

A New Approximation Method for the Systems of Nonlinear Fredholm Integral Equations

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Abstract In this paper, we present a new approximate method for solving systems of nonlinear Fredholm integral equation. This method is based on, first, differentiating both sides of integral equations n times and then substituting the Taylor series the unknown functions in the resulting equation and later, transforming to a matrix equation. By merging these results, a new system which corresponds to a system of linear algebraic equations is obtained. The solution of this system yields the Taylor coefficients of the solution function. Numerical results and comparisons with the exact solution are included to demonstrate the validity and applicability of the technique.

Keywords: nonlinear Fredholm systems, Taylor polynomials and series

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1. Introduction

We are concerned with the systems of s nonlinear integral equations systems of Fredholm type in the form

$$\sum_{j=1}^s a_{mj}(x) y_j(x) = f_m(x) + \int_a^b \sum_{j=1}^s K_{mj}(x,t) [y_j(t)]^p dt, \quad (1a)$$

$$m = 1, 2, \dots, s; a \leq c \leq x$$

where $a_{mj}(x), m, j=1, 2, \dots, s, f_m(x), K_{mj}(x,t)$ are functions having n th derivatives on an interval $a \leq x, c \leq b$; and the solutions is expressed in the form

$$y_m(x) = \sum_{n=0}^N \frac{y_m^{(n)}(c)}{n!} (x-c)^n, \quad (2)$$

$$m = 1, 2, \dots, s, \quad a \leq x, c \leq b$$

which is a Taylor polynomial of degree N at $x=c$, where $y_m^{(n)}(c), n=0, 1, \dots, N$ are the coefficients to be determined.

The literatures of systems of nonlinear integral equation contain few numerical method. So, many different methods have been used to approximate the solution of the integral equation system. On the grounds that few of these equations can be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [1,2]. Furthermore, there are also expansion methods for integral equations such as El-gendi's, Wolfe's and Galerkin methods [3]. Conversely, the solution of integral equations system which occurs in physics [4], biology [5] and engineering [6,7] is based on numerical integration methods such as Euler-Chebyshev [8] and Runge-Kutta [9]

methods, and also in a recent research, the first-order linear Fredholm integral equations system is solved by using rationalized Haar functions method [10] and by Galerkin methods with hybrid functions [11].

Additionally, application of HPM (Homotopy Perturbation Method) and ADM (Adomian Decomposition Method) in nonlinear problems has been undertaken by scientists and engineers. In this paper, we consider Taylor method for solving nonlinear systems of Fredholm equations. This method has been presented by Kanwal and Liu [12] and then it has been extended by Sezer to Volterra integral equations [13] and to differential equations [14]. Similar approach has been used to solve linear Volterra-Fredholm integro-differential equations has been applied by Yalçınbaş and Sezer [15], nonlinear Volterra-Fredholm integral equations by Yalçınbaş [16], high-order linear differential equation system by [17,18] and nonlinear systems of volterra integral equations by Yalçınbaş and Erdem [21]. Thus, the presented method which is an expansion method has been proposed to obtain approximate solution and also analytical solution of systems of higher-order nonlinear integral equations.

2. Fundamental Relations and Solution Method

Let us first write the Eq. (1a) in the form

$$E_m(x) = f_m(x) + F_m(x)$$

or

$$E_m(x) = I_m(x), \quad m = 1, 2, \dots, s \quad (1b)$$

so that

$$E_m(x) = \sum_{j=1}^s a_{mj}(x) y_j(x), m = 1, 2, \dots, s$$

$$I_m(x) = f_m(x) + \int_a^b \sum_{j=1}^s K_{mj}(x,t) [y_j(t)]^p dt,$$

$$m = 1, 2, \dots, s$$

Here the expression $E_m(x)$ and $I_m(x)$, respectively, are called as the first part and second part (or integral part) of equation (1b). To obtain the solution of the given problem in the form of expression (2) we first differentiate equation (1a) equations n times with respect to x to obtain

$$E_m^{(n)}(x) = f_m^{(n)}(x) + F_m^{(n)}(x)$$

or

$$E_m^{(n)}(x) = I_m^{(n)}(x), m = 1, 2, \dots, s \tag{3}$$

and then analyse the expressions $E_m(x)$ and $I_m(x)$.

2.1. Matrix Representation for the First Part

The expression $E_m^{(n)}(x)$ can be more clearly written as

$$E_m^{(n)}(x) = \left[\sum_{j=1}^s a_{mj}(x) y_j(x) \right]^{(n)},$$

$$m = 1, 2, \dots, s; n = 0, 1, 2, \dots, N. \tag{4}$$

Using the Leibnitz's rule (dealing with differentiation of product of functions), simplifying $x=c$ into the resulting relation, we have

$$\mathbf{W}_{11} = \begin{bmatrix} (W_{11})_{00} & (W_{11})_{01} & \dots & (W_{11})_{0s} \\ (W_{11})_{10} & (W_{11})_{11} & \dots & (W_{11})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (W_{11})_{s0} & (W_{11})_{s1} & \dots & (W_{11})_{ss} \end{bmatrix}, \dots, \mathbf{W}_{1s} = \begin{bmatrix} (W_{1s})_{00} & (W_{1s})_{01} & \dots & (W_{1s})_{0s} \\ (W_{1s})_{10} & (W_{1s})_{11} & \dots & (W_{1s})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (W_{1s})_{s0} & (W_{1s})_{s1} & \dots & (W_{1s})_{ss} \end{bmatrix}$$

$$\mathbf{W}_{21} = \begin{bmatrix} (W_{21})_{00} & (W_{21})_{01} & \dots & (W_{21})_{0s} \\ (W_{21})_{10} & (W_{21})_{11} & \dots & (W_{21})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (W_{21})_{s0} & (W_{21})_{s1} & \dots & (W_{21})_{ss} \end{bmatrix}, \dots, \mathbf{W}_{2s} = \begin{bmatrix} (W_{2s})_{00} & (W_{2s})_{01} & \dots & (W_{2s})_{0s} \\ (W_{2s})_{10} & (W_{2s})_{11} & \dots & (W_{2s})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (W_{2s})_{s0} & (W_{2s})_{s1} & \dots & (W_{2s})_{ss} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{W}_{s1} = \begin{bmatrix} (W_{s1})_{00} & (W_{s1})_{01} & \dots & (W_{s1})_{0s} \\ (W_{s1})_{10} & (W_{s1})_{11} & \dots & (W_{s1})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (W_{s1})_{s0} & (W_{s1})_{s1} & \dots & (W_{s1})_{ss} \end{bmatrix}, \dots, \mathbf{W}_{ss} = \begin{bmatrix} (W_{ss})_{00} & (W_{ss})_{01} & \dots & (W_{ss})_{0s} \\ (W_{ss})_{10} & (W_{ss})_{11} & \dots & (W_{ss})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (W_{ss})_{s0} & (W_{ss})_{s1} & \dots & (W_{ss})_{ss} \end{bmatrix} \tag{7}$$

The values $(W_{nm})_{ij}, (n,m=1,2,\dots,1s; i,j=0,1,2,\dots,N)$ are defined by

$$E_m^{(n)}(x) = \left[\sum_{j=1}^s a_{mj}(x) y_j(x) \right]_{x=c}^{(n)}$$

$$= \sum_{j=1}^s \sum_{i=1}^n \binom{n}{i} a_{mj}^{(n-i)}(c) y_j^{(i)}(c),$$

$$m = 1, 2, \dots, s; n = 0, 1, 2, \dots, N. \tag{5}$$

Here the $N+1$ unknown coefficients $y_j^{(0)}(c), y_j^{(1)}(c), \dots, y_j^{(N)}(c); (j=1, 2, \dots, s)$ are Taylor coefficients to be determined and $a_{mj}^{(i)}(c); (m, j = 1, 2, \dots, s)$, respectively, denote the values of the i th derivatives of the functions $a_{mj}^{(i)}(x)$ at $x = c$.

We now write the matrix form of expression (5) as

$$\mathbf{E} = \mathbf{W} \cdot \mathbf{Y} \tag{6}$$

where

$$\mathbf{Y} = \left[y_1^{(0)} y_1^{(1)} \dots y_1^{(N)} y_2^{(0)} y_2^{(1)} \dots y_2^{(N)} \dots y_s^{(0)} y_s^{(1)} \dots y_s^{(N)} \right]^T$$

$$= \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_s \end{bmatrix}$$

and

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \dots & \mathbf{W}_{1s} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \dots & \mathbf{W}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{W}_{s1} & \mathbf{W}_{s2} & \dots & \mathbf{W}_{ss} \end{bmatrix}$$

the elements of which are defined by

$$\begin{aligned}
(W_{11})_{ij} &= \binom{i}{j} a_{11}^{(i-j)}(c), (W_{12})_{ij} = \binom{i}{j} a_{12}^{(i-j)}(c), \dots, (W_{1s})_{ij} = \binom{i}{j} a_{1s}^{(i-j)}(c) \\
(W_{21})_{ij} &= \binom{i}{j} a_{21}^{(i-j)}(c), (W_{22})_{ij} = \binom{i}{j} a_{22}^{(i-j)}(c), \dots, (W_{2s})_{ij} = \binom{i}{j} a_{2s}^{(i-j)}(c) \\
&\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \ddots \qquad \qquad \qquad \vdots \\
(W_{s1})_{ij} &= \binom{i}{j} a_{s1}^{(i-j)}(c), (W_{s2})_{ij} = \binom{i}{j} a_{s2}^{(i-j)}(c), \dots, (W_{ss})_{ij} = \binom{i}{j} a_{ss}^{(i-j)}(c).
\end{aligned} \tag{8}$$

Note in Eq. (8) that for $l < 0$

$$a_{ij}^{(l)}(c) = 0, \quad i, j = 0, 1, 2, \dots, s$$

and for $j < 0$ and $j > i$, $\binom{i}{j} = 0$, where i, j and l are integers. In this case, in Eq. (8), for

$$\begin{aligned}
n < m, (n = 0, 1, 2, \dots, N-1; m = n+1, n+2, \dots, N), \\
(W_{nm})_{ij} = 0.
\end{aligned}$$

Hence, the matrix \mathbf{W} becomes, clearly,

$$\mathbf{W} = \begin{bmatrix}
(W_{11})_{00} & (W_{11})_{01} & \cdots & (W_{11})_{0s} & \cdots & (W_{1s})_{00} & (W_{1s})_{01} & \cdots & (W_{1s})_{0s} \\
(W_{11})_{10} & (W_{11})_{11} & \cdots & (W_{11})_{1s} & \cdots & (W_{1s})_{10} & (W_{1s})_{11} & \cdots & (W_{1s})_{1s} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(W_{11})_{s0} & (W_{11})_{s1} & \cdots & (W_{11})_{ss} & \cdots & (W_{1s})_{s0} & (W_{1s})_{s1} & \cdots & (W_{1s})_{ss} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
(W_{s1})_{00} & (W_{s1})_{01} & \cdots & (W_{s1})_{0s} & \cdots & (W_{ss})_{00} & (W_{ss})_{01} & \cdots & (W_{ss})_{0s} \\
(W_{s1})_{10} & (W_{s1})_{11} & \cdots & (W_{s1})_{1s} & \cdots & (W_{ss})_{10} & (W_{ss})_{11} & \cdots & (W_{ss})_{1s} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(W_{s1})_{s0} & (W_{s1})_{s1} & \cdots & (W_{s1})_{ss} & \cdots & (W_{ss})_{s0} & (W_{ss})_{s1} & \cdots & (W_{ss})_{ss}
\end{bmatrix} \tag{9}$$

2.2. Matrix Representation for the Integral Part

The expression $I_m^{(n)}(x)$ can be more clearly written as

$$I_m^{(n)}(x) = f_m^{(n)}(x) + F_m^{(n)}(x) \tag{10}$$

or

$$I_m^{(n)}(x) = f_m^{(n)}(x) + \sum_{j=1}^s \int_a^b \frac{\partial^n K_{mj}(x,t)}{\partial x^n} Y_j(t) dt,$$

$$m = 1, 2, \dots, s.$$

First, we put $x=c$ in relation (10), thereby in expression (11), become

$$I_m^{(n)}(c) = f_m^{(n)}(c) + \sum_{j=1}^s \int_a^b \frac{\partial^n K_{mj}(x,t)}{\partial x^n} \Big|_{x=c} Y_j(t) dt, \tag{11}$$

$$m = 1, 2, \dots, s.$$

Thereby in expression (11) and then substitute the Taylor expansion of $Y_1(t), Y_2(t), \dots, Y_s(t)$ at $t=c$, i.e.

$$Y_i(t) = \sum_{k=0}^{\infty} \frac{1}{k!} Y_i^{(k)}(c) (t-c)^k, \quad i = 1, 2, \dots, s \tag{12}$$

in the resulting relation. Thus, expression (10) become

$$\begin{aligned}
I_1^{(n)}(c) &= f_1^{(n)}(c) + \sum_{k=0}^{\infty} {}^{11}T_{nk} Y_1^{(k)}(c) + \sum_{k=0}^{\infty} {}^{12}T_{nk} Y_2^{(k)}(c) + \dots + \sum_{k=0}^{\infty} {}^{1s}T_{nk} Y_s^{(k)}(c) \\
I_2^{(n)}(c) &= f_2^{(n)}(c) + \sum_{k=0}^{\infty} {}^{21}T_{nk} Y_1^{(k)}(c) + \sum_{k=0}^{\infty} {}^{22}T_{nk} Y_2^{(k)}(c) + \dots + \sum_{k=0}^{\infty} {}^{2s}T_{nk} Y_s^{(k)}(c) \\
&\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\
I_s^{(n)}(c) &= f_s^{(n)}(c) + \sum_{k=0}^{\infty} {}^{s1}T_{nk} Y_1^{(k)}(c) + \sum_{k=0}^{\infty} {}^{s2}T_{nk} Y_2^{(k)}(c) + \dots + \sum_{k=0}^{\infty} {}^{ss}T_{nk} Y_s^{(k)}(c)
\end{aligned} \tag{13}$$

where

$${}^{mj}T_{nk} = \frac{1}{k!} \int_a^b \frac{\partial^n K_{mj}(x,t)}{\partial x^n} \Big|_{x=c} (t-c)^k dt.$$

The quantities $[Y_j(t)]^{(k)}$ ($k=0, 1, 2, \dots$) can be found from the relation

$$[Y_j(t)]^{(k)} = \sum_{t_1+t_2+\dots+t_p=k} \binom{k}{t_1 t_2 \dots t_p} y^{(t_1)}(c) y^{(t_2)}(c) \dots y^{(t_p)}(c), \quad (14)$$

where $\binom{k}{t_1 t_2 \dots t_p} = \frac{k!}{t_1! t_2! \dots t_p!}$.

Note that the relation (14) can be obtained from the generalised Leibnitz's rule (dealing with differentiation of products of p-functions)

$$[Y_j(t)]^{(k)} = \left[(y_j(c))^p \right]^k = \sum_{t_1+t_2+\dots+t_p=k} \binom{k}{t_1 t_2 \dots t_p} y^{(t_1)}(x) y^{(t_2)}(x) \dots y^{(t_p)}(x).$$

If we take $n, k=0,1,2,\dots,N$, then equation (13) becomes a system of equation which is a system of $s(N+1)$ nonlinear equations for the $s(N+1)$ unknowns $y_1^{(0)}, y_1^{(1)}, \dots, y_1^{(N)}, y_2^{(0)}, y_2^{(1)}, \dots, y_2^{(N)}, \dots, y_s^{(0)}, y_s^{(1)}, \dots, y_s^{(N)}$, where $j=0,1,\dots,s$. These can be solved numerically by standard methods.

In this situation matrix representation for the integral part can be put in a matrix form as

$$I = F + T Y^* \quad (15)$$

where the matrices Y, Y^*, F and T are defined by

$$Y = \begin{bmatrix} y_1^{(0)} & y_1^{(1)} & \dots & y_1^{(N)} & y_2^{(0)} & y_2^{(1)} & \dots & y_2^{(N)} & \dots & y_s^{(0)} & y_s^{(1)} & \dots & y_s^{(N)} \end{bmatrix}^T = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix}$$

$$Y^* = \begin{bmatrix} Y_1^{(0)} & Y_1^{(1)} & \dots & Y_1^{(N)} & Y_2^{(0)} & Y_2^{(1)} & \dots & Y_2^{(N)} & \dots & Y_s^{(0)} & Y_s^{(1)} & \dots & Y_s^{(N)} \end{bmatrix}^T = \begin{bmatrix} Y_1^* \\ Y_2^* \\ \vdots \\ Y_s^* \end{bmatrix}$$

$$F = \begin{bmatrix} f_1^{(0)}(c) & f_1^{(1)}(c) & \dots & f_1^{(N)}(c) & f_2^{(0)}(c) & f_2^{(1)}(c) & \dots & f_2^{(N)}(c) & \dots & f_s^{(0)}(c) & f_s^{(1)}(c) & \dots & f_s^{(N)}(c) \end{bmatrix}^T = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{bmatrix},$$

$$T = \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1s} \\ T_{21} & T_{22} & \dots & T_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ T_{s1} & T_{s2} & \dots & T_{ss} \end{bmatrix}$$

The matrices $T_{nk}, (n, k=1,2,\dots,s)$ are defined by

$$\begin{aligned} T_{11} &= \begin{bmatrix} {}^{11}T_{00} & {}^{11}T_{01} & \dots & {}^{11}T_{0N} \\ {}^{11}T_{10} & {}^{11}T_{11} & \dots & {}^{11}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{11}T_{N0} & {}^{11}T_{N1} & \dots & {}^{11}T_{NN} \end{bmatrix}, T_{12} = \begin{bmatrix} {}^{12}T_{00} & {}^{12}T_{01} & \dots & {}^{12}T_{0N} \\ {}^{12}T_{10} & {}^{12}T_{11} & \dots & {}^{12}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{12}T_{N0} & {}^{12}T_{N1} & \dots & {}^{12}T_{NN} \end{bmatrix}, \dots, T_{1s} = \begin{bmatrix} {}^{1s}T_{00} & {}^{1s}T_{01} & \dots & {}^{1s}T_{0N} \\ {}^{1s}T_{10} & {}^{1s}T_{11} & \dots & {}^{1s}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{1s}T_{N0} & {}^{1s}T_{N1} & \dots & {}^{1s}T_{NN} \end{bmatrix} \\ T_{21} &= \begin{bmatrix} {}^{21}T_{00} & {}^{21}T_{01} & \dots & {}^{21}T_{0N} \\ {}^{21}T_{10} & {}^{21}T_{11} & \dots & {}^{21}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{21}T_{N0} & {}^{21}T_{N1} & \dots & {}^{21}T_{NN} \end{bmatrix}, T_{22} = \begin{bmatrix} {}^{22}T_{00} & {}^{22}T_{01} & \dots & {}^{22}T_{0N} \\ {}^{22}T_{10} & {}^{22}T_{11} & \dots & {}^{22}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{22}T_{N0} & {}^{22}T_{N1} & \dots & {}^{22}T_{NN} \end{bmatrix}, \dots, T_{2s} = \begin{bmatrix} {}^{2s}T_{00} & {}^{2s}T_{01} & \dots & {}^{2s}T_{0N} \\ {}^{2s}T_{10} & {}^{2s}T_{11} & \dots & {}^{2s}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{2s}T_{N0} & {}^{2s}T_{N1} & \dots & {}^{2s}T_{NN} \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ T_{s1} &= \begin{bmatrix} {}^{s1}T_{00} & {}^{s1}T_{01} & \dots & {}^{s1}T_{0N} \\ {}^{s1}T_{10} & {}^{s1}T_{11} & \dots & {}^{s1}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{s1}T_{N0} & {}^{s1}T_{N1} & \dots & {}^{s1}T_{NN} \end{bmatrix}, T_{s2} = \begin{bmatrix} {}^{s2}T_{00} & {}^{s2}T_{01} & \dots & {}^{s2}T_{0N} \\ {}^{s2}T_{10} & {}^{s2}T_{11} & \dots & {}^{s2}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{s2}T_{N0} & {}^{s2}T_{N1} & \dots & {}^{s2}T_{NN} \end{bmatrix}, \dots, T_{ss} = \begin{bmatrix} {}^{ss}T_{00} & {}^{ss}T_{01} & \dots & {}^{ss}T_{0N} \\ {}^{ss}T_{10} & {}^{ss}T_{11} & \dots & {}^{ss}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{ss}T_{N0} & {}^{ss}T_{N1} & \dots & {}^{ss}T_{NN} \end{bmatrix} \end{aligned}$$

2.3. Fundamental Matrix Equations

Substituting the matrix forms (6) and (15) in expression of Eq. (3) at the point $x=c$ we get the matrix form of Eq. (3) as

$$WY = F + TY^* \tag{16}$$

which is a fundamental equation for the ingeral equations system (1a).

Also, if some equations are linear in the system of (1a), the **T** matrix transforms **T₁** matrix and **T₂** matrix for respectively linear part of the system and the nonlinear part of the system.

Similarly, the **W** matrix transforms **W₁** matrix and **W₂** matrix for respectively linear part of the system and the nonlinear part of the system.

So (16) becomes

$$W_1Y + W_2Y^* = F + T_1Y + T_2Y^* \tag{17}$$

From this nonlinear system, the unknown Taylor coefficients $y_m^{(n)}(c)$ ($n=0,1,\dots,N$) are determined and substituted in (2); thus we get the Taylor series solution

$$y_m(x) \cong \sum_{n=0}^N \frac{y_m^{(n)}(c)}{n!} (x-c)^n, \quad m=1,2,\dots,s. \tag{18}$$

3. Accuracy of Solution

We can easily check the accuracy of the solution obtained in the fom (18) as follows. Since the truncated Taylor series (18) or the corresponding polynomial expansion is an approximate solution of Eqs. (1a) and (1b), when the solution $y_m(x)$ are substituted Eqs. (1a) and (1b), resulting equation must be satisfied approximately; that is, for $x = x_r \in [a, b]$, $r = 0, 1, 2, \dots$

$$D(x_r) = |E_m(x_r) - f_m(x_r) - F_m(x_r)| \cong 0$$

or

$$D(x_r) \leq 10^{-k_r} \quad (k_r \text{ is any positive integer}).$$

If $\max(10^{-k_i})=10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $|D(x_r)|$ at each of the points x_r becomes smaller than the prescribed 10^{-k} .

On the other hand, the error function can be estimated by

$$D_N(x) = \sum_{j=1}^s a_{mj}(x) y_j(x) - f_m(x) - \int_a^b K_{mj}(x,t) y_j(t) dt$$

4. Numerical Illustrations

In this section we consider four examples of systems of Fredholm type to illustrate the use of presented method.

Example 1. Let us first consider the nonlinear Fredholm equation system [20] with two unknown

$$\left. \begin{aligned} y_1(x) &= x - \frac{5}{18} + \int_0^1 \frac{1}{3} y_1(t) dt + \int_0^1 \frac{1}{3} y_2(t) dt \\ y_2(x) &= x^2 - \frac{2}{9} + \int_0^1 \frac{1}{3} [y_1(t)]^2 dt + \int_0^1 \frac{1}{3} y_2(t) dt \end{aligned} \right\} \tag{19}$$

and approximate the solution $y_m(x)$ by the Taylor polynomial

$$y_m(x) = \sum_{n=0}^3 \frac{1}{n!} y_m^{(n)}(0) (x)^n, \quad (m=1,2)$$

where $a=0, b=1, c=0, N=3$.

Then, we obtain the matrices **W**, **F**, **T₁** and **T₂** as

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -\frac{5}{18} \\ 1 \\ 0 \\ 0 \\ -\frac{2}{9} \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{18} & \frac{1}{72} & \frac{1}{3} & \frac{1}{6} & \frac{1}{18} & \frac{1}{72} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{18} & \frac{1}{72} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{18} & \frac{1}{72} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using the matrices **W**, **F**, **T₁** and **T₂** we find coefficients $y_m^{(n)}(0)$ are uniquely determined as

$$y_2(x) = \frac{(1)}{0!}x^0 + \frac{(-1)}{1!}x^1 + \frac{(1)}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{(1)}{4!}x^4 + \frac{(-1)}{5!}x^5 + \frac{(1)}{6!}x^6.$$

Now let us find the solution of problem (21) taking $c=0$; $N=6,8,10,12,14$. The comparison of the solutions given above with exact solutions $y_1(x)=e^{2x}, y_2(x)=e^{-x}$ of the problem is given below in Table 1, Table 2 and Figure 1, Figure 4.

Table 1. Comparing the solutions of y_1 and y_2 which has been found for $N=6,8,10,12,14$ at Example 3

i	x_i	EXACT SOLUTION		PRESENT METHOD									
		$y_1(x_i)=e^{2x_i}$	$y_2(x_i)=e^{-x_i}$	$c=0, N=6$		$c=0, N=8$		$c=0, N=10$		$c=0, N=12$		$c=0, N=14$	
				$y_1(x_i)$	$y_2(x_i)$	$y_1(x_i)$	$y_2(x_i)$	$y_1(x_i)$	$y_2(x_i)$	$y_1(x_i)$	$y_2(x_i)$	$y_1(x_i)$	$y_2(x_i)$
0	0	1	1	1	1	1	1	1	1	1	1	1	1
1	0.1	1.221402758	0.904837418	1.221402756	0.9048374181	1.221402758	0.904837418	1.221402758	0.904837418	1.221402758	0.904837418	1.221402758	0.904837418
2	0.2	1.491824698	0.8187307531	1.491824356	0.8187307556	1.491824697	0.8187307531	1.491824698	0.8187307531	1.491824698	0.8187307531	1.491824698	0.8187307531
3	0.3	1.8221188	0.74081822077	1.8221128	0.7408182625	1.822118771	0.7408182207	1.8221188	0.7408182207	1.8221188	0.7408182207	1.8221188	0.7408182207
4	0.4	2.225540928	0.670320046	2.2255404756	0.6703203556	2.225540527	0.6703200467	2.225540926	0.670320046	2.225540928	0.670320046	2.225540928	0.670320046
5	0.5	2.718281828	0.6065306597	2.718055556	0.6065321181	2.718278770	0.6065306648	2.718281801	0.6065306597	2.718281828	0.6065306597	2.718281828	0.6065306597
6	0.6	3.320116923	0.5488116361	3.3192832	0.5488168	3.320100791	0.5488116623	3.320116716	0.5488116362	3.320116923	0.5488116361	3.320116923	0.5488116361
7	0.7	4.055199967	0.4965853038	4.052676356	0.4966003181	4.055133912	0.4965854077	4.055198820	0.4965853043	4.055199953	0.4965853038	4.055199967	0.4965853038
8	0.8	4.953032424	0.4493289641	4.946416356	0.4493667556	4.952807676	0.4493293064	4.953027348	0.4493289661	4.953032343	0.4493289641	4.953032424	0.4493289641
9	0.9	6.049647464	0.4065696597	6.0341032	0.4066548625	6.048983548	0.4065706386	6.049628566	0.4065696671	6.049647081	0.4065696598	6.049647464	0.4065696597
10	1	7.389056099	0.3678794412	7.355555556	0.368055556	7.387301587	0.3678819444	7.388994709	0.3678794643	7.389054567	0.3678794413	7.38905607	0.3678794412

Table 2. Comparison of the error analysis of y_1 and y_2 which has been found for $N=6,8,10,12,14$ at Example 3

i	x_i	Error Analysis									
		$c=0, N=6$		$c=0, N=8$		$c=0, N=10$		$c=0, N=12$		$c=0, N=14$	
		$D_1(x_i)$	$D_2(x_i)$	$D_1(x_i)$	$D_2(x_i)$	$D_1(x_i)$	$D_2(x_i)$	$D_1(x_i)$	$D_2(x_i)$	$D_1(x_i)$	$D_2(x_i)$
0	0	0	0	0	0	0	0	0	0	0	0
1	0.1	2.0E-09	1E-10	0	0	0	0	0	0	0	0
2	0.2	3.42E-07	2.5E-09	1E-09	0	0	0	0	0	0	0
3	0.3	6E-06	4.18E-08	2.9E-08	0	0	0	0	0	0	0
4	0.4	4.6172E-05	3.096E-07	4.01E-07	6.9E-10	2.0E-9	0	0	0	0	0
5	0.5	2.26272E-04	1.4584E-06	3.058E-06	5.1E-09	2.7E-8	0	0	0	0	0
6	0.6	8.33723E-04	5.1639E-06	1.6132E-05	2.62E-08	2.07E-7	1E-10	2E-09	0	0	0
7	0.7	2.523611E-03	1.50143E-05	6.6055E-05	1.039E-07	1.147E-6	5E-10	1.4E-08	0	0	0
8	0.8	6.616068E-03	3.77915E-05	2.24748E-04	3.423E-07	5.076E-6	2E-9	8.1E-08	0	0	0
9	0.9	1.5544264E-02	8.52028E-05	6.63916E-04	9.789E-07	1.8898E-05	7.4E-9	3.83E-07	1E-10	0	0
10	1	3.3500543E-02	1.761144E-04	1.754512E-03	2.5032E-06	6.139E-05	2.31E-8	1.532E-06	1E-10	2.9E-8	0

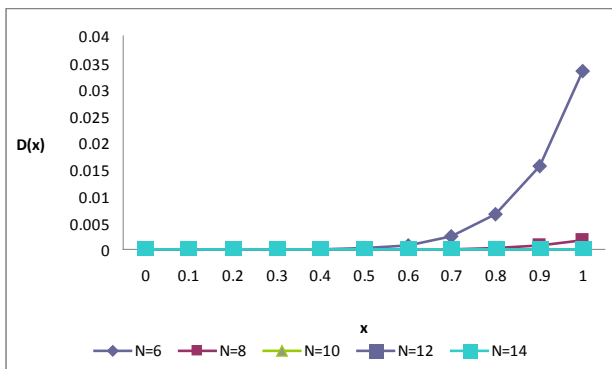


Figure 1. Comparison of the error analysis of y_1 for $N=6,8,10,12,14$

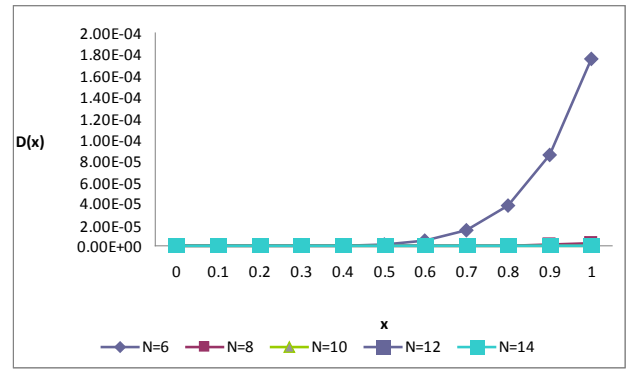


Figure 2. Comparison of the error analysis of y_2 for $N=6,8,10,12,14$

Example 4. Our last example is the nonlinear Fredholm equation system

$$\left. \begin{aligned} e^{-x^2} y_1(x) - e^x y_2(x) &= f_1(x) + \int_0^1 2e^x(y_2(t))dt \\ (\cos x)^{-1} y_2(x) &= f_2(x) + \int_0^1 8x^2(y_2(t))^2 dt \end{aligned} \right\} (22)$$

$$\begin{aligned} f_1(x) &= 1 - e^x + (\cos 1 - \sin 1)e^{x-1} - \cos x \\ f_2(x) &= e^{-x} + x^2((\cos 2 - \sin 2 + 2)e^{-2} - 3). \end{aligned}$$

Applying the previous procedures, we get the approximate solution of problem (22) for $c=0, N=6,8,10$. The comparison of the solutions (for $c=0, N=6,8,10$) with exact solutions $y_1(x)=e^{x^2}$, $y_2(x)=e^{-x} \cos x$ of the problem is given in Table 3, Table 4 and Figure 3, Figure 4.

where

Table 3. Comparing the solutions of y_1 and y_2 which has been found for $N=6,8,10$ at Example 4

i	x_i	EXACT SOLUTION		PRESENT METHOD					
		$y_1(x_i) = e^{x_i^2}$	$y_2(x_i) = e^{-x_i} \cos x_i$	$c=0, N=6$		$c=0, N=8$		$c=0, N=10$	
				$y_1(x_i)$	$y_2(x_i)$	$y_1(x_i)$	$y_2(x_i)$	$y_1(x_i)$	$y_2(x_i)$
0	0	1	1	1	1	1	1	1	1
1	0.1	1.01005016708417	0.900316999845194	1.01008708062198	0.900317	1.01008708103864	0.900316999845238	1.01008708103948	0.900316999845194
2	0.2	1.04081077419239	0.80241064734252	1.04089692213077	0.802410666666667	1.04089702879744	0.802410647365079	1.04089702965077	0.802410647342504
3	0.3	1.09417428370521	0.707730678026351	1.09432351142106	0.707731	1.09432624517106	0.707730678892857	1.09432629437856	0.707730678025
4	0.4	1.17351087099181	0.617405647901646	1.17372201660149	0.617408	1.17374932326815	0.617405659428571	1.17375019708149	0.617405647870194
5	0.5	1.28402541668774	0.532280730215671	1.28420879096151	0.532291666666667	1.28437155137818	0.532280815972222	1.28437968939901	0.532280729855600
6	0.6	1.43332941456034	0.452953789145250	1.43308137725555	0.452992	1.43378121725555	0.452954230857143	1.43383160573555	0.452953786514286
7	0.7	1.63231621995538	0.379809389925154	1.63035851230449	0.379919	1.63276051272116	0.379811155083333	1.63299590876199	0.379809375823765
8	0.8	1.89648087930495	0.31305050400448	1.88944013191458	0.313322666666667	1.89643063858125	0.313056361650794	1.89732542343458	0.313050443761552
9	0.9	2.24790798667647	0.252727753291169	2.22788737611359	0.253333	2.24582350986359	0.252744618892857	2.24872916353109	0.252727536860714
10	1	2.71828182845905	0.198766110346413	2.66832259470441	0.2	2.70998926137107	0.198809523809524	2.71832259470441	0.198765432098765

Table 4. Comparison of the error analysis of y_1 and y_2 which has been found for $N=6,8,10$ at Example 4

i	x_i	Error Analysis					
		$c=0, N=6$		$c=0, N=8$		$c=0, N=10$	
		$D_1(x_i)$	$D_2(x_i)$	$D_1(x_i)$	$D_2(x_i)$	$D_1(x_i)$	$D_2(x_i)$
0	0	0	0	0	0	0	0
1	0.1	3.6914E-05	1.5481E-10	3.6914E-05	4.4E-14	3.6914E-05	0
2	0.2	8.6148E-05	1.9324E-08	8.6255E-05	2.2559E-11	8.6255E-05	1.6E-14
3	0.3	1.4923E-04	3.2197E-07	1.5196E-04	8.6651E-10	1.5201E-04	1.351E-12
4	0.4	2.1115E-04	2.3521E-06	2.3845E-04	1.1527E-08	2.3933E-04	3.1452E-11
5	0.5	1.8337E-04	1.0936E-05	3.4613E-04	8.5757E-08	3.5427E-04	3.6007E-10
6	0.6	2.4804E-04	3.8211E-05	4.518E-04	4.4171E-07	5.0219E-04	2.6310E-09
7	0.7	1.9577E-03	1.0961E-04	2.402E-03	1.7652E-06	2.6374E-03	1.4101E-08
8	0.8	7.0407E-03	2.7216E-04	5.0241E-05	5.8576E-06	8.4454E-04	6.0243E-08
9	0.9	2.0021E-02	6.0525E-04	2.0845E-03	1.6866E-05	8.2118E-04	2.1643E-07
10	1	4.9959E-02	1.2339E-03	8.2926E-03	4.3413E-05	4.0766E-05	6.7825E-07

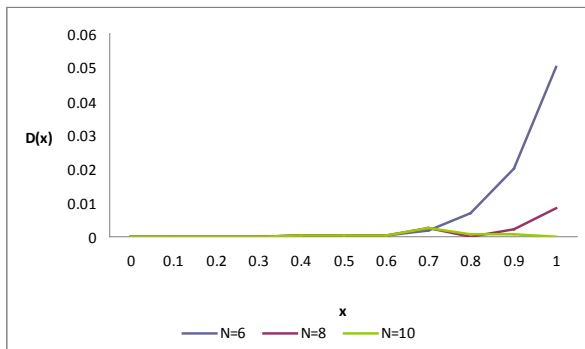


Figure 3. Comparison of the error analysis of y_1 for $N=6,8,10$

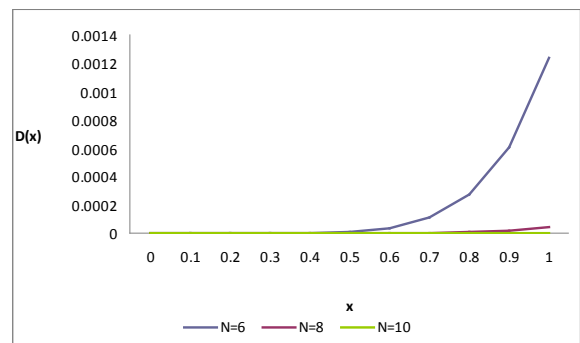


Figure 4. Comparison of the error analysis of y_2 for $N=6,8,10$.

5. Conclusions

In this paper a new approximate method is used to solve nonlinear systems of Fredholm integral equations. We described the method, used in four test problems, and compared the results with their exact solutions in order to demonstrate the validity and applicability of the method. The method can be developed and applied to another high-order linear and nonlinear integro-differential equation systems with variable coefficients.

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