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A Study of an Integral Equation Involving the S-Function as its Kernel with Application

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Abstract The aim of present paper is to study of an integral equation involving the S - function as its kernel. We also define Some Special cases of our main result. At the end, application of our preliminary result by connecting it with Riemann-Liouville type fractional integral operator is given.

Keywords: *Riemann-Liouville type fractional integral operator, S-function, Kampe de Feriet function, Fox's H - function, (2000 Mathematics subject classification: 33C99)*

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1. Introduction

The $\overline{H}$ -function occurring in the paper will be defined and represented as follows:

$$\overline{H}_{P,Q}^{M,N}[z]=\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] \tag{1.1}$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi$$

Where

$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,...,P),\beta_j\geq 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,...,N)$ and $B_j(j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$ -function given by equation (1.1) have been given by (Buschman and Srivastava [1]).

$$\Omega\equiv\sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \tag{1.3}$$

$$\left|\arg(z)\right|<\frac{1}{2}\pi\Omega \tag{1.4}$$

The behavior of the $\overline{H}$ -function for small values of $|z|$ follows easily from a result recently given by (Rathie [4], eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}[z]=0\left(|z|^{\gamma}\right),\gamma=\min_{1\leq j\leq N}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right],|z|\to 0 \tag{1.5}$$

If we take $A_j=1(j=1,...,N)$, $B_j=1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [3].

We shall use the following notation:

$$A^*=\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P}$$
$$B^*=\left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q}$$

Srivastava and Daoust [6] have generalized the Kampe de Feriet function as S -function which is defined and represented as[8]:

$$S[x,y]=S_{q_1:q_2:q_3}^{p_1:p_2:p_3}\left[\begin{matrix}1(a_j,\alpha_j,A_j)_{p_1}:1(c_j,C_j)_{p_2}:1(e_j,E_j)_{p_3};\\1(b_j,\beta_j,B_j)_{q_1}:1(d_j,D_j)_{q_2}:1(f_j,F_j)_{q_3};\end{matrix}x,y\right]$$
$$=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{p_1}\Gamma\left(a_j+\alpha_jm+A_jn\right)\prod_{j=1}^{p_2}\Gamma\left(c_j+C_jm\right)\prod_{j=1}^{p_3}\Gamma\left(e_j+E_jn\right)}{\prod_{j=1}^{q_1}\Gamma\left(b_j+\beta_jm+B_jn\right)\prod_{j=1}^{q_2}\Gamma\left(d_j+D_jm\right)\prod_{j=1}^{q_3}\Gamma\left(f_j+F_jn\right)}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.6}$$

The Beta function is defined as:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0 \tag{1.7}$$

2. Preliminary Result

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\overline{H}_{P,Q}^{M,N}\left[cy(1-x)\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]S[c(1-x),$$

$$d(1-x)dx=\sum_{r,w=0}^{\infty}A_{r,w}c^r d^w \tag{2.1}$$

$$\times\Gamma(\beta+r+w)\overline{H}_{P+1,Q+1}^{M,N+1}\left[cy\left|\begin{matrix}(1-\alpha,\rho;1),A^*\\B^*,(1-\alpha-\beta-r-w,\rho;1)\end{matrix}\right.\right]$$

Where,

$$A_{r,w}=\frac{\prod_{j=1}^{p_1}\Gamma\left(g_j+\delta_j m+G_j n\right)\prod_{j=1}^{p_2}\Gamma\left(c_j+C_j m\right)\prod_{j=1}^{p_3}\Gamma\left(e_j+E_j n\right)}{\prod_{j=1}^{q_1}\Gamma\left(h_j+\eta_j m+H_j n\right)\prod_{j=1}^{q_2}\Gamma\left(d_j+D_j m\right)\prod_{j=1}^{q_3}\Gamma\left(f_j+F_j n\right)r!w!} \tag{2.2}$$

Provided

(i) $\mathrm{Re}(\alpha+\rho\frac{b_j}{\beta_j})>0,\ j=1,2,...,m;\mathrm{Re}(\beta)>0,\rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2}D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2}C_j\geq 0;$

$1+\sum_{j=1}^{q_1}H_j+\sum_{j=1}^{q_3}F_j-\sum_{j=1}^{p_1}G_j-\sum_{j=1}^{p_3}E_j\geq 0;$

(iii) $|\arg z|<\frac{1}{2}\pi\Omega$

Where Ω is given by (1.3).

Proof: Express the S - function on the left hand side of (2.1) as a series uses (1.6) and express the $\overline{H}$ -function as a contour integral using (1.1) and then change the order of integration and summation and evaluate the inner integral using (1.7) to get the required result. The change of order of integration and summation is justified, when the given conditions are satisfied because of the absolute convergence of the integrals involved.

3. Main Result

The Integral Equation

(i) $\int_t^1 (u-t)^{\alpha-1}\overline{H}_{P,Q}^{M,N}\left[z(u-t)^{\rho}\right]g(u,t)du=f(t),t\in K$,

has the solution

(ii)

$$g(u,t)=-\int_u^1\frac{(v-u)^{\beta-1}}{h(v,t)}S[c(v-u),d(v-u)]f'(v)dv \tag{3.1}$$

Where

$$h(v,t)=\sum_{r,w=0}^{\infty}A_{r,w}c^r d^w(v-t)^{\alpha+\beta+r+w-1}\Gamma(\beta+r+w)$$

$$\times\overline{H}_{P+1,Q+1}^{M,N+1}\left[c(v-t)^{\rho}\left|\begin{matrix}(1-\alpha,1;1),A^*\\B^*,(1-\alpha-\beta-r-w,1;1)\end{matrix}\right.\right]$$

$K=\{t:a\leq t\leq 1,a>0\}$ and $A_{r,w}$ is given by (2.2), provided:

(i) $\mathrm{Re}(\alpha+\rho\frac{b_j}{\beta_j})>0,\ j=1,2,...,m;\mathrm{Re}(\beta)>0,\rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2}D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2}C_j\geq 0;$

$1+\sum_{j=1}^{q_1}H_j+\sum_{j=1}^{q_3}F_j-\sum_{j=1}^{p_1}G_j-\sum_{j=1}^{p_3}E_j\geq 0;$

(iii) $|\arg z|<\frac{1}{2}\pi\Omega$

Where Ω is given by (1.3).

(iv) $f(1)=0$ and

(v) $f'(t)$ is continuous in K .

Proof: Substituting for $g(u,t)$ from (ii) of (3.1) and changing the order of integration the left hand side of (i) of (3.1) becomes

$$-\int_t^1\frac{f'(v)}{h(v,t)}\left\{\int_t^v(u-t)^{\alpha-1}\overline{H}_{P,Q}^{M,N}\left[z(u-t)^{\rho}(v-u)^{\beta-1}\right]\right. \tag{3.2}$$

$$\left.S[c(v-u),d(v-u)du\right\}dv$$

On putting $u-t=xy,v-t=y$, the inner integral in (3.2) becomes:

$$y^{\alpha+\beta-1}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\overline{H}_{P,Q}^{M,N}\left[cy^{\rho}x^{\rho}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]S[cy(1-x),dy(1-x)dx$$

Which on using (2.1), reduces to $h(v,t)$.

Now (3.2) becomes:

$$-\int_t^1 f'(v)dv=f(t)$$

4. Special Cases:

When $A_j=1=B_j$, the $\overline{H}$ -function reduces to the Fox's H -function and (3.1) reduces to the result:

The integral equation

(i) $\int_t^1 (u-t)^{\alpha-1}H_{P,Q}^{M,N}\left[z(u-t)^{\rho}\right]g(u,t)du=f(t),t\in K$,

has the solution

(ii)

$$g(u,t)=-\int_u^1\frac{(v-u)^{\beta-1}}{h(v,t)}S[c(v-u),d(v-u)]f'(v)dv \tag{4.1}$$

Where

$$h(v,t)=\sum_{r,w=0}^{\infty}A_{r,w}c^{r}d^{w}(v-t)^{\alpha+\beta+r+w-1}\Gamma(\beta+r+w)$$
$$\times H_{P+1,Q+1}^{M,N+1}\left[c(v-t)^{\rho}\left|\begin{matrix}(1-\alpha,1),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(1-\alpha-\beta-r-w,1)\end{matrix}\right.\right]$$

$K=\{t:a\le t\le 1,a>0\}$ and $A_{r,w}$ is given by (2.2), provided:

(i) $\mathrm{Re}(\alpha+\rho\frac{b_j}{\beta_j})>0,\ j=1,2,...,m;\mathrm{Re}(\beta)>0,\rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2}D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2}C_j\ge 0;$

$1+\sum_{j=1}^{q_1}H_j+\sum_{j=1}^{q_3}F_j-\sum_{j=1}^{p_1}G_j-\sum_{j=1}^{p_3}E_j\ge 0;$

(iii) $|\arg z|<\frac{1}{2}\pi\Delta$

Where

$$\Delta\equiv\sum_{j=1}^{M}|\beta_j|+\sum_{j=1}^{N}|\alpha_j|-\sum_{j=M+1}^{Q}|\beta_j|-\sum_{j=N+1}^{P}|\alpha_j|>0$$

(iv) $f(1)=0$ and

(v) $f'(t)$ is continuous in K.

When $\alpha_j=1=\beta_j$, the H-function reduces to the Meijer's G-function and (4.1) reduces to the result:

The integral equation

(i) $\int_t^1(u-t)^{\alpha-1}G_{P,Q}^{M,N}\left[z(u-t)^{\rho}\right]g(u,t)du=f(t),t\in K$,

has the solution

(ii)

$$g(u,t)=-\int_u^1\frac{(v-u)^{\beta-1}}{h(v,t)}S[c(v-u),d(v-u)]f'(v)dv \quad (4.2)$$

Where

$$h(v,t)=\sum_{r,w=0}^{\infty}A_{r,w}c^{r}d^{w}(v-t)^{\alpha+\beta+r+w-1}\Gamma(\beta+r+w)$$
$$\times G_{P+1,Q+1}^{M,N+1}\left[c(v-t)^{\rho}\left|\begin{matrix}(1-\alpha,1),(a_j,1)_{1,P}\\(b_j,1)_{1,Q},(1-\alpha-\beta-r-w,1)\end{matrix}\right.\right]$$

$K=\{t:a\le t\le 1,a>0\}$ and $A_{r,w}$ is given by (2.2), provided:

(i) $\mathrm{Re}(\alpha+\rho b_j)>0,\ j=1,2,...,m;\mathrm{Re}(\beta)>0,\rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2}D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2}C_j\ge 0;$

$1+\sum_{j=1}^{q_1}H_j+\sum_{j=1}^{q_3}F_j-\sum_{j=1}^{p_1}G_j-\sum_{j=1}^{p_3}E_j\ge 0;$

(iii) $|\arg z|<\frac{1}{2}\pi\Delta'$

Where $\Delta'\equiv\sum_{j=1}^{Q}|b_j|-\sum_{j=1}^{P}|a_j|>0$

(iv) $f(1)=0$ and

(v) $f'(t)$ is continuous in K.

5. Application

We shall define the Riemann-Liouville fractional derivative of function $f(x)$ of order σ (or alternatively, σ^{th} order fractional integral) [2,7] by

$$ {}_aD_x^{\sigma}\{f(x)\}=\begin{cases}\frac{1}{\Gamma(-\sigma)}\int_a^x(x-t)^{\sigma-1}f(t)dt,\ \mathrm{Re}(\sigma)<0\\ \frac{d^q}{dx^q}\,{}_aD_x^{\sigma-q}\{f(x)\},\ q-1\le\mathrm{Re}(\sigma)<q\end{cases}, \quad (5.1)$$

Where q is a positive integer and the integral exists. For simplicity the special case of the fractional derivative operator ${}_aD_x^{\sigma}$, when $a=0$ will be written as D_x^{σ}. Thus we have

$$D_x^{\sigma}={}_0D_x^{\sigma} \quad (5.2)$$

The preliminary result (2.1) can be rewritten as the following fractional integral formula:

$$D_x^{\beta}\left[x^{\alpha-1}\overline{H}_{P,Q}^{M,N}\left[cy(1-x)\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]S[c(1-x),d(1-x)]\right]$$
$$=\frac{1}{\Gamma(\beta)}\sum_{r,w=0}^{\infty}A_{r,w}c^{r}d^{w}\times\Gamma(\beta+r+w)\overline{H}_{P+1,Q+1}^{M,N+1} \quad (5.3)$$
$$\left[cy\left|\begin{matrix}(1-\alpha,\rho;1),A^*\\B^*,(1-\alpha-\beta-r-w,\rho;1)\end{matrix}\right.\right]$$

Where $\mathrm{Re}(\beta)>0$ and all the conditions of validity mentioned with (2.1) are satisfied.

6. Conclusion

Thus, our obtained result the fractional integral formula given by (5.3) is also quite general in nature and can easily yield Riemann-Liouville fractional integrals of a large number of simpler functions and polynomials merely by specializing the parameters $\overline{H},H_{\nu,\eta,\mu}^{\lambda,k}$ occurring in it which may find applications in electromagnetic theory and probability.

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