perturbative theory has led to many important developments. These include both the developments in high energy physics as well as the condensed matter physics. Gauge symmetry plays an important part in most of the quantum field theories describing nature. Gravity can also be considered as a gauge theory [15-34]. Thus, the techniques used to get rid of the unphysical degrees of freedom in other gauge theories can also be used for the same purpose in quantize gravity. However, the problem with perturbative quantum gravity is that it is not renormalizable.

2. De Sitter Spacetime

In quantum gravity not all the degrees of freedom are not physical. This problem can be solved by using BRST quantization. In BRST quantization first a gauge is fixed using the gauge fixing term. Then a ghost term is added to this gauge fixing term. The sum of the original action, the gauge fixing term and the ghost term is invariant under a BRST symmetry. The addition of the ghost term ensures that the theory remains unitary [35-44]. It may be noted that many exotic application of quantum field theory have been recently analysed [46-106]. The Hawking radiation was predicted by analysing quantum field theory in the background of a black hole. It may be noted that de Sitter spacetime has an event horizon. The temperature of de Sitter spacetime is proportional to the Hubble parameter. As de Sitter spacetime is a spacetimes of constant curvature, so we have $2R_{abcd}=R\left[g_{ac}g_{bd}-g_{ad}g_{bc}\right]$. Thus we can write $R_{bd}=\frac{1}{4}Rg_{db}$. Finally, we also have Now the Einstein tensor $-4G_{ab}=g_{ab}R$.

Now we will relate de Sitter spacetime to four sphere. This can be done by writing the metric on de Sitter spacetime as

$$ds^2=-dt^2+\left\{\begin{array}{l} r^2\cosh^2 r^{-1}t \\ \times\left[d\chi^2+\sin^2\chi\left(d^2\theta+\sin^2\theta d\phi\right)\right]\end{array}\right\} \quad (1)$$

This metric can also be expressed in terms of a metric on a three-sphere $d\Omega$,

$$ds^2=r^2\left[-dt^2+\cosh^2 t d\Omega\right]. \quad (2)$$

Now by defining Φ as,

$$\Phi=\frac{\pi}{2}-it, \quad (3)$$

this metric can also be written as

$$ds_4^2=r^2\left[d\Phi^2+\sin\Phi^2 d\Omega\right]. \quad (4)$$

This is the metric on a four sphere. So, we will now analyse different harmonic modes on a four sphere.

3. Scalar Modes on a Four Sphere

In order to study harmonic modes on four sphere, we will need the associated Legendre function $P_L^{-l}(x)$. It is defined as

$$P_L^{-l}(x)=\frac{1}{\Gamma(1+l)}\left(\frac{1-x}{1+x}\right)^{\frac{l}{2}}F\left(-L,L+1,l+1,\frac{1-x}{2}\right). \quad (5)$$

where

$$F(a,b,c,x)=1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{2!c(c+1)}x^2+\cdots. \quad (6)$$

So we have

$$\left(\left(1-x^2\right)\frac{d}{dx}+Lx\right)P_L^{-l}(x)=(L-l)P_{L-1}^{-l}(x), \quad (7)$$

and

$$\left(\left(1-x^2\right)\frac{d}{dx}-(L+1)x\right)P_L^{-l}(x)=-(L+l+1)P_{L+1}^{-l}(x). \quad (8)$$

Now we can write

$$D_m=\frac{d}{d\chi}+m\cot\chi. \quad (9)$$

So we have

$$\begin{aligned}&\left[\frac{d}{d\chi}+m\cot\chi\right](\sin\chi)^n f(\chi)\\ &=\sin^n\chi\left[\frac{d}{d\chi}+(m+n)\cot\chi\right]f(\chi),\end{aligned} \quad (10)$$

and

$$D_m\sin^n\chi f(\chi)=\sin^n\chi D_{m+n}f(\chi). \quad (11)$$

Now we get

$$-\sin\chi D_n=\left[\left(1-\cos\chi^2\right)\frac{d}{d\cos\chi}-n\cos\chi\right]. \quad (12)$$

Finally, we have

$$-\sin\chi D_{-L}P_L^{-l}(\cos\chi)=(L-l)P_{L-1}^{-l}(\cos\chi), \quad (13)$$

$$-\sin\chi D_{L+1}P_L^{-l}(\cos\chi)=-(L+l+1)P_{L+1}^{-l}(\cos\chi). \quad (14)$$

The one dimensions scalar spherical harmonics are given by

$$-\nabla_1^2 Y_m=m^2 Y_m, \quad (15)$$

where

$$Y_m=\frac{1}{\sqrt{2\pi}}\exp(im\phi). \quad (16)$$

In two dimensions we have

$$\begin{aligned}-\nabla_1^2 Y_{pm}&=m^2 Y_{pm},\\ -\nabla_2^2 Y_{pm}&=p(p+1)Y_{pm},\end{aligned} \quad (17)$$

where

$$Y_{pm}=c_2 P_p^{-m}Y_m. \quad (18)$$

In three dimensions, we get

$$\begin{aligned}-\nabla_1^2 Y_{lpm}&=m^2 Y_{lpm},\\ -\nabla_2^2 Y_{lpm}&=p(p+1)Y_{lpm},\\ -\nabla_3^2 Y_{lpm}&=l(l+2)Y_{lpm},\end{aligned} \quad (19)$$

where

$$Y_{lpm}=c_3(\sin\chi)^{1/2}P_{l+1/2}^{-p-1}Y_{pm}. \quad (20)$$

In four dimensions, we get

$$\begin{aligned}
-\nabla_1^2 Y_{Llpm} &= m^2 Y_{Llpm}, \\
-\nabla_2^2 Y_{Llpm} &= p(p+1) Y_{Llpm}, \\
-\nabla_3^2 Y_{Llpm} &= l(l+2) Y_{Llpm}, \\
-\nabla_4^2 Y_{Llpm} &= L(L+3) Y_{Llpm}.
\end{aligned} \tag{21}$$

where

$$Y_{Llpm} = c_4 \sin\chi P_{L+1}^{-l-1} Y_{lpm}. \tag{22}$$

Where

$$\int d^4x \sqrt{g} Y_{Llpm} Y^*_{L'l'p'm'} = \delta_{LL'} \delta_{ll'} \delta_{pp'} \delta_{mm'}. \tag{23}$$

So we have

$$c_4 = \left[\frac{(2L+3)(L+l+2)!}{2(L+l)!} \right]^{\frac{1}{2}}. \tag{24}$$

4. Vector and Tensor Modes on a Four Sphere

The vector spherical harmonics on S^4 are given by

$$\begin{aligned}
-\nabla_n^2 A_a &= \left[L(L+3)-1\right] A_a, \\
\nabla^a A_a &= 0.
\end{aligned} \tag{25}$$

The two kinds of solutions to these equations are written as A^n where $n = 0,1$.

$$A_\chi^1 = 0, \tag{26}$$

$$A_i^1 = n_1 P_{L+1} Y_i, \tag{27}$$

$$A_\chi^0 = n_2 (\sin\chi)^{-2} P_{L+1} Y, \tag{28}$$

$$A_i^1 = n_2 \frac{1}{\ell(\ell+2)} D_1 P_{L+1} \nabla_i Y. \tag{29}$$

Finally, we also have

$$\int d^4x \sqrt{g} g^{ab} A_a^L A_b^{*L'} = \delta^{LL'}. \tag{30}$$

Tensor spherical harmonics $E_{ab}^{Llp\cdots}$ on S^n are defined by the following equations

$$\begin{aligned}
-\nabla_n^2 E_{ab}^{Llp\cdots} &= \left[L(L+n-1)-2\right] E_{ab}^{Llp\cdots}, \\
\nabla_n^a E_{ab}^{Llp\cdots} &= 0, \\
E_{ab}^{a\,Llp\cdots} &= 0.
\end{aligned} \tag{31}$$

So the tensor spherical harmonics on S^4 will be defined by the following equations,

$$\begin{aligned}
-\nabla_n^2 E_{ab} &= \left[L(L+3)-2\right] B_{ab}, \\
\nabla_n^a E_{ab} &= 0, \\
E_a^a &= 0.
\end{aligned} \tag{32}$$

There are three kind of solutions to these equations. They will be denoted by E_{ab}^n where $n = 0,1,2$. They are given by

$$E_{\chi\chi}^2 = E_{\chi i}^2 = 0, \tag{33}$$

$$E_{ij}^2 = n_3 (\sin\chi)^{-1} P_{L'+1} Y_{ij}, \tag{34}$$

$$E_{\chi\chi}^1 = 0, \tag{35}$$

$$E_{\chi i}^1 = n_4 (\sin\chi)^{-1} P_{L+1} Y_i, \tag{36}$$

$$E_{ij}^1 = n_4 \frac{1}{(\ell-1)(\ell+3)} \sin\chi D_2 P_{L'+1} \left(\nabla_i Y_j + \nabla_j Y_i \right), \tag{37}$$

$$E_{\chi\chi}^0 = n_5 (\sin\chi)^{-3} P_{L'+1} Y, \tag{38}$$

$$E_{\chi i}^0 = n_5 \frac{1}{\ell(\ell+2)} (\sin\chi)^{-1} D_1 P_{L'+1} \nabla_i Y, \tag{39}$$

$$E_{ij}^0 = n_5 \left(T_{ij} b_1 + \eta_{ij} b_2 \right), \tag{40}$$

$$b_1 = \frac{3}{2(\ell+3)(\ell-1)} \begin{pmatrix} -\frac{1}{3}(\sin\chi)^{-1} \\ +\frac{1}{\ell(\ell+2)} \sin\chi D_2 D_1 \end{pmatrix} P_{L'+1}, \tag{41}$$

$$b_2 = \frac{1}{3} \sin^{-1}\chi P_{L'+1}, \tag{42}$$

$$T_{ij} = \nabla_i \nabla_j + \frac{\ell(\ell+2)}{3} \eta_{ij}. \tag{43}$$

where n_3, n_4 and n_5 are normalization constants, chosen such that

$$\int d^4x \sqrt{g} g^{bd} g^{ac} E_{ab}^L E_{cd}^{*L'} = \delta^{LL'}. \tag{44}$$

The solutions $n = 0,1$ here are some what similar to the two kind of solution in the vector case.

5. Conclusion

In this paper, we analysed the scalar harmonics on a four sphere. These scalar harmonics where used to construct two types of vector harmonics on a four sphere. They were also used to construct three types of tensor harmonic on a four sphere. The results of this paper, can be used for calculating the mode expansion of various fields on de Sitter spacetime. The results thus obtained can be used to study two-point function on de Sitter spacetime. This can have important applications in inationary cosmology.

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