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A Preconditioned ELMRES Implementation

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Abstract In this paper we review the ELMentary RESidual(ELMRES) algorithm for solving linear system of equations. ELMRES is a krylov subspace method which uses the Hessenberg transformation as the projection technique for reducing the dimension of original matrix A. We apply some preconditioned techniques for this algorithm. At the end of this paper, some numerical examples have been shown to compare the preconditioned ELMRES with the original version.

Keywords: *ELMRES, preconditioned technique, Hessenberg algorithm, least square problems*

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1. Introduction

Many applications require the solution of several sparse systems of linear equations

$$Ax = b \qquad (1)$$

with the coefficient matrix $A \in R^{n\times n}$, the right hand side vector $b \in R^n$ and the unknown solution vector $x \in R^n$.

There are several different iterative algorithms for computing the solution of (1). Among them, krylov subspace methods, like algorithms from [1,3,5,6], are more popular solvers for this sort of equations. The krylov subspace $K_k(A, r_0)$ is defined as

$$K_k(A, r_0) = span\left\{ r_0, Ar_0, \ldots, A^{k-1}r_0 \right\}$$

which is a practical subspace for modification of new approximations with $r_0 = b - Ax_0$ where x_0 is the initial guess. ELMRES was proposed by G. Howell and D. Stephens [3] in 2000. The implementation of ELMRES is approximately similar to GMRES so that in both methods the original problem (1) is transferred to an upper Hessenberg least square problem with dimension k which $K \ll n$ whilst GMRES uses the Arnoldi orthogonalization algorithm for reduction and ELMRES uses the Hessenberg decomposition algorithm. The construction of this paper is as follows: In section 2, the ELMRES is discussed. In section 3, the left and right preconditioner methods are described shortly and some popular applicable matrices for these methods which are also practical for ELMRES are shown. In the next section, the proposed preconditioned matrices are applied and then they are tested by some numerical examples.

2. Elmres

Here the ELMRES algorithm is described briefly. In ELMRES method a basis constructs for the krylov subspace by the Hessenberg algorithm. Suppose $l_1 = \frac{r_0}{r_0(1)}$.

Now, the procedure for extending the dimension of the subspace is shown in the following algorithm [3].

2.1. Algorithm 1: Hessenberg Decomposition Algorithm

1. $w = Al_k$ vector r and set $\beta = r(1)$ and $v_1 = r/\beta$.
2. for $i = 1, \cdots, k$
   1. $h_{i,k} = w(i)$,
   2. $w = w - h_{i,k}\, l_i$,

   End
3. $h_{k+1,k} = w(k+1)$, $l_{k+1} = \frac{w}{h_{k+1,k}}$.

In the algorithm 1, it is shown that the ith step of the for loop zeros the ith entry of w where $i = 1, \cdots, k$. At the end of algorithm, the $k+1st$ entry of l_{k+1} becomes 1. In this case, $l_{k+1} \perp \{ e_1, e_2, \ldots, e_k \}$. If the partial pivoting is employed, the relation

$$l_{k+1} \in span\left\{ l_1, Al_1, \ldots, A^k l_1 \right\}, \qquad (2)$$

is concluded,(see [3]).

An elementary similarity transformation is of the form

$$L_i = I_n + \hat{l}_i e_i^T, \; L_i^{-1} = I_n - \hat{l}_i e_i^T, \qquad (3)$$

where $l_i = [0 \ldots 0\; 1\; \hat{l}_i]^T$ is a column vector with zeros as its first $i-1$ entries and e_i is a vector with its only nonzero entry 1 in the ith position.

Without pivoting condition, reduction by elementary similarity transformations to upper hessenberg form is accomplished by

$$H = L_{n-1}^{-1} L_{n-2}^{-1} \cdots L_2^{-1} A L_2 \cdots L_{n-1}, \quad (4)$$

which form the structures of L_i s and upper hessenberg matrix H we have

$$H(:,k) = L_{k+1}^{-1} L_k^{-1} \cdots L_2^{-1} \left(A L_k e_k \right).$$

By definition of $\tilde{V}_k = [\, l_1, l_2, \ldots, l_k \,]$ and $V_k = \left[\, \tilde{V}_k \,|\, I_{n-k} \,\right]$ the following relation is obtained

$$A\tilde{V}_k = \tilde{V}_{k+1} \bar{H}_k . \quad (5)$$

By using this important relation, we reach to the reduced problem

$$V_{k+1}^{-1} \left(b - Ax \right) = V_{k+1}^{-1} r_0 - V_{k+1}^{-1} A \tilde{V}_k y,$$

$$= \begin{pmatrix} \beta e_1 \\ 0 \end{pmatrix} - \begin{pmatrix} \bar{H}_k \\ 0 \quad \alpha \hat{l}_{k+1} \end{pmatrix} y.$$

The kth approximate solution is $x_k = x_0 + \tilde{V}_k \, \bar{y}$ where $\bar{y} = \arg\min \left\| \beta e_1 - \bar{H}_k y \right\|$.

Wilkinson [8] suggested Hessenberg reduction with implicit pivoting avoids the breakdown in decomposition process. Here, the ELMRES with implicit pivoting [3] is shown.

2.2. Algorithm 2: ELMRES with Implicit Pivoting

1. Choose x_0, set $r_0 = b - Ax_0$.
   set p so that $\left| r_0(p) \right| = \max \left| r_0(1 \cdots n) \right|$.
   $p_{ivnew}(1) = p$, $\beta = r_0(p), l_1 = r_0 / \beta$,
   $l_{temp}(1 \cdots n) = l_1$, $l_1(1) \leftrightarrow l_1(p)$,
   $w = A l_{temp}$, $w_1(1) \leftrightarrow w_1(p), h_{2,1} = w_1(2)$,
   $l_2(2 \cdots n) = w_1(2 \cdots n) / h_{2,1}$,
2. For $j = 1, \ldots, k$, until satisfied do
   $l_{temp} = l_{j+1}$,
   For $k_k = j+1 : -1 : 1$,
   $l_{temp}(p_{ivnew}(k_k)) \leftrightarrow l_{temp}(k_k)$,
   End
   $w_{j+1} = A l_{temp}$,
   For $i = 1, \ldots, j+1$ do
   $w_{j+1}(p_{ivnew}(i)) \leftrightarrow w_{j+1}(i)$, $h_{i,j+1} = w_{j+1}(i)$,
   $w_{j+1} = w_{j+1} - h_{i,j+1} l_i$,
   End
   set p so that $\left| w_{j+1}(p) \right| = \max \left| w_{j+1}(j+2, \ldots, n) \right|$.
   $p_{ivnew}(j+2) = p$, $w_{j+1}(j+2) \leftrightarrow w_{j+1}(p)$,
   $h_{j+2,j+1} = w_{j+1}(j+2)$, $l_{j+2} = w_{j+1} / h_{j+2,j+1}$.
   End
   Choose $\bar{y}$ to minimize $\left\| \beta e_1 - \bar{H}_k y \right\|$.
3. From the approximate solution $z = x_0 + \tilde{V}_k \, \bar{y}$.
   For $i = k : -1 : 0$,
   $z(i+2, \cdots, n) = z(i+2, \cdots, n) + z(i+1) \hat{l}_{j+1}$,
   $z(p_{ivnew}(i+1)) \leftrightarrow z(i+1)$
   End
   $x_k = x_0 + z$

For more information about ELMRES and its properties see [3].

3. Preconditioned ELMRES

The structure of GMRES is approximately similar to ELMRES. So, some prediction GMRES techniques may be useful for ELMRES. Here, GMRES with right preconditioners are shown and some suitable matrices for preconditioned ELMRES are suggested.

Let M is a preconditioned matrix. In left preconditioned GMRES, the original equation $Ax = b$ is replaced with $M^{-1}Ax = M^{-1}b$ and the orthogonalization procedure is run so that a normalized basis vectors is obtained for the preconditioned krylov subspace

$$span \left\{ r_0, M^{-1} A r_0, \ldots, \left(M^{-1} A \right)^{k-1} r_0 \right\}$$

by the Arnoldi algorithm. So, we have to apply GMRES algorithm just for the new linear system of equations.

Furthermore, the right preconditioned GMRES algorithm is based on solving

$$AM^{-1} u = b \;,\; u = M\,x . \quad (6)$$

As the right preconditioned GMRES is more practical, its algorithm is written below [6].

2.3. Algorithm 3: Right Preconditioned GMRES

1. Choose x_0, set $r_0 = b - Ax_0$, $\beta = \left\| r_0 \right\|, v_1 = r_0 / \beta$.
2. For $j = 1 : k$
   $v_{j+1} = AM^{-1} v_j$,
   For $i = 1 : j$
   do $h_{i,j} = \left(v_{j+1}, v_i \right)$, $v_{j+1} = v_{j+1} - h_{i,j} v_i$,
   End
   $h_{j+1,j} = \left\| v_{j+1} \right\|$, $v_{j+1} = v_{j+1} / h_{j+1,j}$,
   $J_j \left(J_{j-1} \cdots J_1 \bar{H}_j \right) = \begin{pmatrix} R_j \\ 0 \end{pmatrix}$ and $J_j \left(J_{j-1} \cdots J_1 \left(\beta e_1 \right) \right) = \begin{pmatrix} g_j \\ \gamma_j \end{pmatrix}$,
   If $\gamma_j < eps$ set $k = j$ and goto 3,
   End
3. $y_k = R_k^{-1} g_k$ and $x_k = x_0 + M^{-1} V_k y_k$.

From the above algorithm, the vector u is not computed explicitly and the Arnoldi process generates a basis vector set for the preconditioned subspace

$$span \left\{ r_0, AM^{-1} r_0, \ldots, \left(AM^{-1} \right)^{k-1} r_0 \right\}.$$

In this case, the residual norm is now relative to the initial system (1).

Gauss-Seidel and SOR are two popular classical iterative methods for solving (1) while using the matrices regard to the structure of these methods as some preconditioners are popular techniques for GMRES. Now, we apply the preconditioned matrices M_{GS} and M_{SOR} obtained from Gauss-Seidel and SOR respectively. Suppose $A = L + D + U$ where L, D and U are the elements lower than the diagonal, the diagonal elements and the elements upper than the diagonal of matrix A respectively. So, M_{GS} and M_{SOR} are defined as follows [6],

$$M_{GS} = L + D, \qquad M_{SOR} = D + \omega L. \tag{7}$$

4. Numerical Tests

The In this section, some well-known numerical examples are applied to test the effect of preconditioned techniques on the convergence speed of ELMRES. For simplicity, $x_0 = 0$ is chosen as the initial guess in these experiments. These algorithms are implemented in Matlab 7.3 on a Pentium 4 PC with 1 GB of RAM. The stopping criterion $\| r_k \| < tol = 10^{-12}$ is used for all implementations.

***Example 1*:** For construction of linear system of equations (1), Let M be an even integer and denote by 1 and 0, respectively, the $M/2 \times M/2$ identity and zero matrices. Define also the $M/2 \times M/2$ matrices T_1 and T_2 as in

$$T_1 = \begin{pmatrix} -2 & 0 & \dots & \dots & 0 \\ -1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 0 \\ 0 & \dots & \dots & -1 & -1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} -1 & 0 & \dots & \dots & 0 \\ -1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 0 \\ 0 & \dots & 0 & -1 & -2 \end{pmatrix}.$$

Finally, construct the $M^2 \times M^2$ non-symmetric matrix A as in

$$A = \left(\begin{array}{cccccccccccccccc}
4I & 0 & \dots & \dots & \dots & \dots & \cdots & 0 & T_1 & -2I & 0 & \dots & \dots & \dots & \dots & 0 \\
0 & 4I & & & & & & \vdots & -I & T_2 & -I & 0 & & & & \vdots \\
\vdots & & \ddots & & & & & \vdots & 0 & -I & T_1 & -I & 0 & & & \vdots \\
\vdots & & & \ddots & & & & \vdots & \vdots & 0 & -I & T_2 & -I & 0 & & \vdots \\
\vdots & & & & \ddots & & & \vdots & \vdots & & & & \ddots & & & \vdots \\
\vdots & & & & & \ddots & & \vdots & \vdots & & & & & \ddots & & \vdots \\
\vdots & & & & & & 4I & 0 & \vdots & & & & 0 & -I & T_1 & -I \\
0 & \dots & \dots & \dots & \dots & \dots & 0 & 4I & 0 & \dots & \dots & \dots & \dots & 0 & -2I & T_2 \\
T_2 & -2I & 0 & \dots & \dots & \dots & \dots & 0 & 4I & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
-I & T_1 & -I & 0 & & & & \vdots & 0 & 4I & & & & & & \vdots \\
0 & -I & T_2 & -I & & & & \vdots & \vdots & & \ddots & & & & & \vdots \\
\vdots & 0 & -I & T_1 & -I & 0 & & \vdots & \vdots & & & \ddots & & & & \vdots \\
\vdots & & & & \ddots & & & \vdots & \vdots & & & & \ddots & & & \vdots \\
\vdots & & & & & \ddots & & \vdots & \vdots & & & & & \ddots & & \vdots \\
\vdots & & & & 0 & -I & T_2 & -I & \vdots & & & & & & 4I & 0 \\
0 & \dots & \dots & \dots & \dots & 0 & -2I & T_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 4I
\end{array}\right).$$

The matrix A is obtained by first discretizing the Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) = f(x, y)$$

for

$$(x, y) \in \Omega = [0,1] \times [0,1]$$

with Neumann boundary conditions

$$\frac{\partial}{\partial x} u(x, y) = \phi(x, y) \text{ on } \partial\Omega,$$

on a uniform grid of mesh size $\mathrm{h} = 1/(\mathrm{M}\text{-}1)$ via central differences, and then by taking the unknowns in the red-black ordering. This problem was also considered by Saberi Najafi and Zareamoghaddam in [7] for testing another GMRES implementation. Note that A is singular with a 1D null space spanned by the vector $e = (1, \ldots, 1)^T$.

In our numerical experiments, for the first exam we took M=30 which the corresponding square matrix A is in order $\mathrm{n}{=}900$. Table1shows the numerical results of different ELMRES implementations applied for the linear problem with order 900.

Table 1. Numerical results of example1

Example 1	Error	Iterate
ELMRES	1.2701e-012	23
ELMRES-GS	4.2935e-012	15
ELMRES-SOR	6.8397e-012	15

Table 2. Numerical results of example2

Example 2	Error	Iterate
ELMRES	5.3971e-012	59
ELMRES-GS	9.0382e-012	42
ELMRES-SOR	6.7512e-012	40

***Example 2*:** There are many different numerical examples for system of linear equations which among them, "Regularization Tools" of Prof. Hansen [2] is one popular package for ill-posed linear system of equations. In this example, the main source code of one discrete image deblurring problem is selected from this package [2]. The Matlab code *"blure"* with detailed parameters $N = 10$, $band = 3$ and $sigma = 0.7$, i.e. is used to generate the square matrix $A \in R^{100 \times 100}$ with the right hand side vector $b \in R^{100}$. This problem has been contaminated by blur and noise. Numerical results of this test have been illustrated in Table 2.

5. Conclusion

Elementary residual method is rather a new krylov subspace method for solving linear system of equations. In this paper, some right preconditioned matrices have been tested for ELMRES to solve different linear system of equations and the results show that the convergence of ELMRES has also been influenced by some appropriate preconditioning matrices.

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References

[1] Freund R., Nachtigal N., QMR: A Quasi-Minimal Residual Method for Non-Hermitian Linear Systems, Numer. Math. 60(1991) 315-339.

[2] Hansen P. C., Regularization Tools Version 4.0 for Matlab 7.3, Numerical Algorithms, 46 (2007) 189-194.

[3] Howell G., Stephens D., ELMRES: an oblique projection solver for sparse systems of linear equations, technical report, Florida Institute of Technology, 2000.

[4] Lewis B., Reichel L., Arnoldi-Tikhonov regularization methods, J. Comput. Appl. Math., 226 (2009) 92-102.

[5] Paige C. C., Saunders M. A., LSQR: an algorithm for sparse linear equations and sparse least squares, ACM Trans. Math. Software 8 (1982) 43-71.

[6] Saad Y., Iterative method for sparse linear systems. PWS publishing company, a division of international Thomson publishing Inc, U.S.A, 1996.

[7] Saberi Najafi H., Zareamoghaddam H., A new computational GMRES method, Appl. Math. Comput. 2 (2008) 527-534.

[8] Wilkinson J. H., the Algebraic Eigenvalue Problem. Clarendon Press, Oxford, England, 1965.