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Numerical Solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs

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Abstract The main purpose of this paper is to develop the graph theoretic polynomial to solve numerical problems. We present a new method for the solution of Fredholm integral equations using Hosoya polynomials obtained from one of the standard class of graphs called as path. Proposed algorithm expands the desired solution in terms of a set of continuous polynomials over a closed interval [0,1]. However, accuracy and efficiency are dependent on the size of the set of Hosoya polynomials and compared with the existing method.

Keywords: *Fredholm integral equations, Hosoya polynomial, path*

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1. Introduction

In graph theory, as in discrete mathematics in general, not only the existence, but also the counting of objects with some given properties, is of main interest. Each area introduces its own special terms for shared concepts in discrete mathematics. The only way to keep from reinventing the wheel from area to area is to know the precise mathematical ideas behind the concept being applied by these various fields. Graph theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include biochemistry, electrical engineering (Communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling) [1,2].

Integral equations have motivated a large amount of research work in recent years. Integral equations find its applications in various fields of mathematics, science and technology has been studied extensively both at the theoretical and practical level. In particular, integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc. In most of the cases, it is difficult to solve them, especially analytically [3]. Analytical solutions of integral equations, however, either does not exist or are difficult to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations.

Consider the Fredholm integral equation:

$$y(x) = f(x) + \int_0^1 k(x,t)\, y(t)\, dt,\ 0 \le x,t \le 1 \qquad (1)$$

where $f(x)$ and the kernels $k(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x,t \le 1$. We assume that Eq.(1) has a unique solution y to be determined. There are several numerical methods for approximating the solution of Fredholm integral equations are known and many different basic functions have been used. Such as, Lepik et al. [4] applied the Haar Wavelets. Maleknejad et al. [5] applied a combination of Hybrid taylor and block-pulse functions, Rationalized Haar wavelet [6], Hermite Cubic splines [7]. Muthuvalu et al. [8] applied Half-sweep arithmetic mean method with composite trapezoidal scheme for the solution of Fredholm integral equations. In this paper, we proposed a numerical method for the solution of Fredholm integral equations using Hosoya polynomial of paths as basis.

2. Properties of Hosoya Polynomial

A *graph* G consists of a finite nonempty set V of n points (vertices) together with a prescribed set X of m unordered pairs of distinct points of V. Each pair $x = (u, v)$ of points in X is an edge of G. If the points u and v are joined by an edge, then we say that u and v are adjacent points. Let $v_1, v_2, \dots, v_n$ be the vertices of G. The *path* P_n is a graph with n vertices $v_1, v_2, \dots, v_n$, where v_i is adjacent to v_{i+1}, $i = 1, 2, \dots, n-1$. The length of a path is the number of edges in it. A graph G is said to be connected if every pair of points of G is joined by some path. The *distance* between the vertices v_i and v_j in G is equal to the length of the shortest path joining them and is denoted by $d(u_i, v_j)$. For more details about the graph theory one can refer the book [9].

The *Wiener index* $W(G)$ of a connected graph G is defined as the sum of the distances between all unordered pairs of vertices of G, that is,

$$W(G) = \sum_{1 \le i < j \le n} d(u_i, v_j).$$

This index was put forward by Harold Wiener [10] in 1947 for approximation of the boiling points of alkanes. The effect of approximation was surprisingly good. From that point forward, the Wiener index has attracted the attention of chemists.

The Hosoya polynomial of a graph is a generating function about distance distributing, introduced by Hosoya [11] in 1988. For a connected graph G, the *Hosoya polynomial* denoted by $H(G,\lambda)$ is defined as

$$H(G,\lambda)=\sum_{k\geq 0} d(G,k)\lambda^{k} \quad (2)$$

where $d(G,k)$ is the number of pairs of vertices of G that are at distance k and λ is the parameter.

The connection between the Hosoya polynomial and the Wiener index is elementary [11,12]:

$$W(G)=H'(G,1),$$

where $H'(G,\lambda)$ is the first derivative of $H(G,\lambda)$.

Hosoya polynomial of tress [13,14], composite graphs [15], benzenoid graphs [16,17], tori [18], zig-zag open-ended nanotubes [19], armchair open-ended nanotubes [20], zigzag polyhexnanotorus [21], Fibonacci and Lucas cubes [22] are reported in the literature.

The paths P_1, P_2 and P_3 are depicted in Figure 1.

Figure 1. Path graphs P_1, P_2 and P_3

The Hosoya polynomial of a path P_n is:

$$H(P_n,\lambda)=n+(n-1)\lambda+(n-2)\lambda^2+\dots$$
$$+[n-(n-2)]\lambda^{n-2}+[n-(n-1)]\lambda^{n-1}.$$

In particular,

$$H(P_1,\lambda)=1$$
$$H(P_2,\lambda)=\lambda+2$$
$$H(P_3,\lambda)=\lambda^2+2\lambda+3.$$

3. Function Approximation

A function $u(0)=f(0),\ \ u'(1)=f'(1)$ is expanded as:

$$f(x)=\sum_{i=1}^{n} c_i H(P_i,x)=C^T H_P(x), \quad (3)$$

where C and $H_P(x)$ are $n\times 1$ matrices given by:

$$C=[c_1,c_2,\dots,c_n]^T, \quad (4)$$

and

$$H_P(x)=[H(P_1,x),H(P_2,x),\dots,H(P_n,x),]^T. \quad (5)$$

4. Hosoya Polynomial Method (HPM)

Consider the Fredholm integral equation,

$$y(x)=f(x)+\int_0^1 k(x,t)y(t)dt, \quad 0\leq x,\ t\leq 1 \quad (6)$$

To solve Eq. (6), the procedure is as follows:

Step 1: We first approximate $y(x)$ as truncated series defined in Eq. (3). That is,

$$y(x)=C^T H_P(x) \quad (7)$$

where C and $H_P(x)$ are defined as in Eqs. (4) and (5).

Step 2: Substituting Eq. (7) in Eq. (6), we get,

$$C^T H_P(x)=f(x)+\int_0^1 k(x,t)[C^T H_P(t)]dt\,. \quad (8)$$

Step 3: Substituting the collocation point $x_i=\dfrac{i-0.5}{n}$, $i=1,2,\dots,n$ in Eq. (8), we obtain,

$$C^T H_P(x_i)=f(x_i)+\int_0^1 k(x_i,t)[C^T H_P(t)]dt. \quad (9)$$

That is,

$$C^T(H_P(x_i)-Z)=f,\ where\ Z=\int_0^1 k(x_i,t)H_P(t)dt.$$

Step 4: Now, we get the system of algebraic equations with unknown coefficients.

$$C^T K=f,\ \text{ where }\ K=(H_P(x_i)-Z).$$

Solving the above system of equations, we get the Hosoya coefficients 'C' and then substituting these coefficients in Eq. (7), we get the required approximate solution of Eq. (6).

5. Numerical Results

In this section, we consider the some illustrative examples from the literature to demonstrate the capability of the method and error function is presented to verify the accuracy and efficiency of the numerical results:

$$E_{Max}=Errorfunction$$
$$=y_e(x_i)-y_a(x_i)_\infty=\sqrt{\sum_{i=1}^{n}\left(y_e(x_i)-y_a(x_i)\right)^2}.$$

where, y_e and y_a are the exact and approximate solution respectively.

We considered the examples given in [7]. Using the present technique, the numerical solutions with exact solutions presented in Table 1 and the maximum error analysis compared with existing method [7] given in Table 2.

Example 1. Consider Fredholm integral equations,

$$y(x)=\sin(2\pi x)+\int_0^1 \cos(x)y(t)dt, \quad (10)$$

which has the exact solution $y(x)=\sin(2\pi x)$.

Table 1. Comparison of HPM solutions with exact solutions, for *n* = 8

x	Example 1		Example 2		Example 3	
	HPM	Exact	HPM	Exact	HPM	Exact
0.0625	0.3827	0.3827	0.3827	0.3827	-0.0513	-0.0513
0.1875	0.9239	0.9239	0.9239	0.9239	-0.0952	-0.0952
0.3125	0.9239	0.9239	0.9239	0.9239	-0.0806	-0.0806
0.4375	0.3827	0.3827	0.3827	0.3827	-0.0308	-0.0308
0.5625	-0.3827	-0.3827	-0.3827	-0.3827	0.0308	0.0308
0.6875	-0.9239	-0.9239	-0.9239	-0.9239	0.0806	0.0806
0.8125	-0.9239	-0.9239	-0.9239	-0.9239	0.0952	0.0952
0.9375	-0.3827	-0.3827	-0.3827	-0.3827	0.0513	0.0513

Table 2. Maximum error analysis of HPM with the existing method [7]

Method [7]				Hosoya Polynomial Method (HPM)			
2^{n+1}	Example 1	Example 2	Example 3	*n*	Example 1	Example 2	Example 3
4	2.84e-02	2.84e-02	1.33e-10	3	1.55e-15	1.23e-14	8.88e-16
8	2.38e-03	2.38e-03	3.79e-10	4	2.33e-13	1.19e-13	3.55e-15
16	2.09e-04	2.10e-04	3.26e-10	6	1.47e-12	6.82e-13	6.94e-14
32	1.20e-04	2.00e-04	4.83e-10	8	9.18e-11	9.01e-10	1.44e-10

Solution: First we substitute $y(x)=C^T H_P(x)$ in Eq. (10) we get

$$C^T H_P(x)=\sin(2\pi x)+\int_0^1 \cos(x)[C^T H_P(t)]\,dt.$$

Therefore for n = 3

$$C_1\left[H_1(x)-\int_0^1 \cos(x)H_1(t)\,dt\right]+C_2\left[H_2(x)-\int_0^1 \cos(x)H_2(t)\,dt\right]+C_3\left[H_3(x)-\int_0^1 \cos(x)H_3(t)\,dt\right]=\sin(2\pi x).$$

Next, we substitute the Hosoya polynomials as

$$C_1\left[1-\int_0^1 \cos(x)\,dt\right]+C_2\left[(x+2)-\int_0^1 \cos(x)(t+2)\,dt\right]+C_3\left[(x^2+2x+3)-\int_0^1 \cos(x)(t^2+2t+3)\,dt\right]=\sin(2\pi x).$$

Next,

$$C_1[1-\cos(x)]+C_2\left[(x+2)-\frac{5\cos(x)}{2}\right]+C_3\left[(x^2+2x+3)-\frac{13\cos(x)}{3}\right]=\sin(2\pi x).$$

Substituting the collocation points, we get the system of three equations with three unknowns as,

$$C_1[1-\cos(x_1)]+C_2\left[(x_1+2)-\frac{5\cos(x_1)}{2}\right]+C_3\left[(x_1^2+2x_1+3)-\frac{13\cos(x_1)}{3}\right]=\sin(2\pi x_1),$$

$$C_1[1-\cos(x_2)]+C_2\left[(x_2+2)-\frac{5\cos(x_2)}{2}\right]+C_3\left[(x_2^2+2x_2+3)-\frac{13\cos(x_2)}{3}\right]=\sin(2\pi x_2),$$

$$C_1[1-\cos(x_3)]+C_2\left[(x_3+2)-\frac{5\cos(x_3)}{2}\right]+C_3\left[(x_3^2+2x_3+3)-\frac{13\cos(x_3)}{3}\right]=\sin(2\pi x_3).$$

Solving these systems we obtain the three unknown Hosoya coefficients

$$C_1=6.4952, C_2=-2.5981, C_3=0.$$

Substituting these coefficients in the approximation,

$$y(x)=C_1[H_1(x)]+C_2[H_2(x)]+C_3[H_3(x)],$$

we get the approximate values

$$y_1=0.8660, y_2=0, y_3=-0.8660.$$

Maximum Error analyzed for n = 3 is 1.55e-15 and for n = 4, 6 & 8 are shown in the Table 2.

Example 2. Consider Fredholm integral equations,

$$y(x)=\sin(2\pi x)+\int_0^1\left(x^2-x-t^2+t\right)y(t)dt, \tag{11}$$

has the exact solution $y(t)=\sin(2\pi t)$.

Example 3. Consider Fredholm integral equations,

$$y(x)=-2x^3+3x^2-x+\int_0^1\left(x^2-x-t^2+t\right)y(t)dt, \tag{12}$$

which has the exact solution $y(x)=-2x^3+3x^2-x$.

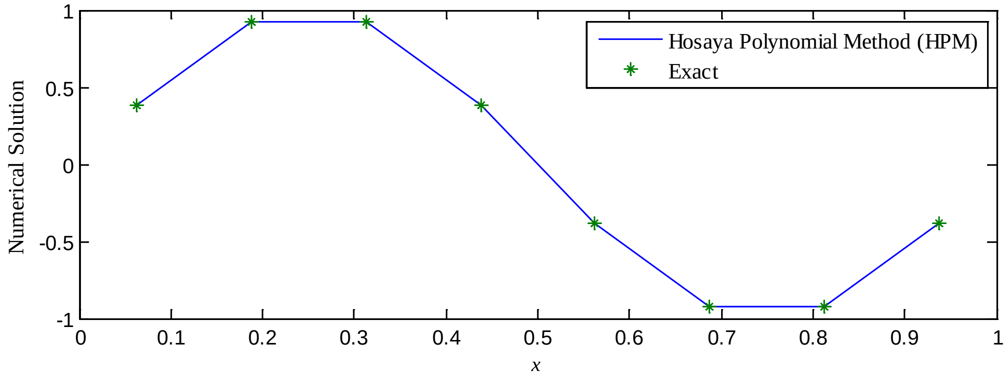

Figure 2. Numerical solution of exact and HPM Example 1, for n = 8

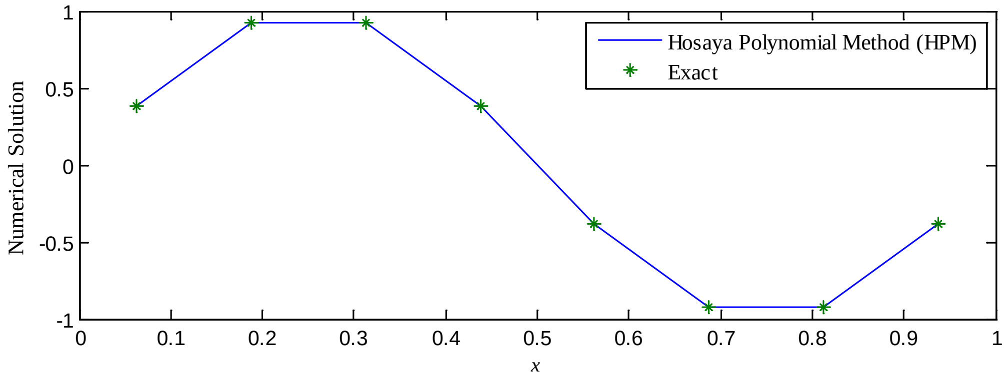

Figure 3. Numerical solution of exact and HPM Example 2, for n = 8

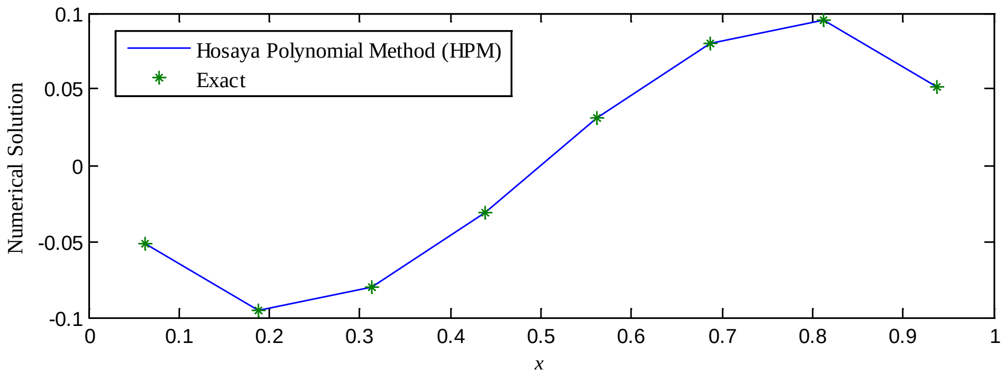

Figure 4. Numerical solution of exact and HPM Example 3, for n = 8

6. Conclusion

The Hosoya polynomial method is applied for the numerical solution of Fredholm integral equations. The present method reduces an integral equation into a set of algebraic equations. For instance in Example 1, our results are higher accuracy with exact ones and existing method [7]. Subsequently other examples are also same in the nature. The numerical result shows that the accuracy improves with increasing of n, the order of a path P_n, for better accuracy. Error analysis justifies the accuracy, efficiency and validity of the present technique.

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