American Journal of Numerical Analysis, 2017, Vol. 5, No. 1, 1-10
Available online at http://pubs.sciepub.com/ajna/5/1/1

DOI:10.12691/ajna-5-1-1

Numerical Solution of Singularly Perturbed Delay Reaction-Diffusion Equations with Layer or Oscillatory Behaviour

Gemechis File[1], Gashu Gadisa[1], Tesfaye Aga[1], Y. N. Reddy[2,*]

[1]Department of Mathematics, Jimma University, Jimma, P. O. Box 378, Ethiopia
[2]Department of Mathematics, National Institute of Technology, Warangal-506 004, India
*Corresponding author: ynreddy@nitw.ac.in

Abstract In this paper, we presented numerical method for solving singularly perturbed delay differential equations with layer or oscillatory behaviour for which a small shift (δ) is in the reaction term. First, the given singularly perturbed delay reaction-diffusion equation is converted into an asymptotically equivalent singularly perturbed two point boundary value problem and then solved by using fourth order finite difference method. The stability and convergence of the method has been investigated. The numerical results have been tabulated and further to examine the effect of delay on the boundary layer and oscillatory behavior of the solution, graphs have been given for different values of δ. Both theoretical and numerical rate of convergence have been established and are observed to be in agreement for the present method. Briefly, the present method improves the findings of some existing numerical methods in the literature.

Keywords: *singularly perturbed problem, delay reaction-diffusion equation, oscillatory behaviour*

Cite This Article: Gemechis File, Gashu Gadisa, Tesfaye Aga, and Y. N. Reddy, "Numerical Solution of Singularly Perturbed Delay Reaction-Diffusion Equations with Layer or Oscillatory Behaviour." *American Journal of Numerical Analysis*, vol. 5, no. 1 (2017): 1-10. doi: 10.12691/ajna-5-1-1.

1. Introduction

Delay differential equations play an important role in the mathematical modeling of various practical phenomena in the biosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. A delay differential equation is of the retarded type if the delay argument does not occur in the highest order derivative term. If we restrict this class to a class in which the highest derivative term is multiplied by a small parameter, then it is said to be singularly perturbed delay differential equation of the retarded type. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. In recent years, there has been a growing interest in the numerical study of singularly perturbed delay differential equations because of their applications in many scientific and technical fields like micro scale heat transfer, hydrodynamics of liquid helium, second-sound theory, thermo elasticity, diffusion in polymers, reaction-diffusion equations, stability, control of chaotic systems, a variety of models for physiological processes, Gemechis and Reddy [1].

Lange and Miura [2] gave an asymptotic approach for a class of boundary-value problems for linear second-order singularly perturbed differential-difference equations. Ramesh and Kadalbajoo [3] presented the numerical approximation of singularly perturbed linear second order reaction-diffusion boundary value problems with a small shift (δ) in the reaction term. Swamy [4] presented the quantitative analysis of delay reaction-diffusion equations with layer or oscillatory behaviour by employing the numerical integration. Soujanya and Reddy [5] presented a computational technique for solving singularly perturbed delay differential equations with layer or oscillatory behaviour in which the small delay is in the reaction term. The treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions, Doolan et al. [6], Kadalbajoo and Reddy [7] and Roos et al. [8]. Kadalbajoo and Ramesh [9] states that, the accuracy of the problem increased by increasing the resolution of the grid which might be impractical in some cases like higher dimensions. Pratima and Sharma [10] states that, till date ε-uniformly convergent methods have not been sufficiently developed for a wide class of singularly perturbed delay differential equations. In this paper, we present numerical method for solving singularly perturbed delay reaction-diffusion equations with layer or oscillatory behaviour via fourth order finite difference method which is uniformly convergent and more accurate than the others.

2. Description of the Method

Consider singularly perturbed delay reaction-diffusion equation of the form:

$$\varepsilon^2 y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x), \quad 0 < x < 1 \tag{1}$$

with the interval and boundary conditions,

$$y(x) = \phi(x), \; -\delta \le x \le 0 \text{ and } y(1) = \beta \tag{2}$$

where ε is small parameter, $0 < \varepsilon << 1$ and δ is also small delay parameter, $0 < \delta << 1$; $a(x), b(x), f(x)$ and $\phi(x)$ are bounded smooth functions in $(0,1)$ and β is a given constant. The layer or oscillatory behaviour of the problem under consideration is maintained for $\delta \ne 0$ but sufficiently small, depending on the sign of $a(x)+b(x)$, for all $x \in (0,1)$. If $a(x)+b(x) < 0$, the solution of the problem in Eqs. (1) and (2) exhibits layer behaviour, and if $a(x)+b(x) > 0$, it exhibits oscillatory behaviour. Therefore, if the solution exhibits layer behaviour, there will be two boundary layers which will occur at both the end points $x = 0$ and $x = 1$.

The solution $y(x)$ should be continuous on $[0,1]$, continuously differentiable on $(0,1)$ and also satisfies Eqs. (1) and (2).

By using Taylor series expansion in the neighborhood of the point x, we have:

$$y(x-\delta) \approx y(x) - \delta y'(x) + \mathrm{o}(\delta^2) \tag{3}$$

Substituting Eq. (3) into Eq. (1), we obtain an asymptotically equivalent singularly perturbed two point boundary value problem of the form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{4}$$

under the boundary conditions,

$$y(0) = \phi_0 \text{ and } y(1) = \beta. \tag{5}$$

where, $p(x) = \dfrac{-\delta a(x)}{\varepsilon^2}$, $q(x) = \dfrac{a(x)+b(x)}{\varepsilon^2}$ and $r(x) = \dfrac{f(x)}{\varepsilon^2}$.

The transition from Eq. (1) to Eq. (4) is admitted, because of the condition that $0 < \delta << 1$ is sufficiently small. Further details on the validity of this transition can be found in Elsgolt's and Norkin [11].

Now, divide the interval [0,1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then, we have $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, N$.

By using Taylor series expansion, we obtain:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i^{(3)} + \frac{h^4}{4!}y_i^{(4)} + O(h^5) \tag{6}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i^{(3)} + \frac{h^4}{4!}y_i^{(4)} + O(h^5) \tag{7}$$

Subtracting Eq. (6) from Eq. (7), we obtain:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6}y_i^{(3)} + \ldots \tag{8}$$

Hence, the second order finite difference approximation $N+1$ for the first derivative of N is:

$$\delta_c^1 y_i = \frac{y_{i+1} - y_{i-1}}{2h} + T_1 \tag{9}$$

where,

$$T_1 = -\frac{h^2}{6}y_i^{(3)}.$$

Similarly, adding Eqs. (6) and (7), we obtain:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12}y_i^{(4)} + \ldots \tag{10}$$

Hence, the second order finite difference approximation $i = 2, \ldots., N.$ for the second derivative of $a(x_i) = a_i$, is:

$$\delta_c^2 y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + T_2 \tag{11}$$

where,

$$T_2 = -\frac{h^2}{12}y_i^{(4)}.$$

Substituting Eqs. (6) and (7) into Eq. (9) yields:

$$\delta_c^1 y_i = y_i' + \frac{h^2}{6}y_i^{(3)} + T_3 \tag{12}$$

where,

$$T_3 = \frac{h^4}{120}y_i^{(5)} + T_1 = \frac{h^4}{120}y_i^{(5)} - \frac{h^2}{6}y_i^{(3)}.$$

Again, substituting Eqs. (6) and (7) into Eq. (11), we obtain:

$$\delta_c^2 y_i = y_i'' + \frac{h^2}{12}y_i^{(4)} + T_4 \tag{13}$$

where,

$$T_4 = \frac{h^4}{360}y_i^{(6)} + T_2 = \frac{h^4}{360}y_i^{(6)} - \frac{h^2}{12}y_i^{(4)}$$

Applying δ_c^2 to $y_i^{(3)}$ in Eq. (9), we obtain:

$$y_i^{(3)} = \delta_c^2 y_i' - T_1^{(2)}. \tag{14}$$

Substituting Eq. (14) into Eq. (12), we obtain:

$$\delta_c^1 y_i = y_i' + \frac{h^2}{6}\delta_c^2 y_i' + T_5 \tag{15}$$

where,

$$T_5 = T_3 - \frac{h^2}{6}T_1^{(2)} = -\frac{h^2}{6}y_i^{(3)} + \frac{39h^4}{1080}y_i^{(5)}.$$

Applying δ_c^2 to y_i'' in Eq. (11), we obtain a fourth order finite difference scheme for Eq. (4) as:

$$y_i^{(4)} = \delta_c^2 y_i'' - T_2^{(2)}. \tag{16}$$

Substituting Eq. (16) into Eq. (13), we obtain:

$$\delta_c^2 y_i = y_i'' + \frac{h^2}{12}\delta_c^2 y_i'' + T_6 \tag{17}$$

where,

$$T_6 = T_4 - \frac{h^2}{12}T_2^{(2)} = -\frac{h^2}{12}y_i^{(4)} + \frac{101h^4}{12{,}960}y_i^{(6)}.$$

From Eqs. (15) and (17), we have:

$$y_i' = \frac{\delta_c^1 y_i - T_5}{1+\frac{h^2}{6}\delta_c^2} \text{ and } y_i'' = \frac{\delta_c^2 y_i - T_6}{1+\frac{h^2}{12}\delta_c^2}. \tag{18}$$

Evaluating Eq. (4) at x_i and using Eq. (18), we obtain:

$$\frac{\delta_c^2 y_i - T_6}{1+\frac{h^2}{12}\delta_c^2} + p_i\left(\frac{\delta_c^1 y_i - T_5}{1+\frac{h^2}{6}\delta_c^2}\right) + q_i y_i = r_i. \tag{19}$$

Simplifying Eq.(19), we get:

$$\begin{aligned}&\delta_c^2 y_i + \frac{h^2}{6}\delta_c^4 y_i + p_i\delta_c^1 y_i + \frac{h^2}{12}p_i\delta_c^3 y_i\\ &+q_i\left(1+\frac{h^2}{4}\delta_c^2+\frac{h^4}{72}\delta_c^4\right)y_i\\ &=\left(1+\frac{h^2}{4}\delta_c^2+\frac{h^4}{72}\delta_c^4\right)r_i + T_7\end{aligned} \tag{20}$$

where,

$$T_7 = \left(1+\frac{h^2}{6}\delta_c^2\right)T_6 + q_i\left(1+\frac{h^2}{12}\delta_c^2\right)T_5.$$

By successively differentiating both sides of Eq. (4), evaluating at x_i, and using into Eq. (20), we obtain:

$$\begin{aligned}&\delta_c^2 y_i + \frac{h^2}{6}\Big\{r_i'' - \left(2p_i' + q_i - p_i^2\right)\delta_c^2 y_i\\ &-\left(p_i'' + 2q_i' - p_i(p_i'+q_i)\right)\delta_c^1 y_i\\ &+\left(p_i q_i' - q_i''\right)y_i - p_i r_i'\Big\} + p_i\delta_c^1 y_i\\ &+\frac{h^2}{12}p_i\left\{r_i' - p_i\delta_c^2 y_i - (p_i'+q_i)\delta_c^1 y_i - q_i' y_i\right\}\\ &+q_i y_i + \frac{h^2}{4}q_i\delta_c^2 y_i + \frac{h^4}{72}q_i\Big\{r_i'' - \left(2p_i'+q_i-p_i^2\right)\delta_c^2 y_i\\ &-\left(p_i''+2q_i' - p_i(p_i'+q_i)\right)\delta_c^1 y_i\\ &+\left(p_i q_i' - q_i''\right)y_i - p_i r_i'\Big\}\\ &= r_i + \frac{h^2}{4}\delta_c^2 r_i + \frac{h^4}{72}\delta_c^2 r_i'' + T_7\end{aligned} \tag{21}$$

Substituting Eqs. (9) and (11) into Eq. (21) for $\delta_c^1 y_i$ and $\delta_c^2 y_i$ and making use of $\delta_c^2 r_i = \frac{r_{i-1} - 2r_i + r_{i+1}}{h^2}$ and $\delta_c^2 r_i'' = \frac{r_{i-1}'' - 2r_i'' + r_{i+1}''}{h^2}$, we obtain:

$$\begin{aligned}&\Big\{\frac{1}{h^2} - \frac{1}{6}\left(2p_i' + q_i - p_i^2\right)\\ &+\frac{h}{12}\left(p_i'' + 2q_i' - p_i(p_i'+q_i)\right) - \frac{p_i}{2h} - \frac{p_i^2}{12}\\ &+\frac{h}{24}p_i(p_i'+q_i) + \frac{q_i}{4} + \frac{h^3}{144}q_i\left(p_i'' + 2q_i'\right.\\ &\left.-p_i(p_i'+q_i)\right) - \frac{h^2}{72}q_i\left(2p_i'+q_i-p_i^2\right)\Big\}y_{i-1}\\ &-\Big\{\frac{2}{h^2} - \frac{1}{3}\left(2p_i'+q_i-p_i^2\right) - \frac{h^2}{6}\left(p_i q_i' - q_i''\right)\\ &-\frac{p_i^2}{6} + \frac{h^2}{12}p_i q_i' - \frac{q_i}{2} - \frac{h^2}{36}q_i\left(2p_i'+q_i-p_i^2\right)\\ &-\frac{h^4}{72}q_i\left(p_i q_i' - q_i''\right)\Big\}y_i + \Big\{\frac{1}{h^2} - \frac{1}{6}\left(2p_i'+q_i-p_i^2\right)\\ &-\frac{h}{12}\left(p_i''+2q_i' - p_i(p_i'+q_i)\right) + \frac{p_i}{2h} - \frac{p_i^2}{12}\\ &-\frac{h}{24}p_i(p_i'+q_i) + \frac{q_i}{4} - \frac{h^3}{144}q_i\left(p_i''+2q_i'\right.\\ &\left.-p_i(p_i'+q_i)\right) - \frac{h^2}{72}q_i\left(2p_i'+q_i-p_i^2\right)\Big\}y_{i+1}\\ &= r_i - \frac{h^2}{6}r_i'' + \frac{h^2}{12}p_i r_i' - \frac{h^4}{72}q_i r_i'' + \frac{h^4}{72}p_i q_i r_i'\\ &+\frac{1}{4}\left(r_{i-1} - 2r_i + r_{i+1}\right) + \frac{h^2}{72}\left(r_{i-1}'' - 2r_i'' + r_{i+1}''\right) + T\end{aligned} \tag{22}$$

where, $T = \frac{h^4}{45}q_i y_i^{(5)} - \frac{79h^4}{12{,}960}y_i^{(6)} + O(h^5)$ is the local truncation error.

Multiplying both sides of Eq. (22) by h^2, we get the three-term recurrence relation of the form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i,\ \text{for } i = 1,2,\cdots,N-1 \tag{23}$$

where,

$$\begin{aligned}E_i &= 1 - \frac{h^2}{6}\left(2p_i' + q_i - p_i^2\right)\\ &+\frac{h^3}{12}\left(p_i'' + 2q_i' - p_i(p_i'+q_i)\right)\\ &-\frac{h}{2}p_i - \frac{h^2}{12}p_i^2 + \frac{h^3}{24}p_i(p_i'+q_i) + \frac{h^2}{4}q_i\\ &+\frac{h^5}{144}q_i\left(p_i''+2q_i' - p_i(p_i'+q_i)\right)\\ &-\frac{h^4}{72}q_i\left(2p_i'+q_i-p_i^2\right)\end{aligned}$$

$$F_i = 2 - \frac{h^2}{3}\left(2p_i' + q_i - p_i^2\right) - \frac{h^4}{6}\left(p_i q_i' - q_i''\right) - \frac{h^2}{6}p_i^2 + \frac{h^4}{12}p_i q_i' - \frac{h^2}{2}q_i - \frac{h^4}{36}q_i\left(2p_i' + q_i - p_i^2\right) - \frac{h^6}{72}q_i\left(p_i q_i' - q_i''\right)$$

$$G_i = 1 - \frac{h^2}{6}\left(2p_i' + q_i - p_i^2\right) - \frac{h^3}{12}\left(p_i'' + 2q_i' - p_i(p_i' + q_i)\right) + \frac{h}{2}p_i - \frac{h^2}{12}p_i^2 - \frac{h^3}{24}p_i(p_i' + q_i) + \frac{h^2}{4}q_i - \frac{h^5}{144}q_i\left(p_i'' + 2q_i' - p_i(p_i' + q_i)\right) - \frac{h^4}{72}q_i\left(2p_i' + q_i - p_i^2\right)$$

$$H_i = h^2 r_i - \frac{h^4}{6}r_i'' + \frac{h^4}{12}p_i r_i' - \frac{h^6}{72}q_i r_i'' + \frac{h^6}{72}p_i q_i r_i' + \frac{h^2}{4}\left(r_{i-1} - 2r_i + r_{i+1}\right) + \frac{h^4}{72}\left(r_{i-1}'' - 2r_i'' + r_{i+1}''\right).$$

The tri-diagonal system in Eq. (23) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

Since Eq. (23) holds for $i = 1, 2, \cdots, N-1$, we have $N-1$ linear equations in the $N-1$ unknowns $y_1, y_2, \cdots, y_{N-1}$. The matrix of this set of linear equations is denoted as A_N.

3. Stability and Convergence Analysis

Lemma 1: For all $\varepsilon > 0$ and sufficiently small h, the matrix A_N is an irreducible and diagonally dominant matrix.

Proof:

Clearly, A_N is a tridiagonal matrix. A_N is irreducible if its co-diagonals contain non-zero elements only. The co-diagonal contains E_i, G_i. It is easily seen that, for sufficiently small h (*i.e.* $h \to 0$),

$$E_i \neq 0 \text{ and } G_i \neq 0, \forall i = 1, 2, \cdots, N-1.$$

Hence, A_N is irreducible.

Again one can observe that, $E_i > 0$ and $G_i > 0$ and in each row of A_N, the sum of the two off-diagonal elements less than or equal to the modulus of the diagonal element. This proves the diagonal dominant of A_N.

Under these conditions the discrete imbedding algorithm is stable, Kadalbajoo and Reddy [12].

Lemma 2: Let $y(x)$ be the analytical solution of the problem in Eqs. (4) and (5) and $y^N(x)$ be the numerical solution of the discretized problem of Eq. (23). Then, $\left\| y - y^N \right\| \le C h^4$ for sufficiently small h and C is positive constant.

Proof:

Multiplying both sides of Eq. (23) by -1, we get:

$$\left(-1 + u_i\right)y_{i-1} + \left(2 + v_i\right)y_i + \left(-1 + w_i\right)y_{i+1} + g_i + T_i = 0 \quad (24)$$

where,

$$u_i = \frac{h^2}{6}\left(2p_i' + q_i - p_i^2\right) - \frac{h^3}{12}\left(p_i'' + 2q_i' - p_i(p_i' + q_i)\right) + \frac{h}{2}p_i + h^2\frac{p_i^2}{12} - \frac{h^3}{24}p_i(p_i' + q_i) - \frac{h^2}{4}q_i - \frac{h^5}{144}q_i\left(p_i'' + 2q_i' - p_i(p_i' + q_i)\right) + \frac{h^4}{72}q_i\left(2p_i' + q_i - p_i^2\right)$$

$$v_i = -\frac{h^2}{3}\left(2p_i' + q_i - p_i^2\right) - \frac{h^4}{6}\left(p_i q_i' - q_i''\right) - \frac{h^2}{6}p_i^2 + \frac{h^4}{12}p_i q_i' - \frac{h^2}{2}q_i - \frac{h^4}{36}q_i\left(2p_i' + q_i - p_i^2\right) - \frac{h^6}{72}q_i\left(p_i q_i' - q_i''\right)$$

$$w_i = \frac{h^2}{6}\left(2p_i' + q_i - p_i^2\right) + \frac{h^3}{12}\left(p_i'' + 2q_i' - p_i(p_i' + q_i)\right) - \frac{h}{2}p_i + \frac{h^2}{12}p_i^2 + \frac{h^3}{24}p_i(p_i' + q_i) - \frac{h^2}{4}q_i + \frac{h^5}{144}q_i\left(p_i'' + 2q_i' - p_i(p_i' + q_i)\right) + \frac{h^4}{72}q_i\left(2p_i' + q_i - p_i^2\right)$$

$$g_i = h^2 r_i - \frac{h^4}{6}r_i'' + \frac{h^4}{12}p_i r_i' - \frac{h^6}{72}q_i r_i'' + \frac{h^6}{72}p_i q_i r_i' + \frac{h^2}{4}\left(r_{i-1} - 2r_i + r_{i+1}\right) + \frac{h^4}{72}\left(r_{i-1}'' - 2r_i'' + r_{i+1}''\right)$$

$$T_i(h) = \frac{h^6}{45}q_i y_i^{(5)} - \frac{79h^6}{12{,}960}y_i^{(6)} + O(h^7)$$

is a local truncation error, for $i = 1, 2, \ldots, N-1$.

Incorporating the boundary conditions $y_0 = \phi(x_0) = \phi_0$, $y_N = y(1) = \beta$ in Eq. (24), we get the systems of equations of the form:

$$\begin{bmatrix} (2+v_1) & (-1+w_1) & 0 & \cdots & 0 \\ (-1+u_2) & (2+v_2) & (-1+w_2) & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & (-1+u_{N-1}) & (2+v_{N-1}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \end{bmatrix} + \begin{bmatrix} g_1 + (-1+u_1)\phi(0) \\ g_2 \\ g_3 \\ \vdots \\ g_{N-1} + (-1+w_{N-1})\beta \end{bmatrix} + \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-1} \end{bmatrix} = \overline{0}$$

$$\Rightarrow (D+P)y + M + T(h) = 0 \tag{25}$$

where,

$$D = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & -1 & 2 \end{bmatrix},$$

$$P = \begin{bmatrix} v_1 & w_1 & 0 & \cdots & 0 \\ u_2 & v_2 & w_2 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & u_{N-1} & v_{N-1} \end{bmatrix}$$

are tri-diagonal matrices of order $N-1$, and

$$M = \left[\left(g_1 + (-1+u_1)\phi(0)\right),\ g_2,\ g_3, \cdots, \left(g_{N-1} + (-1+w_{N-1})\beta\right)\right]^T,$$

$T(h) = O\left(h^6\right)$ and

$$y = [y_1, y_2, \cdots, y_{N-1}]^T,\ T(h) = [T_1, T_2, \cdots, T_{N-1}]^T,$$
$$\bar{0} = [0, 0, \cdots, 0]^T$$

are the associated vectors of Eq. (25).

Let $y^N = \left[y_1^N, y_2^N, \cdots, y_{N-1}^N\right]^T \cong y$ be the solution which satisfies the Eq. (25), we have:

$$(D+P)y^N + M = 0. \tag{26}$$

Let $e_i = y_i - y_i^N$, for $i = 1, 2, \cdots, N-1$ be the discretization error, then,

$$y - y^N = [e_1, e_2, \cdots, e_{N-1}]^T.$$

Subtracting Eq. (25) from Eq. (26), we get:

$$(D+P)\left(y^N - y\right) = T(h). \tag{27}$$

Let $|p_i| \le C_1$, $|p_i'| \le C_2$, $|p_i''| \le C_3$, $|q_i| \le K_1$, $|q_i'| \le K_2$, $|q_i''| \le K_3$

Let t_{ij} be the $(i,j)^{th}$ element of the matrix P, then:

For, $i = 1, 2, \cdots, N-2$

$$|t_{i,i+1}| = |w_i| \le h\left\{\frac{C_1}{2} + \frac{h}{12}\left(4C_2 - K_1 - C_1^2\right) + \frac{h^2}{12}\left(C_3 + 2K_2 - \frac{C_1}{2}(C_2 + K_1)\right) + \frac{h^3}{72}K_1\left(2C_2 + K_1 - C_1^2\right) + \frac{h^4}{144}K_1\left(C_3 + 2K_2 - C_1(C_2 + K_1)\right)\right\}$$

For, $i = 2, 3, \cdots, N-1$

$$|t_{i,i-1}| = |u_i| \le h\left\{\frac{C_1}{2} + \frac{h}{12}\left(4C_2 - K_1 - C_1^2\right) + \frac{h^2}{12}\left(C_3 + 2K_2 + \frac{C_1}{2}(C_2 + K_1)\right) + \frac{h^3}{72}K_1\left(2C_2 + K_1 - C_1^2\right) + \frac{h^4}{144}K_1\left(C_3 + 2K_2 - C_1(C_2 + K_1)\right)\right\}.$$

Thus, for sufficiently small h, we have:

$$-1 + |t_{i,i+1}| \ne 0, i = 1, 2, \cdots, N-2,$$
$$-1 + |t_{i,i-1}| \ne 0, i = 2, 3, \cdots, N-1.$$

Hence, the matrix $(D+P)$ is irreducible, Varga [13].

Let S_i be the sum of the elements of the i^{th} row of the matrix $(D+P)$, then:

$$S_i = 1 + v_i + w_i, \quad \text{for } i = 1$$
$$= 1 + h\left(-\frac{p_i}{2}\right) + h^2\left(\frac{1}{12}\left(-4p_i' - 11q_i + p_i^2\right)\right) + h^3\left(\frac{1}{12}\left(p_i'' + 2q_i'\right) - \frac{p_i}{24}\left(p_i' + q_i\right)\right) + O(h^4)$$

$$S_i = u_i + v_i + w_i, \quad \text{for } i = 2, 3, \cdots, N-2$$
$$= h^2(-q_i) + O(h^4)$$

$$S_i = 1 + u_i + v_i, \quad \text{for } i = N-1$$
$$= 1 + h\left(\frac{p_i}{2}\right) + h^2\left(\frac{1}{12}\left(-4p_i' - 11q_i + p_i^2\right)\right) + h^3\left(\frac{p_i}{24}\left(p_i' + q_i\right) - \frac{1}{12}\left(p_i'' + 2q_i'\right)\right) + O(h^4).$$

Let

$$C_{1*} = \min_{1\le i\le N-1}|p_i|,\quad C_1^* = \max_{1\le i\le N-1}|p_i|,$$
$$K_{1*} = \min_{1\le i\le N-1}|q_i|,\quad K_1^* = \max_{1\le i\le N-1}|q_i|,$$

then:

$$0 < C_{1*} \le C_1 \le C_1^* \text{ and } 0 < K_{1*} \le K_1 \le K_1^*.$$

For sufficiently small h, $(D+P)$ is monotone, Varga [13] and Young [14].

Hence, $(D+P)^{-1}$ exists and $(D+P)^{-1} \ge 0$.

From the error Eq. (27), we have:

$$\left\|y - y^N\right\| \le \left\|(D+P)^{-1}\right\|\left\|T(h)\right\|. \tag{28}$$

For sufficiently small h, we have:

$$S_i > \frac{11}{12}h^2 K_{1*}, \text{ for } i = 1$$

$$S_i > h^2 K_{1*}, \text{ for } i = 2,3,\cdots,N-2$$
$$S_i > \frac{11}{12} h^2 K_{1*}, \text{ for } i = N-1,$$
$$\text{where, } K_{1*} = \min_{1\le i\le N-1} |q_i|$$

Let $(D+P)_{i,k}^{-1}$ be the $(i,k)^{th}$ element of $(D+P)^{-1}$ and we define,

$$\left\|(D+P)^{-1}\right\| = \max_{1\le i\le N-1} \sum_{k=1}^{N-1} (D+P)_{i,k}^{-1} \text{ and}$$
$$\|T(h)\| = \max_{1\le i\le N-1} |T_i| \tag{29}$$

Since $(D+P)_{i,k}^{-1} \ge 0$, then from the theory of matrices, we have:

$$\sum_{k=1}^{N-1} (D+P)_{i,k}^{-1}.S_k = 1, \; i = 1,2,\cdots,N-1.$$

Hence,

$$(D+R)_{i,1}^{-1} \le \frac{1}{S_1} < \frac{12}{11}\left(\frac{1}{h^2 K_{1*}}\right), \text{ for } k = 1 \tag{30}$$

$$(D+R)_{i,N-1}^{-1} \le \frac{1}{S_{N-1}} < \frac{12}{11}\left(\frac{1}{h^2 K_{1*}}\right), \text{ for } k = N-1. \tag{31}$$

Further,

$$\sum_{k=2}^{N-2} (D+R)_{i,k}^{-1} \le \frac{1}{\min_{2\le k\le N-2} S_k} \le \frac{1}{h^2 K_{1*}}, \tag{32}$$
for $k = 2,3,\cdots,N-2$.

Now, from Eqs. (28) – (32), we get:

$$\left\|y - y^N\right\| < \frac{35}{11Q_1}\left(\frac{Q_2}{45} y_i^{(5)} + \frac{79}{12,960} y_i^{(6)}\right) h^4 = C h^4 \tag{33}$$
since $0 < \varepsilon << 1$.

where,

$$Q_1 = \min_{1\le i\le N-1} |a_i + b_i|, \quad Q_2 = |a_i + b_i| \text{ and}$$
$$C = \frac{35}{11Q_1}\left(\frac{Q_2}{45} y_i^{(5)} + \frac{79}{12,960} y_i^{(6)}\right)$$

which is independent of perturbation parameter ε and mesh size h.

This establishes that the method is of fourth order uniformly convergent.

4. Numerical Examples

To demonstrate the applicability of the method, we implemented the method on four numerical examples, two with boundary layers and two with oscillatory behaviour. Since, those examples have no exact solution, so the numerical solutions are computed using double mesh principle. The maximum absolute errors are computed using double-mesh principle given by:

$$Z_h = \max_i \left| y_i^h - y_i^{h/2} \right|, \; i = 1,2,\ldots,N-1 \tag{34}$$

where y_i^h is the numerical solution on the mesh $\{x_i\}_1^{N-1}$ at the nodal point x_i and $x_i = x_0 + ih$, $i = 1,2,\ldots,N-1$ and $y_i^{h/2}$ is the numerical solution on a mesh, obtained by bisecting the original mesh with N number of mesh intervals, Doolan et al. [6].

Example 1. Consider the singularly perturbed delay reaction-diffusion equation with layer behaviour,

$$\varepsilon^2 y''(x) - 2y(x-\delta) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \le x \le 0 \; and \; y(1) = 0.$$

The maximum absolute errors are presented in Table 1 and Table 5 for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.1$ and different values of δ is also given in Figure 1.

Example 2. Consider the singularly perturbed delay reaction-diffusion equation with layer behaviour,

$$\varepsilon^2 y''(x) + 0.25y(x-\delta) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \le x \le 0 \; and \; y(1) = 0.$$

The maximum absolute errors are presented in Table 2 and Table 6 for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.1$ and different values of δ is also given in Figure 2.

Example 3. Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behaviour,

$$\varepsilon^2 y''(x) + 0.25y(x-\delta) + y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \le x \le 0 \; and \; y(1) = 0.$$

The maximum absolute errors are presented in Table 3 for different values of δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Figure 3.

Example 4. Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behaviour,

$$\varepsilon^2 y''(x) + y(x-\delta) + 2y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \le x \le 0 \; and \; y(1) = 0.$$

The maximum absolute errors are presented in Table 4 for different values of δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Figure 4.

5. Numerical Results

Table 1. The maximum absolute errors of Example 1, for different values of δ with $\varepsilon = 0.1$

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Our Method					
0.03	3.3211e-06	2.0765e-07	4.1021e-08	1.2982e-08	5.3183e-09
0.05	3.3265e-06	2.0860e-07	4.1236e-08	1.3045e-08	5.3441e-09
0.09	3.9208e-06	2.4744e-07	4.8834e-08	1.5467e-08	6.3334e-09
Soujanya and Reddy [5]					
0.03	3.2676e-03	1.6475e-03	1.1015e-03	8.2735e-04	6.6245e-04
0.05	3.2657e-03	1.6526e-03	1.1062e-03	8.3136e-04	6.6593e-04
0.09	3.5460e-03	1.7987e-03	1.2051e-03	9.0609e-04	7.2594e-04
Swamy [4]					
0.03	9.3352e-03	4.9360e-03	3.3540e-03	2.5398e-03	-
0.05	8.7514e-03	4.7344e-03	3.2355e-03	2.4561e-03	-
0.09	7.2037e-03	4.1449e-03	2.8840e-03	2.2111e-03	-

Table 2. The maximum absolute errors of Example 2, for different values of δ with $\varepsilon = 0.1$

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Our Method					
0.03	4.2909e-07	2.6820e-08	5.2979e-09	1.6763e-09	6.8659e-10
0.05	4.4265e-07	2.7686e-08	5.4687e-09	1.7304e-09	7.0877e-10
0.09	4.7354e-07	2.9598e-08	5.8468e-09	1.8501e-09	7.5778e-10
Soujanya and Reddy [5]					
0.03	2.1226e-03	1.0639e-03	7.0985e-04	5.3259e-04	4.2617e-04
0.05	2.1099e-03	1.0574e-03	7.0543e-04	5.2928e-04	4.2351e-04
0.09	2.0816e-03	1.0426e-03	6.9547e-04	5.2178e-04	4.1750e-04
Swamy [4]					
0.03	8.9194e-03	8.8966e-03	3.0511e-03	2.2959e-03	-
0.05	8.9177e-03	4.5440e-03	3.0482e-03	2.2934e-03	-
0.09	8.8966e-03	4.5252e-03	3.0345e-03	2.2825e-03	-

Table 3. The maximum absolute errors of Example 3, for different values of δ with $\varepsilon = 0.1$

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Our Method					
0.03	1.0936e-05	6.8420e-07	1.3515e-07	4.2761e-08	1.7516e-08
0.05	1.0707e-05	6.6910e-07	1.3217e-07	4.1826e-08	1.7132e-08
0.09	1.0117e-05	6.3260e-07	1.2499e-07	3.9546e-08	1.6197e-08
Soujanya and Reddy [5]					
0.03	2.4582e-03	1.2196e-03	8.1096e-04	6.0742e-04	4.8554e-04
0.05	2.5127e-03	1.2472e-03	8.2948e-04	6.2134e-04	4.9669e-04
0.09	2.6198e-03	1.3016e-03	8.6589e-04	6.4872e-04	5.1863e-04
Swamy [4]					
0.03	7.1024e-02	3.5558e-02	2.3661e-02	1.7721e-02	-
0.05	6.9203e-02	3.4790e-02	2.3181e-02	1.7373e-02	-
0.09	6.6055e-02	3.3490e-02	2.2377e-02	1.6794e-02	-

Table 4. The maximum absolute errors of Example 4, for different values of δ with $\varepsilon = 0.1$

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Our Method					
0.03	9.5107e-05	5.9430e-06	1.1739e-06	3.7154e-07	1.5220e-07
0.05	1.6132e-04	1.0082e-05	1.9930e-06	6.3066e-07	2.5831e-07
0.09	3.7561e-03	2.3545e-04	4.6501e-05	1.4716e-05	6.0275e-06
Soujanya and Reddy [5]					
0.03	1.8682e-02	9.0640e-03	5.9795e-03	4.4608e-03	3.5572e-03
0.05	1.4987e-02	7.2328e-03	4.7631e-03	3.5505e-03	2.8299e-03
0.09	2.1346e-02	1.0306e-02	6.7863e-03	5.0577e-03	4.0307e-03
Swamy [4]					
0.03	1.9740e-01	1.0467e-01	7.0844e-02	5.3521e-02	-
0.05	2.5749e-01	1.3585e-01	9.2035e-02	6.9554e-02	-
0.09	1.5004e+00	7.1504e-01	4.6444e-01	3.4319e-01	-

Table 5. The maximum absolute errors of Example 1, for different values of ε with $\delta = 0.03$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Our Method					
2^{-1}	6.2873e-09	3.9296e-10	7.7611e-11	2.4565e-11	1.0067e-11
2^{-2}	9.1057e-08	5.6925e-09	1.1245e-09	3.5581e-10	1.4575e-10
2^{-3}	1.3770e-06	8.6125e-08	1.7022e-08	5.3861e-09	2.2060e-09
2^{-4}	2.1676e-05	1.3569e-06	2.6910e-07	8.5189e-08	3.4888e-08
2^{-5}	4.2477e-04	2.6670e-05	5.2734e-06	1.6692e-06	6.8381e-07
Soujanya and Reddy [5]					
2^{-1}	9.2363e-04	4.6407e-04	3.0991e-04	2.3263e-04	1.8619e-04
2^{-2}	1.6390e-03	8.2516e-04	5.5141e-04	4.1404e-04	3.3146e-04
2^{-3}	2.7044e-03	1.3653e-03	9.1315e-04	6.8602e-04	5.4937e-04
2^{-4}	4.1751e-03	2.1168e-03	1.4178e-03	1.0658e-03	8.5380e-04
2^{-5}	6.2518e-03	3.1866e-03	2.1382e-03	1.6088e-03	1.2895e-03
Swamy [4]					
2^{-1}	1.7734e-03	8.9248e-04	5.9631e-04	4.4773e-04	-
2^{-2}	4.0485e-03	2.0629e-03	1.3839e-03	1.0412e-03	-
2^{-3}	7.7050e-03	4.0273e-03	2.7243e-03	2.0579e-03	-
2^{-4}	1.3031e-02	7.2790e-03	5.0468e-03	3.8619e-03	-
2^{-5}	8.5398e-03	9.1073e-03	7.2501e-03	5.9025e-03	-

Table 6. The maximum absolute errors of Example 2, for different values of ε with $\delta = 0.03$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Our Method					
2^{-1}	6.8226e-10	3.8033e-11	7.5153e-12	2.3742e-12	9.5853e-13
2^{-2}	1.3051e-08	8.1568e-10	1.6112e-10	5.0953e-11	2.0934e-11
2^{-3}	1.7639e-07	1.1025e-08	2.1780e-09	6.8914e-10	2.8226e-10
2^{-4}	2.8836e-06	1.8027e-07	3.5626e-08	1.1273e-08	4.6172e-09
2^{-5}	5.0014e-05	3.1282e-06	6.1801e-07	1.9561e-07	8.0141e-08
Soujanya and Reddy [5]					
2^{-1}	5.0597e-04	2.5321e-04	1.6886e-04	1.2667e-04	1.0134e-04
2^{-2}	9.6556e-04	4.8357e-04	3.2250e-04	2.4194e-04	1.9357e-04
2^{-3}	1.7698e-03	8.8657e-04	5.9149e-04	4.4377e-04	3.5508e-04
2^{-4}	3.0307e-03	1.5201e-03	1.0145e-03	7.6132e-04	6.0926e-04
2^{-5}	4.7379e-03	2.3810e-03	1.5901e-03	1.1937e-03	9.5544e-04

$\varepsilon \downarrow$	$N=100$	$N=200$	$N=300$	$N=400$	$N=500$
Swamy [4]					
2^{-1}	9.9866e-04	5.0025e-04	3.3369e-04	2.5033e-04	-
2^{-2}	3.1466e-03	1.5822e-03	1.0568e-03	7.9335e-04	-
2^{-3}	7.1647e-03	3.6370e-03	2.4372e-03	1.8327e-03	-
2^{-4}	1.3971e-02	7.1917e-03	4.8431e-03	3.6513e-03	-
2^{-5}	2.6436e-02	1.3924e-02	9.4601e-03	7.1615e-03	-

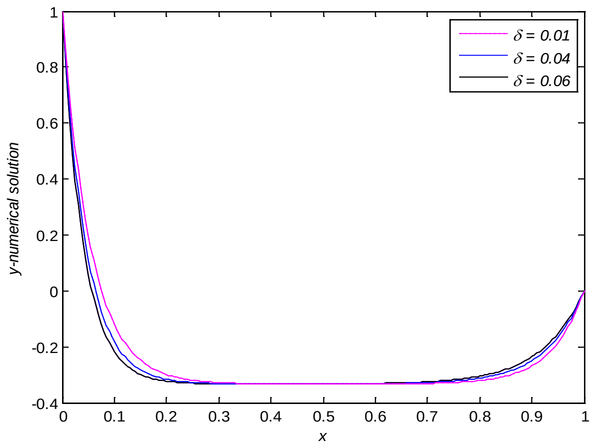

Figure 1. The numerical solution of Example 1 with $\varepsilon = 0.1$

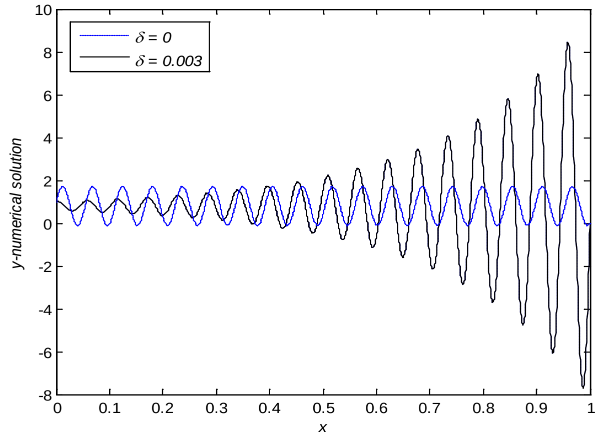

Figure 3. The numerical solution of Example 3 with $\varepsilon = 0.01$

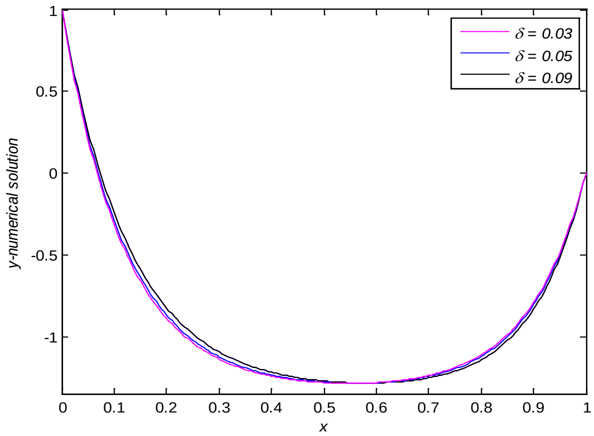

Figure 2. The numerical solution of Example 2 with $\varepsilon = 0.1$

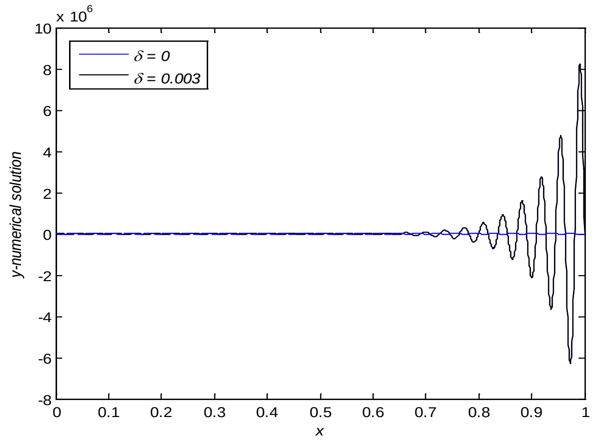

Figure 4. The numerical solution of Example 4 with $\varepsilon = 0.01$

Table 7. Rate of Convergence ρ for $\varepsilon = 0.1$ and $\delta = 0.05$

	h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	ρ
Example 1						
	1/100	1/200	3.3265e-06	1/400	2.0805e-07	3.9990
	1/200	1/400	2.0860e-07	1/800	1.3040e-08	3.9998
	1/300	1/600	4.1236e-08	1/1200	2.5774e-09	3.9999
Example 2						
	1/100	1/200	4.4265e-07	1/400	2.7667e-08	3.9999
	1/200	1/400	2.7686e-08	1/800	1.7304e-09	4.0000
	1/300	1/600	5.4687e-09	1/1200	3.4180e-10	4.0000
Example 3						
	1/100	1/200	1.0707e-05	1/400	6.6910e-07	4.0001
	1/200	1/400	6.6910e-07	1/800	4.1818e-08	4.0000
	1/300	1/600	1.3217e-07	1/1200	8.2610e-09	4.0000
Example 4						
	1/100	1/200	1.6132e-04	1/400	1.0082e-05	4.0001
	1/200	1/400	1.0082e-05	1/800	6.3013e-07	4.0000
	1/300	1/600	1.9930e-06	1/1200	1.2456e-07	4.0000

The Effect of Delay Parameter on the Solution Profile

The following graphs (Figure 1 – Figure 4) show the numerical solutions obtained by the present method for different values of delay parameter δ.

The Rate of Convergence (ρ)

In the same way in Eq. (34) one can define $Z_{h/2}$ by replacing h by $h/2$ and $N-1$ by $2N-1$, that is:

$$Z_{h/2} = \max_i \left| y_i^{h/2} - y_i^{h/4} \right|, \text{ for } i = 1, 2, ..., 2N-1.$$

The computational rate of convergence ρ is also obtained by using the double mesh principle defined as, Doolan et al. [6]:

$$\rho = \frac{\left(\log\left(Z_h\right) - \log\left(Z_{h/2}\right)\right)}{\log 2}.$$

6. Discussion and Conclusion

Numerical solution of second order singularly perturbed delay reaction-diffusion equations with boundary layer or oscillatory behaviour via finite difference method has been presented. To demonstrate the efficiency of the method, four model examples without exact solutions have been considered for different values of the perturbation parameter ε and delay parameter δ. The numerical solutions are tabulated (Table 1 - Table 6) in terms of maximum absolute errors and observed that the present method improves the findings of Soujanya and Reddy [5] and Swamy [4]. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. The stability and convergence of the method are investigated and established well. The results presented in Table 7 confirmed that computational rate of convergence as well as theoretical estimates indicate that method is a fourth order convergent.

Further, to investigate the effect of delay on the solution of the problem, numerical solutions have been presented using graphs. Accordingly, when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour (Figure 2). When the delay parameter is of $o(\varepsilon)$, the solution maintains layer behaviour although the coefficient of the delay term in the equation is of $O(1)$ and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases (Figure 1).

To demonstrate the effect on the oscillatory behavior, we consider the examples 3 and 4 when the solution of the problem exhibits oscillatory behaviour for delay parameter equal to zero and different from zero. We observe that, if the coefficient of the delay term is of $o(1)$, the amplitude of the oscillations increases slowly as the delay increases provided the delay parameter is of $o(\varepsilon)$ (Figure 3) and if the coefficient of the delay term is of $O(1)$, there is no oscillation in the left half of the interval [0,1] while the amplitude of the oscillations increases as the delay increases in the right half of the interval [0,1] provided the delay parameter is of $o(\varepsilon)$ (Figure 4).

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank Jimma University for the financial and material support.

References

[1] Gemechis File and Y.N. Reddy, *Numerical integration of a class of singularly perturbed delay differential equations with small shift*, International Journal of Differential Equation, 572723 (2012), 1-12.

[2] C. G. Lange and R. M. Miura, *Singular perturbation analysis of boundary-value problems for differential-difference equations very small shifts with layer behavior*, SIAM J. Appl. Math., 54 (1994), 249-272.

[3] V.P. Ramesh and M.K. Kadalbajoo, *Numerical Algorithm for Singularly Perturbed Delay Differential Equations with Layer and Oscillatory Behavior*, Neural, Parallel, and Scientific Computations, 19 (2011), 21-34.

[4] D. K. Swamy, *Quantative analysis of delay differential equations with layer*, Advance Research and Innovations in Mechanical, Material Science, Industrial Engineering and Management - ICARMMIEM- (2014), 145-150.

[5] G. BSL. Soujanya and Y.N. Reddy, *Computational method for singularly perturbed delay differential equations with layer or oscillatory behaviour*, Appl. Math. Inf. Sci. 10 (2), (2016), 527-536.

[6] E.R. Doolan, J.J.H. Miller and W.H.A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.

[7] M.K. Kadalbajoo and Y.N. Reddy, *Asymptotic and numerical analysis of singular perturbation problems*, Applied Mathematics and Computation, 30 (1989), 223-259.

[8] H.G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Eqns, Convection-Diffusion and Flow Problems*, Springer Verlag, Berlin, 1996.

[9] M.K. Kadalbajoo and V.P. Ramesh, *Numerical methods on Shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy*, Applied Mathematics and Computation, 188 (2007), 1816-1831.

[10] R. Pratima and K.K. Sharma, *Numerical analysis of singularly perturbed delay differential turning point problem*, Applied Mathematics and Computation, 218 (2011), 3483-3498.

[11] L.E. Elsgolt's and S.B. Norkin, *Introduction to the Theory and Applications of Differential Equations with Deviating Arguments*, Academic Press, New York, 1973.

[12] M. K. Kadalbajoo and Y.N. Reddy, *A non-asymptotic method for general singular perturbation problems*, Journal of Optimization Theory and Applications, 55 (1986), 256 269.

[13] R.S. Varga, *Matrix Iterative Analysis*. New Jersey: Prentice-Hall, Englewood Cliffs, 1962.

[14] D.M. Young, *Iterative Solution of Large Linear Systems*, Academic Press, *Inc.* New York, 1971.