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Two Numerical Methods to Solve the Second Order Multi-pantograph Equation with Boundary Conditions

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Abstract In this article, we present two numerical methods to solve the second order multi-pantograph equation with boundary conditions. The multi-pantograph equation is converted to an integral equation then the integral equation is solved by two projective methods. Some properties of Chebyshev polynomials are employed to prove the convergence analysis of the two proposed methods. Finally, numerical examples also are given to illustrate the efficiency and validity of the two proposed methods.

Keywords: *Multi-pantograph equation, Integral equation, Chebyshev polynomials*

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1. Introduction

The following second order delay differential

$$\begin{cases} y''(t)+p(t)y'(t)+\sum_{i=1}^{m} g_i(t)y(q_i t) = f(t), \quad t\in[0,b] \\ y(0)+y(b)=0, \\ y'(0)+y'(b)=0, \end{cases} \tag{1}$$

is called multi-pantograph equation with anti-periodic boundary conditions when $0<q_i<1$, $i=1,\dots,m-1$, $q_m=1$ and $m\in\mathbb{N}$. The first order of the multi-pantograph equation has been solved by many methods such as Bessel collection method [12] by Yüzbaşi et al. the Taylor approximation method [6,7,8] by Sezer et al. and θ-method [4] by Liu and Li. Saadatmandi and Dehghan applied the variational iteration method in [5] and Yu in [15] as well. Homotopy method has been employed in [10] by Yusufoğlu. Brunner at al. [2] applied discontinuous Galerkin method to solve the delay differential equations of pantograph type and explained convergence analysis of the method completely. Some other cases of the neutral delay differential equation have been studied in [13,14] by Yüzbaşi et al. The system of multi-pantograph equation of the first order has been solved by Bessel collocation method in [11]. The present paper introduces two projective methods to solve the second order of multi-pantograph equation with anti periodic boundary conditions. We convert multi-pantograph equation (1) to an integral equation and solve it by two projective methods. The rest of the paper is arranged as fallows: Section 2 introduces preliminaries of the method and describes the method which is to convert problem (1) to an integral equation also the integral equation is solved by two projective methods. Section 3 proves the convergence analysis of the two projective methods. The last section illustrates numerical examples to confirm the theory.

2. Preliminaries and the Method

Multi-pantograph equation (1) with anti-periodic boundary conditions can be converted to the following integral equation, for convenience assume that $m=3$ and $0<q_1<q_2<q_3=1$. The value m is fixed throughout the paper, $m=3$. Consider

$$u(t) = f(t) + \int_0^b K(s,t)u(s)ds, \tag{2}$$

when

$$\begin{cases} K(s,t) = Z(s,t)+H(s,t), \\ H(s,t) = \dfrac{p(t)}{2} + \sum_{i=1}^{3} \dfrac{g_i(t)}{4}(-b+2s-2q_i t), \end{cases}$$

and

$$Z(s,t) = \begin{cases} \sum_{i=1}^{3} g_i(t)(q_i t - s), & 0<s<q_1 t, \\ \sum_{i=2}^{3} g_i(t)(q_i t - s), & q_1 t<s<q_2 t, \\ & q_2 t<s<t, \\ t-s, & \\ -p(t), & t<s<b. \end{cases}$$

If $u(t)$ is the solution of integral equation (2) then the following formula yields the solution of multi-pantograph equation(1).

$$y(t) = \frac{b+2t}{-4}\int_0^b u(s)ds + \frac{1}{2}\int_0^b su(s)ds + \int_0^b (t-s)u(s)ds. \quad (3)$$

To obtain the solution of multi-pantograph equation (1) with anti-periodic boundary conditions, it is sufficient to solve integral equation (2) and considering (3). Therefore, we solve integral equation (2) by two projective methods and obtain the approximate solution of (1) by approximate solution (2). Integral equation (2) can be converted to the same integral equation on $[-1,1]$ by changing variable, then, without loss of generality, assume that $[0,b]=[-1,1]$. To present the two projective methods, we have to introduce Chebyshev polynomials. Let T_j denotes Chebyshev polynomial of degree j as follows:

$$\begin{aligned} &T_j(t) = \cos(j\arccos(t)), \quad -1 \le t \le 1, \\ &t_i = \cos\left(\frac{(2i-1)\pi}{2j}\right), \quad i = 1,\ldots,j. \end{aligned} \quad (4)$$

Note that t_i's are the zeros of $T_j(t)$. Let V_n be the polynomial space of degree n. Consider the two projections as follows:

$$\begin{cases} \prod_n^G : C[-1,1] \to V_n, \\ \prod_n^G f(t) = \sum_{i=0}^n \frac{2}{\pi}\langle f, T_i\rangle T_i(t), \\ \prod_n^I : C[-1,1] \to V_n, \\ \prod_n^I f(t) = \sum_{i=0}^n f(t_i) l_i(t), \end{cases}$$

when t_i's are the zeros of T_n introduced in (4) and l_i is Lagrange polynomial

$$l_i(x) = \prod_{j=0, j\ne i}^{n} \frac{(x-t_j)}{(t_i-t_j)}, \quad 0 \le i \le n.$$

Conside $u_n(t) := \sum_{i=0}^n c_i T_i(t)$ as an approximation solution of (2) in the finite dimensional space V_n. We will obtain the unknown coefficients c_i, $0 \le i \le n$, by two projective methods. Integral equation (2) yields

$$\sum_{i=0}^n c_i T_i(t) = f(t) + \int_0^b K(t,s) \sum_{i=0}^n c_i T_i(s) ds. \quad (5)$$

Taking projection $\prod_n^G$ on both sides of (5) implies

$$\begin{aligned} &\prod\nolimits_n^G \Big(\sum_{i=0}^n c_i T_i(t)\Big) \\ &= \prod\nolimits_n^G (f(t)) + \prod\nolimits_n^G \Big(\int_0^b K(t,s)\sum_{i=0}^n c_i T_i(s) ds\Big). \end{aligned} \quad (6)$$

Both sides of the above equation are two polynomials of degree n, this equality of polynomials yields a system of $(n+1)$ equations with $(n+1)$ unknowns $c_i, i=0,\ldots,n$ then we can obtain unknowns $c_i, i=0,\ldots,n$ by solving a linear system. This method is called Galerkin method. We have another linear system with unknowns $c_i, i=0,\ldots,n$ if we apply the projection $\prod_n^I$ instead of $\prod_n^G$ in (6). This projective method is called collocation method. Both projective methods solve integral equation (2) approximately and the approximate solution converges to the exact solution of (2). By considering (3) and the approximate solution of (2), we will obtain the approximate solution of (1), which converges to the exact solution (1). We will prove this fact in the next section.

3. Convergence Analysis

This section proves that the approximate solutions of two projective methods converge to the exact solution of integral equation (2). This fact will be presented by theorem 3.1 and corollary 3.2. The approximate solution of the integral equation yields an approximate solution of multi-pantograph equation with anti-periodic boundary conditions (1) which converges to the exact solution of (1), it is the result of theorem 3.3.

Theorem 3.1. Assume that $\mathcal{K}: V \to V$, is bounded operator

$$\mathcal{K}u(t) = f(t) + \lambda\int_a^b K(s,t)u(s)ds,$$

and assume $\lambda - \mathcal{K}: V \to V$ is one to one and onto. Further assume $\|\mathcal{K} - P_n\mathcal{K}\| \to 0$ *as* $n \to \infty$, where P_n is a projection $P_n : V \to V_n$ and V_n is a finite dimensional space. Then for all sufficiently large n, say $n \ge N$, the operator $(\lambda - P_n\mathcal{K})^{-1}$ exists as a bounded operator. Moreover, it is uniformly bounded:

$$\sup_{n\ge N} \left\|(\lambda - P_n\mathcal{K})^{-1}\right\| < \infty.$$

For the approximate solution u_n and u of

$$(\lambda - P_n\mathcal{K})u_n = P_n f, u_n \in V, \quad (7)$$

and $(\lambda - \mathcal{K})u = f$ respectively, we have

$$\frac{|\lambda|}{\|\lambda - P_n\mathcal{K}\|}\|u - P_n u\| \le \|u - u_n\| \le |\lambda| \left\|(\lambda - P_n\mathcal{K})^{-1}\right\| \|u - P_n u\|.$$

Proof: This theorem has been presented by Atkinson and Han [1] page 479.

Corollary 3.2. Assume that $[a,b]=[-1,1]$, $V = C^2[-1,1]$ and V_n is the polynomial space of degree n. If u_n is the approximate solution of (7) and u is the exact solution of (2), then u_n converges to u in the two following cases:

i) $P_n = \prod_n^G$,

ii) $P_n = \prod_n^I$,

where $\prod_n^G$ and $\prod_n^I$ introduced in the previous section.

Proof: Theorem 3.1 yields

$$\|u_n - u\| \le M \|P_n u - u\|,$$

it is sufficient to show that $\|P_n u - u\|$ converges zero in two cases $P_n = \prod_n^I$ and $P_n = \prod_n^G$. Trefethen proved

$\left\|\prod_n^G u-u\right\|$ converges zero in the recent paper [9]. It is obvious that $\prod_n^I$ is interpolation projection at zeros of Chebyshev polynomials, then, $\left\|\prod_n^I u-u\right\|$ converges to zero when the sequence $\left\|u^{(n)}\right\|$ is uniformly bounded (See Burden and Faires [3] page 524).

Theorem 3.3. Let u be the solution of integral equation (2) then the function y in (3) is a solution of the multi-pantograph equation with anti-periodic boundary conditions (1). Moreover, if u_n is an approximate solution of integral equation (2) and y_n is approximate solution (1) which is obtained by substituting u_n in (3), then the approximate solution y_n converges to the exact solution (1), when

$$\|u_n-u\|\to 0, n\to\infty.$$

Proof. Let u be the exact solution of integral equation (2). It is clear that the defined function y in (3) satisfies the multi-pantograph equation with anti-periodic boundary conditions (1). If y_n is an approximate solution (1), which is obtained by (3) and u_n (approximate solution (2)) then the following inequality is clear by considering (3),

$$\|y_n-y\|\le C\|u_n-u\|,$$

where C is a constant independent of n. The above inequality proves that the approximate solution y_n converges to the exact solution of the multi-pantograph equation with anti-periodic boundary conditions.

Let $u_n=\sum_{i=0}^{n} c_i T_i$ be the solution of the following system:

$$u_n(t)-P_n\mathcal{K}u_n(t)=P_n f(t),$$

where $P_n=\prod_n^I$ or $P_n=\prod_n^G$, then corollary 3.2. and theorem 3.3. imply that y_n converges to the exact solution of multi-pantograph equation (1).

4. Numerical Examples

This section confirms the theory of the two proposed methods by illustrating numerical examples. The tables show the error of the methods ($\|y_n-y\|_\infty$) and CPU time. All computations of the following examples have been run by Maple 15.Software.

Example 4.1. Consider the following multi-pantograph equation with anti-periodic boundary conditions,

$$\begin{cases} y''(t)+\sin(t)y'(t)+\sum_{i=0}^{3} g_i(t)y(q_i t)=f(t), & t\in[0,\pi],\\ y(0)+y(\pi)=0, \\ y'(0)+y'(\pi)=0, \end{cases}$$

where

$$\begin{cases} g_1(t)=\exp(t), & q_1=0.5,\\ g_2(t)=t^2, & q_2=0.75,\\ g_3(t)=\frac{1}{4}\cos(\frac{2t}{5}), & q_3=1, \end{cases}$$

and

$$\begin{aligned} f(t)=&1/32(-8\cos(t)+4t^2-4\pi t)\cos(2t/5)\\ &-t^2\cos(3t/4)-\exp(t)\cos(t/2)\\ &-\cos^2(t)+\cos(t)+t/8(t-2\pi)\exp(t)\\ &+(t-\pi/2)\sin(t)+9t^4/4+3\pi t^3/8+2 \end{aligned}$$

The exact solution is $y(t)=-\cos(t)+\frac{t^2}{2}-\frac{-\pi t}{2}$. Table 1 presents the error of the methods and CPU time. Figures 1. and figure 2. illustrate the error of Galerkin method and collocation method respectively.

Table 1. Error and CPU time

N	Collocation error	Collocation CPU time	Galerkin error	Galerkin CPU time
4	4.90 e-02	0.125	2.86 e-02	5.491
8	3.89 e-06	0.109	2.59 e-06	28.751
16	5.47 e-16	0.672	4.50 e-16	317.571
32	3.63 e-40	3.642	1.46 e-36	7161.235
64	1.53 e-97	35.32	-	-
128	5.03 e-231	305.29	-	-

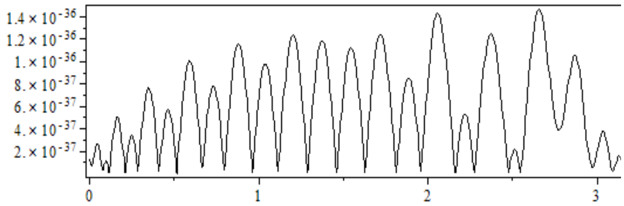

Figure 1. Ex.4.1. Galerkin method N=32

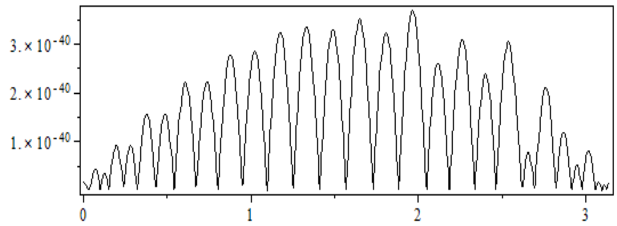

Figure 2. Ex.4.1. Collocation method N=32

Example 4.2. Consider the following multi-pantograph equation with anti-periodic boundary conditions**,**

$$\begin{cases} y''(t)+\sin(2t)y'(t)+\sum_{i=0}^{3} g_i(t)y(q_i t)=f(t), & t\in[0,\pi],\\ y(0)+y(\pi)=0, \\ y'(0)+y'(\pi)=0. \end{cases}$$

where

$$\begin{cases} g_1(t) = 0.25 + 0.5t^2, & q_1 = 0.15, \\ g_2(t) = \exp(t), & q_2 = 0.45, \\ g_3(t) = t^2, & q_3 = 1. \end{cases}$$

and

$$f(t) = ((\cos(t) + \sin(t))\exp(t) + 7.04t)\sin(2t)$$
$$-26.06t^2 + 2.70 + \exp(t)\sin(0.45t)\exp(0.45t)$$
$$+(0.25 + 0.5t^2)\sin(0.15t)\exp(0.15t) +$$
$$(2\cos(t) + t^2\sin(t) - 17.38 + 0.71t^2)\exp(t) + 3.56t^4$$

The exact solution of the multi-pantograph with boundary condition is $y(t) = \sin(t)\exp(t) + (\pi/4 - t/2\pi)(1 - \exp(\pi))$.

Table 2 shows the error of the methods and CPU time. Figures.3 and figures.4 show the error of Galerkin method and collocation method respectively.

Table 2. Error and CPU time

N	Collocation error	Collocation CPU time	Galerkin error	Galerkin CPU time
4	4.90 e-01	0.078	2.95 e-01	3.61
8	7.20 e-04	0.107	3.86 e-04	15.437
16	1.36 e-12	0.547	8.49 e-13	137.406
32	2.29 e-34	3.485	1.24 e-34	393.594
64	3.20 e-87	31.730	-	-
128	4.50 e-211	307.141	-	-

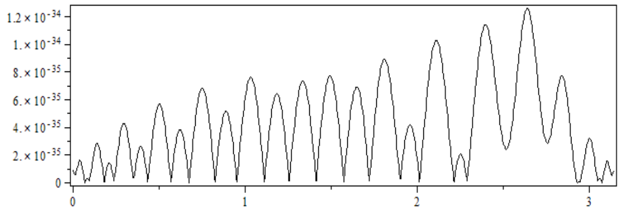

Figure 3. Ex. 4.2. Galerkin method N=32

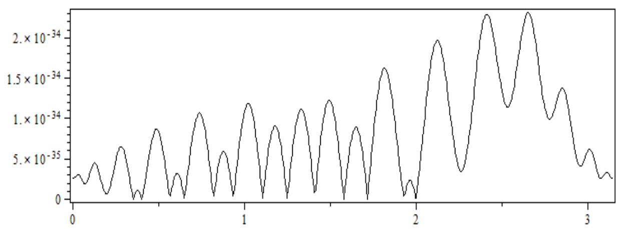

Figure 4. Ex. 4.2. Collocation method N=32

Figure 5 and Figure 6 illustrate the error of the collocation method for Example 4.1-4.2 and $N = 128$.

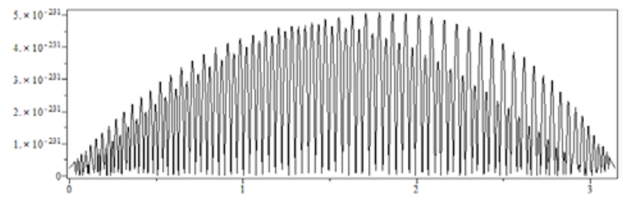

Figure 5. Ex.4.1. Collocation method N=128

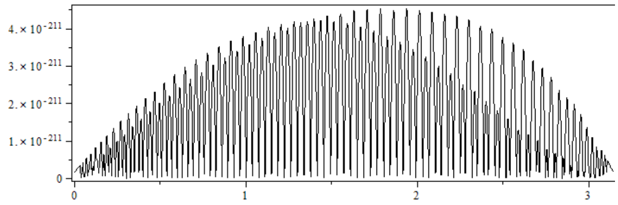

Figure 6. Ex.4.2. Collocation method N=128

5. Conclusion

In this study, we convert the second order multi-pantograph equation with anti-periodic boundary conditions to an integral equation then two projective methods are proposed to solve the integral equation. Some properties of interpolation and Chebyshev polynomials prove the convergence analysis of the two proposed methods. The numerical examples show that the errors of the two methods are same approximately, but the collocation method spends CPU time less than Galerkin method.

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