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Solution of Singularly Perturbed Differential Difference Equations Using Higher Order Finite Differences

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Abstract In this paper, we discuss the solution of singularly perturbed differential-difference equations exhibiting dual layer using the higher order finite differences. First, the second order singularly perturbed differential-difference equations is replaced by an asymptotically equivalent second order singular perturbed ordinary differential equation. Then, fourth order stable finite difference scheme is applied to get a three term recurrence relation which is easily solved by Thomas algorithm. Some numerical examples have been solved to validate the computational efficiency of the proposed numerical scheme. To analyze the effect of the parameters on the solution, the numerical solution has also been plotted using graphs. The error bound and convergence of the method have also been established.

Keywords: *differential-difference equations, delay parameter, advance parameter, dual layer*

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1. Introduction

A class of functional differential equations which have the characteristics of both classes i.e., delay and/or advanced and singularly perturbed behaviour are known as Singularly Perturbed Differential-Difference Equations (SPDDEs). These differential-difference equation models have richer mathematical framework for the analysis of bio system dynamics, compared with ordinary differential equations. Also, they exhibit better consistency with the nature of the underlying process and predictive results. SPDDEs provide mathematical models in optics, physiology, mechanics, epidemiology, neural networks as well as many other applications Ref. [1,2,3,4]. As a result, numerical treatment of SPDDEs has received a great deal of attention. Stein [5] gave stochastic effects due to neuronal excitation with help of a model that represents a differential-difference equation. Lange and Miura [6,7,8,9,10] published a series of papers to study the numerical solution of SPDDEs. They extended the matched asymptotic expansions, initially developed for ordinary differential equations, to obtain approximate solutions to differential difference equations. The same authors [7] extended the problems involving boundary and interior layer phenomenon, rapid oscillations and resonance behaviour in [6] and [10] to the problems having solutions which exhibit turning point behaviour i.e., transition regions between rapid oscillations and exponential behaviour. Kadalbajoo and Sharma [11] devised a finite difference scheme to obtain the solution for singularly perturbed differential difference equations of mixed type i.e., which contain negative as well as positive shifts. The same authors [12] described a numerical approach based on finite difference method to solve a mathematical model arising from neuronal variability. Sharma and Kaushik [13] discussed two approaches, namely, fitted operator and fitted mesh method to solve SPDDEs with small delay as well as advanced, with layer behaviour. Amiraliyev and Cimen [14] used an exponentially fitted difference scheme on uniform mesh, accomplished by the method of integral identities, for linear second order delay differential equations. Prathima and Sharma [15] presented a numerical method to solve SPDDEs with negative shift and isolated turning point at x=0. Chakravarthy and Rao [16] presented a modified fourth order Numerov method for solving SPDDEs of mixed type.

With this motivation, in this paper, we discuss the solution of singularly perturbed differential-difference equations exhibiting dual layer using the higher order finite differences. First, the second order singularly perturbed differential-difference equations is replaced by an asymptotically equivalent second order singular perturbed ordinary differential equation. Then, fourth order stable finite difference scheme is applied to get a three term recurrence relation which is easily solved by Thomas algorithm. Some numerical examples have been solved to validate the computational efficiency of the proposed numerical scheme. To analyze the effect of the parameters on the solution, the numerical solution has also been plotted using graphs. The error bound and convergence of the method have also been established.

2. Description of the Method

Consider the singularly perturbed differential difference equation of mixed type, i.e., with delay as well as advance parameters of the form:

$$\varepsilon^2 y''(x) + a(x)y(x-\delta) + c(x)y(x) + b(x)y(x+\eta) = f(x) \tag{1}$$

$\forall\, x \in (0,1)$ and subject to the interval and boundary conditions

$$y(x) = \varphi(x)\,, \quad \text{on} \quad -\delta \le x \le 0 \tag{2}$$

$$y(x) = \gamma(x), \quad on \quad 1 \le x \le 1+\eta \tag{3}$$

where $a(x)$, $b(x)$, $c(x)$, $f(x)$, $\varphi(x)$ *and* $\gamma(x)$ are bounded and continuously differentiable functions on (0,1), $0 < \varepsilon << 1$ is the singular perturbation parameter; and $0 < \delta = o(\varepsilon)$ and $0 < \eta = o(\varepsilon)$ are the delay and the advance parameters respectively.

Using Taylor series expansion in the neighborhood of the point *x*, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) \tag{4}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) \tag{5}$$

Substituting (4) and (5) into Eq. (1), we obtain an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon^2 y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{6}$$

$$y(0) = \varphi(x) = \varphi_0 \tag{7}$$

$$y(1) = \gamma(x) = \gamma_1 \tag{8}$$

where $p(x) = \eta b(x) - \delta a(x)$

and $q(x) = a(x) + b(x) + c(x)$.

The transition from Eq.(1) to Eq. (6) is admitted, because of the condition that $0 < \delta << 1$ and $0 < \eta << 1$ are sufficiently small. Further details on the validity of this transition can be found in Elsgolt's and Norkin [17]. If $a(x) + b(x) + c(x) \le 0$ on the interval $[0, 1]$, then the solution of Eq. (1) exhibits boundary layers at both ends of the interval $[0,1]$, whereas it exhibits oscillatory behaviour for $a(x) + b(x) + c(x) > 0$. Here, we have considered the first case where the solutions of the problem exhibit dual layers. Dividing the interval [0, 1] into N equal parts with constant mesh length h, with mesh points: $0 = x_0,\ x_1,\ x_2,\ \ldots,\ x_N = 1$, we have $x_i = ih,\quad i = 0,\ 1,\ 2,\ \ldots,\ N$. Assuming that $y(x)$ has continuous fourth derivatives on [0,1] and by making use of Taylor's expansion, we have the fourth order central differences:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) + T_1 \tag{9}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''(\varsigma) + T_2 \tag{10}$$

where $T_1 = -2h^4 y^{(6)}(\xi)\big/6!$ and $T_2 = -h^4 y^{(5)}(\varsigma)\big/5!$ for $\xi, \varsigma \in [x_i - h, x_i + h]$.

Substituting (9) and (10) into Eq. (6) we can write the central difference approximation of Eq.(6) in the form that includes all the $o(h^2)$ error terms as follows:

$$\left[\frac{\varepsilon^2}{h^2} - \frac{p_i}{2h}\right] y_{i-1} + \left[q_i - \frac{2\varepsilon^2}{h^2}\right] y_i + \left[\frac{\varepsilon^2}{h^2} + \frac{p_i}{2h}\right] y_{i+1} - \frac{h^2}{12}\left[2 p_i y_i''' + \varepsilon^2 y_i^{(4)}\right] + T = f_i \tag{11}$$

where $T = \varepsilon^2 T_1 + p_i T_2$.

Now, from Eq.(6) we have

$$\varepsilon^2 y_i'' = -p_i y_i' - q_i y_i + f_i \tag{12}$$

Differentiating both sides of (12), we get:

$$\varepsilon^2 y_i''' = -\left(p_i y_i'' + p_i' y_i' + q_i' y_i + q_i y_i'\right) + f_i' \tag{13}$$

$$\varepsilon^2 y_i^{(4)} = -\begin{bmatrix} p_i y_i''' + \left(2p_i' + q_i\right) y_i'' \\ +\left(p_i'' + 2q_i'\right) y_i' + q_i'' y_i \end{bmatrix} + f_i'' \tag{14}$$

Using (13) and (14) into (11) we obtain

$$\begin{aligned}
&\left(\frac{\varepsilon^2}{h^2} - \frac{p_i}{2h}\right) y_{i-1} + \left(q_i - \frac{2\varepsilon^2}{h^2}\right) y_i \\
&+\left(\frac{\varepsilon^2}{h^2} + \frac{p_i}{2h}\right) y_{i+1} + \frac{h^2}{12}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_i'' \\
&+\frac{h^2}{12}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right) y_i' \\
&+\frac{h^2}{12}\left(\frac{p_i q_i'}{\varepsilon^2} + q_i''\right) y_i + T = f_i + \frac{h^2}{12}\left(\frac{p_i f_i'}{\varepsilon^2} + f_i''\right)
\end{aligned} \tag{15}$$

Now, approximating the converted error term, which has a stabilizing effect, in (15) by using the central difference formulas (9) and (10) for y_i'' and y_i' we obtain

$$\begin{aligned}
&\left(\frac{\varepsilon^2}{h^2} - \frac{p_i}{2h}\right) y_{i-1} + \left(q_i - \frac{2\varepsilon^2}{h^2}\right) y_i \\
&+\left(\frac{\varepsilon^2}{h^2} + \frac{p_i}{2h}\right) y_{i+1} + \frac{1}{12}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_{i+1} \\
&-\frac{1}{6}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_i + \frac{1}{12}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_{i-1} \\
&+\frac{h}{24}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right) y_{i+1} \\
&-\frac{h}{24}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right) y_{i-1} \\
&+\frac{h^2}{12}\left(\frac{p_i q_i'}{\varepsilon^2} + q_i''\right) y_i = f_i + \frac{h^2}{12}\left(\frac{p_i f_i'}{\varepsilon^2} + f_i''\right) + \tilde{T}
\end{aligned} \tag{16}$$

Where

$$\tilde{T} = \left(\frac{q_i^2}{\varepsilon^2} + 2q_i' + r_i\right)\frac{h^4}{144} y_i^{(4)} + \left(\frac{q_i q_i'}{\varepsilon^2} + \frac{q_i r_i}{\varepsilon^2} + q_i'' + 2r'\right)\frac{h^4}{72} y''' - T$$

is the truncation error and $T = \varepsilon^2 T_1 + q_i T_2 = O\left(h^4\right)$.

Rearranging Eq. (16) we obtain the three term recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{17}$$

for i = 1 (1) N-1.
where

$$E_i = \frac{\varepsilon^2}{h^2} - \frac{p_i}{2h} + \frac{1}{12}\left(\frac{p_i{}^2}{\varepsilon^2} + 2p_i' + q_i\right) - \frac{h}{24}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right)$$

$$F_i = \frac{2\varepsilon^2}{h^2} - q_i + \frac{1}{6}\left(\frac{p_i{}^2}{\varepsilon^2} + 2p_i' + q_i\right) - \frac{h^2}{12}\left(\frac{p_i q_i'}{\varepsilon^2} + q_i''\right)$$

$$G_i = \frac{\varepsilon^2}{h^2} + \frac{p_i}{2h} + \frac{1}{12}\left(\frac{p_i{}^2}{\varepsilon^2} + 2p_i' + q_i\right) + \frac{h}{24}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right)$$

$$H_i = f_i + \frac{h^2}{12}\left(\frac{p_i f_i'}{\varepsilon^2} + f_i''\right)$$

This gives us the tri-diagonal system which can easily be solved by Thomas Algorithm.

3. Thomas Algorithm

We give a brief description of Thomas algorithm for solving the tri-diagonal system. Consider the system:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{18}$$

$$i = 1,\ 2,\ \ldots,\ N-1$$

subject to the boundary conditions

$$y_0 = y(0) = \varphi_0 \tag{19}$$

$$y_N = y(1) = \gamma_1 \tag{20}$$

We set

$$y_i = W_i y_{i+1} + T_i \tag{21}$$

for $i = N-1,\ N-2,\ \ldots,\ 2,\ 1$,
where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

From (21), we have:

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{22}$$

By substituting (22) in (18) and comparing with (21) we get the recurrence relations:

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) \tag{23}$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right) \tag{24}$$

To solve these recurrence relations for $i = 1,\ 2,\ \ldots,\ N-1$, we need the initial conditions for W_0 and T_0. For this we take $y_0 = \varphi_0 = W_0 y_1 + T_0$. We choose $W_0 = 0$ so that the value of $T_0 = \varphi_0$. With these initial values, we compute W_i and T_i for $i = 1, 2, \ldots N-1$ from (23) and (24) in forward process, and then obtain y_i in the backward process from (20) and (21).

4. Error Analysis

In this section we discuss the error analysis of the method. Writing the tri-diagonal system (18) in matrix-vector form, we get

$$AY = C \tag{26}$$

where, $A = \left(m_{ij}\right)$, $1 \le i, j \le N\text{-}1$ is a tri diagonal matrix of order N-1, with

$$m_{ii+1} = -\varepsilon^2 + \frac{h}{2} p_i - \frac{h^2}{12}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) + \frac{h^3}{24}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right)$$

$$m_{ii} = 2\varepsilon^2 - h^2 q_i + \frac{h^2}{6}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) - \frac{h^4}{12}\left(\frac{p_i q_i'}{\varepsilon^2} + q_i''\right)$$

$$m_{ii-1} = -\varepsilon^2 - \frac{h}{2} p_i - \frac{h^2}{12}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) - \frac{h^3}{24}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right)$$

and $C = \left(d_i\right)$ is a column vector with $d_i = -h^2 f_i - \frac{h^4}{12}\left(\frac{p_i f_i'}{\varepsilon^2} + f_i''\right)$, where $i = 1,2,\ldots,N\text{-}1$ with local truncation error

$$T_i(h) = h^6 K + o(h^7) \tag{27}$$

where

$$K = -\frac{1}{144}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_i^{(4)} - \frac{1}{72}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right) y_i'''$$

We also have

` $A\,\overline{Y} - T(h) = C$ (28)

where $\overline{Y} = \left(\overline{y_0}, \overline{y_1}, \overline{y_2}, \ldots, \overline{y_N}\right)^T$ denotes the actual solution and

$T(h) = (T_1(h), T_2(h), \ldots, T_N(h))^T$ is the local truncation error.

From (26) and (28), we get

$$A\left(\overline{Y} - Y\right) = T(h) \quad (29)$$

Thus, we obtain the error equation

$$AE = T(h) \quad (30)$$

where $E = \overline{Y} - Y = (e_0, e_1, e_2, \ldots, e_N)^T$.

Let S_i be the sum of elements of the i^{th} row of A, then we have

$$S_i = \varepsilon^2 + \frac{h}{2} p_i + \frac{h^2}{6}\left(\frac{p_i}{2\varepsilon^2} + p_i' + \frac{p_i q_i}{4\varepsilon^2} - \frac{11}{2} q_i\right) + \frac{h^3}{12}\left(\frac{p_i p_i'}{2\varepsilon^2} + \frac{1}{2} p_i'' + q_i'\right) - \frac{h^4}{12}\left(\frac{p_i q_i'}{\varepsilon^2} - q_i''\right) + O\left(h^5\right) \text{ for i = 1}$$

$$S_i = -h^2 q_i - \frac{h^4}{12}\left(\frac{p_i q_i'}{\varepsilon^2} - q_i''\right) + O\left(h^3\right) \quad \text{for i = 2,3,..N-2}$$

$$S_i = \varepsilon^2 - \frac{h}{2} p_i + \frac{h^2}{6}\left(p_i' + \frac{1}{2\varepsilon} p_i^2 - \frac{11}{2} h^2 q_i\right) - \frac{h^3}{12}\left(\frac{1}{2\varepsilon} p_i\, p_i' + \frac{1}{2\varepsilon^2} p_i\, q_i + \frac{1}{2} p_i'' - q_i'\right) + O\left(h^4\right) \text{ for i = N-1}$$

and $C = (d_i)$ is a column vector with $d_i = -h^2 f_i - \frac{h^4}{12}\left(\frac{p_i f_i'}{\varepsilon^2} + f_i''\right)$, where $i = 1,2 \ldots N\text{-}1$ with local truncation error

$$T_i(h) = h^6\left[-\frac{1}{144}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_i^{(4)} - \frac{1}{72}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q_i'\right) y_i'''\right] + o(h^7)$$

Since $0 < \varepsilon << 1$ and $\delta = o(\varepsilon)$, for sufficiently small h the matrix A is irreducible and monotone Ref. [18]. Then it follows that A^{-1} exists and its elements are non negative.

Hence, from Eq. (30), we get

$$E = A^{-1} T(h) \quad (31)$$

and

$$\|E\| \le \left\|A^{-1}\right\| . \left\|T(h)\right\| \quad (32)$$

Let $\overline{m}_{k,i}$ be the $(k,i)^{th}$ element of A^{-1}. Since $\overline{m}_{k,i} \ge 0$, from the theory of matrices we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i}\ S_i = 1, \quad k = 1,2,\ldots N\text{-}1 \quad (33)$$

Therefore,

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \le \frac{1}{\min\limits_{1 \le i \le N-1} S_i} = \frac{1}{h^2 |q_i|} \quad (34)$$

We define $\left\|A^{-1}\right\| = \max\limits_{1 \le k \le N-1} \sum_{i=1}^{N-1} \left|\overline{m}_{k,i}\right|$ and $\left\|T(h)\right\| = \max\limits_{1 \le i \le N-1} \left|T_i(h)\right|$.

From (23), (27), (28) and (30), we get

$$e_j = \sum_{i=1}^{N-1} \overline{m}_{k.i}\ T_i(h), \quad j = 1,2,3..N\text{-}1$$

which implies

$$e_j \le \frac{h^6 K}{h^2 |q_i|} = \frac{h^2 K}{\left|B_{i_0}\right|}, \quad j = 1,2\ldots N\text{-}1 \quad (35)$$

where

$$K = -\frac{1}{144}\left(\frac{p_i^2}{\varepsilon^2} + 2p_i' + q_i\right) y_i^{(4)} - \frac{1}{72}\left(\frac{p_i p_i'}{\varepsilon^2} + \frac{p_i q_i}{\varepsilon^2} + p_i'' + 2q'\right) y_i'''.$$

Therefore, $\|E\| = o\left(h^4\right)$

Hence, the method gives a fourth order convergence for uniform mesh as given in Ref. [19].

5. Numerical Examples

To validate the computational efficiency of the scheme, we have applied it to two cases of the problem (1)-(2,3). These cases have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

$$\varepsilon^2 y''(x) + a(x) y(x-\delta) + c(x) y(x) + b(x) y(x+\eta) = f(x)$$

$\forall x \in (0,1)$ and subject to the conditions

$$y(x) = \varphi(x), \quad \text{on} \quad -\delta \le x \le 0$$

$$y(x) = \gamma(x), \quad \text{on} \quad 1 \le x \le 1+\eta$$

The exact solution of such boundary value problems having constant coefficients (i.e. $a(x) = a$, $b(x) = b$, $c(x) = c$, $f(x) = f$, $\varphi(x) = \varphi$ and $\gamma(x) = \gamma$ are constants) is given by:

$$y(x) = \left(\left[(1-a-b-c)\exp(m_2) - 1\right]\exp(m_1 x) - \left[(1-a-b-c)\exp(m_1) - 1\right]\exp(m_2 x)\right) / \left[(1-a-b-c)\exp(m_1) - 1\right] \exp(m_2 x)\left[(a+b+c)\left(\exp(m_1) - \exp(m_2)\right)\right] + 1/(a+b+c) \quad (36)$$

where

$$m_1 = \frac{\left[(a\delta - b\eta) + \sqrt{(b\eta - a\delta)^2 - 4\varepsilon^2(a+b+c)}\right]}{2\varepsilon^2}$$

$$m_2 = \frac{\left[(a\delta - b\eta) - \sqrt{(b\eta - a\delta)^2 - 4\varepsilon^2(a+b+c)}\right]}{2\varepsilon^2}$$

Example 1: Consider the model problem given by equations (1)-(3) with

$$a(x) = -2,\ b(x) = -2,\ c(x) = -1,$$
$$f(x) = 1,\ \ \varphi(x) = 1,\ \ \gamma(x) = 0.$$

The exact solution of the problem is given by Eq. (36). The numerical results are given in Table 1-Table 4 for ε =0.1, 0.01 and different values of the delay and advance parameters.

Table 1. Numerical solution of Example 1 for $\varepsilon = 0.1$; $\delta = 0.07$ and N=100

x	$\eta = 0.00$		$\eta = 0.03$		$\eta = 0.06$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.6852094	0.6851608	0.7187086	0.7186640	0.7498358	0.7497953
0.03	0.2816988	0.2816195	0.3384816	0.3384032	0.3950906	0.3950146
0.05	0.0621230	0.0620510	0.1156196	0.1155430	0.1728358	0.1727564
0.07	-0.0573621	-0.0574169	-0.0150061	-0.0150690	0.0335889	0.0335193
0.09	-0.1223815	-0.1224199	-0.0915697	-0.0916171	-0.0536519	-0.0537080
0.20	-0.1972676	-0.1972706	-0.1942572	-0.1942628	-0.1888177	-0.1888273
0.40	-0.1999833	-0.1999833	-0.1999698	-0.1999699	-0.1998952	-0.1998954
0.60	-0.1997206	-0.1997202	-0.1998878	-0.1998877	-0.1999605	-0.1999604
0.80	-0.1925249	-0.1925204	-0.1952673	-0.1952640	-0.1972243	-0.1972220
0.90	-0.1613347	-0.1613231	-0.1692343	-0.1692234	-0.1764387	-0.1764290
0.92	-0.1462890	-0.1462761	-0.1552637	-0.1552511	-0.1638619	-0.1638500
0.94	-0.1253886	-0.1253752	-0.1349493	-0.1349354	-0.1445717	-0.1445581
0.96	-0.0963554	-0.0963429	-0.1054101	-0.1053967	-0.1149846	-0.1149706
0.98	-0.05602462	-0.0560159	-0.0624573	-0.0624476	-0.0696041	-0.0695934
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table 2. Numerical solution of Example 1for $\varepsilon = 0.1$; $\eta = 0.05$ and N=100

x	$\delta = 0.00$		$\delta = 0.03$		$\delta = 0.06$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.8032296	0.8031978	0.7781023	0.7780662	0.7498358	0.7497953
0.03	0.5011945	0.5011280	0.4498160	0.4497440	0.3950906	0.3950146
0.05	0.2900910	0.2900134	0.2317144	0.2316346	0.1728358	0.1727564
0.07	0.1425429	0.1424670	0.0868155	0.0867413	0.0335889	0.0335193
0.09	0.0394160	0.0393478	-0.0094500	-0.0095134	-0.0536519	-0.0537080
0.20	-0.1666140	-0.1666352	-0.1798978	-0.1799126	-0.1888177	-0.1888273
0.40	-0.1990711	-0.1990723	-0.1996631	-0.1996636	-0.1998952	-0.1998954
0.60	-0.1999713	-0.1999713	-0.1999830	-0.1999830	-0.1999605	-0.1999604
0.80	-0.1992474	-0.1992467	-0.1984970	-0.1984956	-0.1972243	-0.1972220
0.90	-0.1877378	-0.1877314	-0.1826627	-0.1826546	-0.1764387	-0.1764290
0.92	-0.1785681	-0.1785593	-0.1717258	-0.1717152	-0.1638619	-0.1638500
0.94	-0.1625414	-0.1625297	-0.1538895	-0.1538765	-0.1445717	-0.1445581
0.96	-0.1345297	-0.1345162	-0.1248013	-0.1247873	-0.1149846	-0.1149706
0.98	-0.0855707	-0.0855589	-0.0773634	-0.0773519	-0.0696041	-0.0695934
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table 3. Numerical solution of Example 1 for $\varepsilon = 0.01$; $\eta = 0.005$ and N=100

x	$\delta = 0.000$		$\delta = 0.003$		$\delta = 0.006$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	-0.1717803	-0.1666352	0.1909078	0.1888273	-0.1836011	-0.1799126
0.04	-0.1993363	-0.1990723	0.1999311	0.1998959	-0.1997758	-0.1996637
0.06	-0.1999843	-0.1999742	0.1999994	0.1999990	-0.1999969	-0.1999943
0.08	-0.1999996	-0.1999992	0.1999999	0.1999999	-0.1999999	-0.1999999
0.20	-0.2000000	-0.2000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.40	-0.2000000	-0.2000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.60	-0.2000000	-0.2000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.80	-0.2000000	-0.2000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.90	-0.1999999	-0.1999999	0.1999999	0.1999999	-0.1999999	-0.1999999
0.92	-0.1999999	-0.1999999	0.1999999	0.1999999	-0.1999999	-0.1999999
0.94	-0.1999999	-0.1999999	0.1999997	0.1999994	-0.1999999	-0.1999999
0.96	-0.1999975	-0.1999971	0.1999759	0.1999614	-0.1999927	-0.1999886
0.98	-0.1993018	-0.1992474	0.1978053	0.1972220	-0.1987979	-0.1984956
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table 4. Numerical solution of Example 1 for $\varepsilon = 0.01$; $\delta = 0.007$ and N=100

x	$\eta = 0.000$		$\eta = 0.003$		$\eta = 0.006$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	-0.1610390	-0.1551229	0.1762764	0.1715840	-0.1864881	-0.1833323
0.04	-0.1987350	-0.1983217	0.1995309	0.1993271	-0.1998478	-0.1997684
0.06	-0.1999589	-0.19993723	0.1999907	0.1999840	-0.1999982	-0.1999967
0.08	-0.1999986	-0.19999765	0.1999998	0.1999996	-0.1999999	-0.1999999
0.20	-0.1999999	-0.19999999	0.2000000	0.2000000	-0.2000000	-0.2000000
0.40	-0.2000000	-0.20000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.60	-0.2000000	-0.20000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.80	-0.2000000	-0.20000000	0.2000000	0.2000000	-0.2000000	-0.2000000
0.90	-0.1999999	-0.19999999	0.1999999	0.1999999	-0.1999999	-0.1999999
0.92	-0.1999999	-0.19999999	0.1999999	0.1999999	-0.1999999	-0.1999999
0.94	-0.1999999	-0.19999999	0.1999999	0.1999999	-0.1999999	-0.1999998
0.96	-0.1999985	-0.19999896	0.1999966	0.1999954	-0.1999892	-0.1999826
0.98	-0.1994603	-0.19954517	0.1991755	0.1990438	-0.1985317	-0.1981378
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Example 2: Consider the model problem given by equations (1)-(3) with

$$a(x) = 0.25,\ b(x) = 0.25,\ c(x) = -1,$$
$$f(x) = 1,\ \ \varphi(x) = 1,\ \ \gamma(x) = 0.$$

The exact solution of the problem is given by Eq. (36). The numerical results are given in Table 5- Table 8 for ε =0.1, 0.01 and different values of the delay and advance parameters.

Table 5. Numerical solution of Example 2 for $\varepsilon = 0.1$; $\delta = 0.07$ and N=100

x	$\eta = 0.00$		$\eta = 0.03$		$\eta = 0.06$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.8183456	0.8183354	0.8088641	0.8088533	0.7988828	0.7988714
0.03	0.4874103	0.4873831	0.4624101	0.4623817	0.4362809	0.4362513
0.05	0.1953953	0.1953554	0.1587787	0.1587372	0.1207850	0.1207420
0.07	-0.0622659	-0.0623151	-0.1073082	-0.1073590	-0.1537044	-0.1537566
0.09	-0.2896025	-0.2896583	-0.3404761	-0.3405333	-0.3924940	-0.3925523
0.20	-1.1373021	-1.1373637	-1.1923861	-1.1924467	-1.2463575	-1.2464166
0.40	-1.7373567	-1.7373880	-1.7639703	-1.7639980	-1.7871364	-1.7871602
0.60	-1.8481739	-1.8481741	-1.8465416	-1.8465377	-1.8413608	-1.8413530
0.80	-1.5771988	-1.5771666	-1.5471171	-1.5470824	-1.5150297	-1.5149930
0.90	-1.0931658	-1.0931264	-1.0576093	-1.0575696	-1.0218891	-1.0218494
0.92	-0.9389593	-0.9389219	-0.9054360	-0.9053988	-0.8720932	-0.8720563
0.94	-0.7576246	-0.7575915	-0.7280374	-0.7280047	-0.6988914	-0.6988593
0.96	-0.5444978	-0.5444717	-0.5213136	-0.5212881	-0.4986875	-0.4986627
0.98	-0.2941007	-0.2940853	-0.2804886	-0.2804737	-0.2673248	-0.2673104
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table 6. Numerical solution of Example 2 for $\varepsilon = 0.1$; $\eta = 0.05$ and N=100

x	$\delta = 0.00$		$\delta = 0.03$		$\delta = 0.06$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.7774611	0.7774487	0.7884102	0.7883983	0.7988828	0.7988714
0.03	0.3808618	0.3808299	0.4090750	0.4090442	0.4362809	0.4362513
0.05	0.0411538	0.0411084	0.0815297	0.0814854	0.1207850	0.1207420
0.07	-0.2497876	-0.2498419	-0.2012692	-0.2013226	-0.1537044	-0.1537566
0.09	-0.4989222	-0.4989818	-0.4454036	-0.4454627	-0.3924940	-0.3925523
0.20	-1.3492323	-1.3492871	-1.2987820	-1.2988392	-1.2463575	-1.2464166
0.40	-1.8224426	-1.8224580	-1.8066673	-1.8066869	-1.7871364	-1.7871602
0.60	-1.8208802	-1.8208652	-1.8327478	-1.8327364	-1.8413608	-1.8413530
0.80	-1.4462457	-1.4462061	-1.4812865	-1.4812481	-1.5150297	-1.5149930
0.90	-0.9510154	-0.9509765	-0.9862750	-0.9862355	-1.0218891	-1.0218494
0.92	-0.8068149	-0.8067794	-0.8391531	-0.8391167	-0.8720932	-0.8720563
0.94	-0.6425650	-0.6425346	-0.6703526	-0.6703212	-0.6988914	-0.6988593
0.96	-0.4555107	-0.4554875	-0.4767242	-0.4767001	-0.4986875	-0.4986627
0.98	-0.2425137	-0.2425005	-0.2546547	-0.2546409	-0.2673248	-0.2673104
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table 7. Numerical solution of Example 2 for $\varepsilon = 0.01$; $\eta = 0.007$ and N=100

x	$\delta = 0.000$		$\delta = 0.003$		$\delta = 0.006$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	-1.3940899	-1.3943104	-1.3422222	-1.3423828	-1.2886972	-1.2888146
0.04	-1.8776243	-1.8777133	-1.8557761	-1.8558465	-1.8313494	-1.8314051
0.06	-1.9752837	-1.9753107	-1.9683775	-1.9684007	-1.9600128	-1.9600326
0.08	-1.9950080	-1.9950153	-1.9930664	-1.9930732	-1.9905189	-1.9905252
0.20	-1.9999996	-1.9999996	-1.9999992	-1.9999992	-1.9999983	-1.9999983
0.40	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999
0.60	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999
0.80	-1.9999926	-1.9999925	-1.9999962	-1.9999962	-1.9999981	-1.9999981
0.90	-1.9961664	-1.9961390	-1.9972690	-1.9972484	-1.9980923	-1.9980773
0.92	-1.9866006	-1.9865241	-1.9897845	-1.9897228	-1.9923334	-1.9922850
0.94	-1.9531648	-1.9529645	-1.9617878	-1.9616149	-1.9691888	-1.9690430
0.96	-1.8362966	-1.8358300	-1.8570631	-1.8566322	-1.8761730	-1.8757825
0.98	-1.4278053	-1.4269904	-1.4653283	-1.4645230	-1.5023516	-1.5015676
1.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table 8. Numerical solution of Example 2 for $\varepsilon = 0.01$; $\delta = 0.005$ and N=100

x	$\eta = 0.000$		$\eta = 0.003$		$\eta = 0.006$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	-1.1780109	-1.1780821	-1.2338439	-1.2339324	-1.2886972	-1.2888146
0.04	-1.7747779	-1.7748170	-1.8043349	-1.8043801	-1.8313494	-1.8314051
0.06	-1.9382899	-1.9383060	-1.9500300	-1.9500473	-1.9600128	-1.9600326
0.08	-1.9830916	-1.9830975	-1.9872383	-1.9872442	-1.9905189	-1.9905252
0.20	-1.9999928	-1.9999928	-1.9999964	-1.9999964	-1.9999983	-1.9999983
0.40	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999
0.60	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999	-1.9999999
0.80	-1.9999996	-1.9999996	-1.9999991	-1.9999991	-1.9999981	-1.9999981
0.90	-1.9991227	-1.9991155	-1.9986935	-1.9986829	-1.9980923	-1.9980773
0.92	-1.9958817	-1.9958546	-1.9943366	-1.9942997	-1.9923334	-1.9922850
0.94	-1.9806673	-1.9805718	-1.9754492	-1.9753294	-1.9691888	-1.9690430
0.96	-1.9092450	-1.9089464	-1.8935728	-1.8932266	-1.8761730	-1.8757825
0.98	-1.5739602	-1.5732598	-1.5386385	-1.5378889	-1.5023516	-1.5015676
1.00	0.0000000	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000

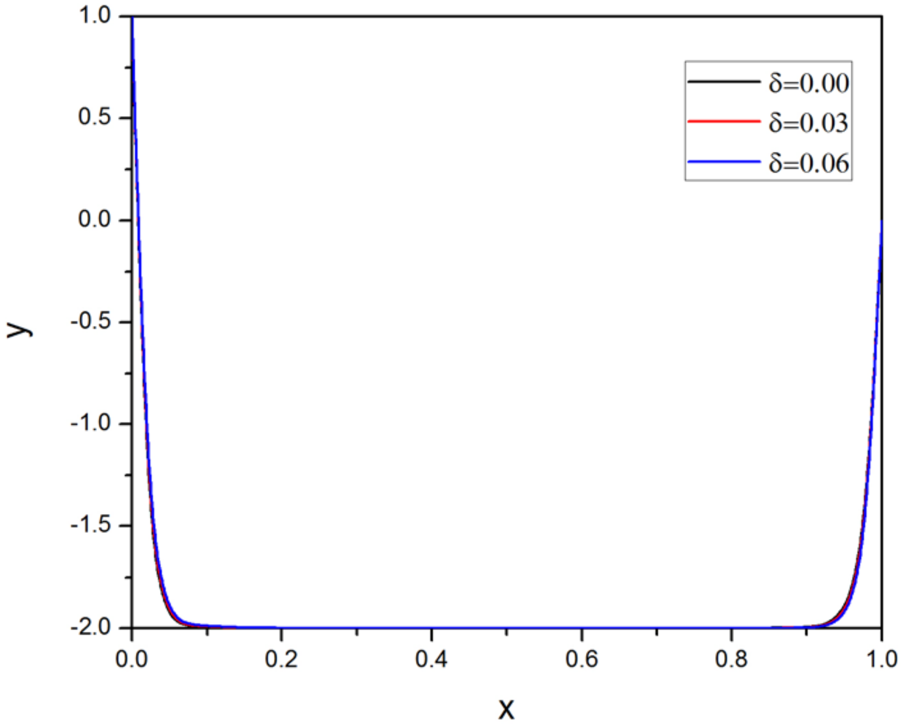

Figure 1. Numerical solution of Example 1 for $\varepsilon = 0.1$ and $\delta = 0.07$

6. Discussion and Conclusion

In this paper, we have discussed the solution of singularly perturbed differential-difference equations exhibiting dual layer using the higher order finite differences. First, the second order singularly perturbed differential-difference equations is replaced by an asymptotically equivalent second order singular perturbed ordinary differential equation. Then, fourth order stable finite difference scheme is applied to get a three term recurrence relation which is easily solved by Thomas algorithm. The error bound and convergence of the method have also been established. Some model examples (*i.e.*, examples in which both the negative and positive shifts are non zero) have been solved to demonstrate the applicability of the proposed approach, by taking various values of the delay, advanced and perturbation parameters. These cases have been chosen because they have been widely discussed in literature Ref. [20,21,22] and also because the exact solutions are available for comparison. The numerical solutions are presented in tables and compared with the exact solution. To analyze the effect of the parameters on the solution, the numerical solution has also been plotted using graphs. From the graphs, it is observed that by varying either δ or η the thickness of the left boundary layer decreases and that of the right boundary layer increases. The effect of negative shift is more dominant than the positive shift.

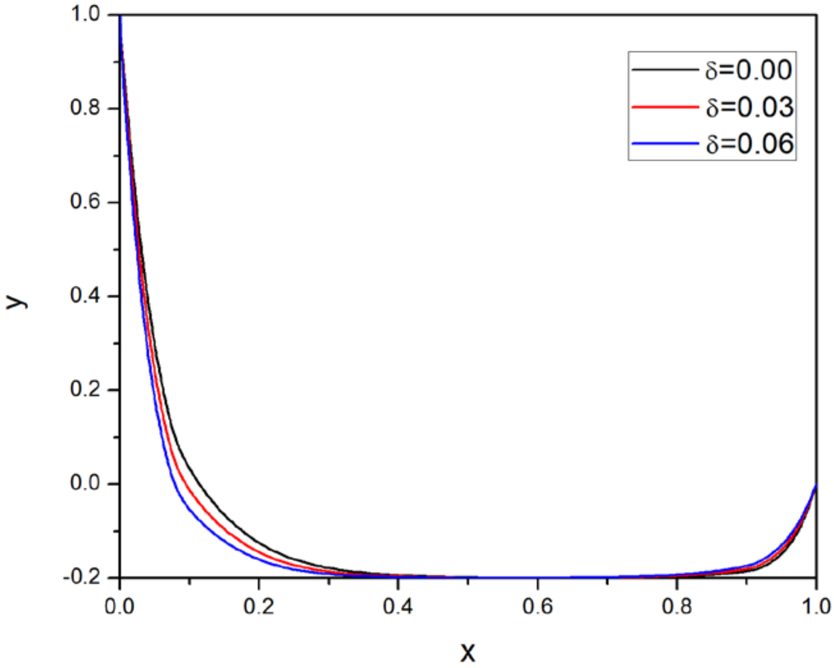

Figure 2. Numerical solution of Example 1 for $\varepsilon = 0.1$ and $\eta = 0.05$

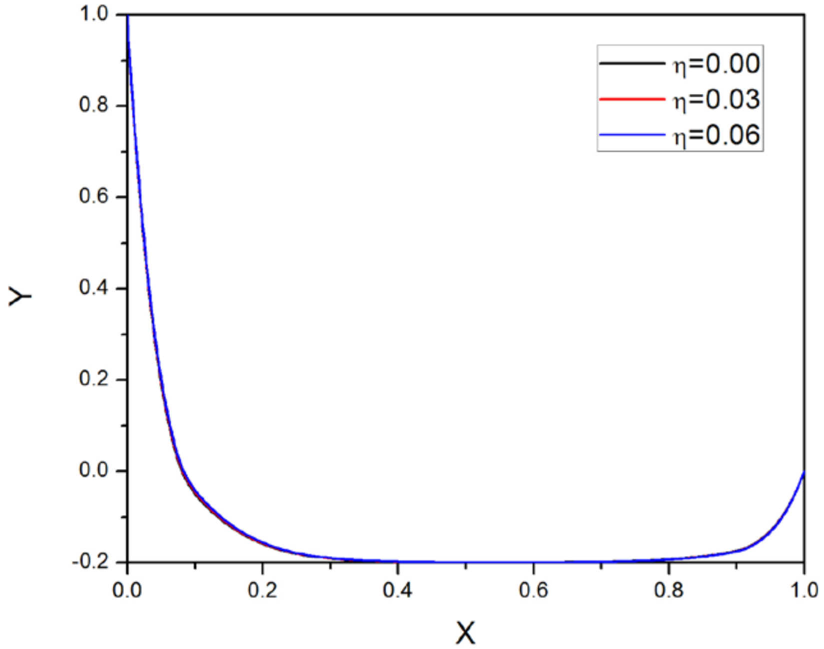

Figure 3. Numerical solution of Example 1 for $\varepsilon = 0.01$ and $\delta = 0.007$

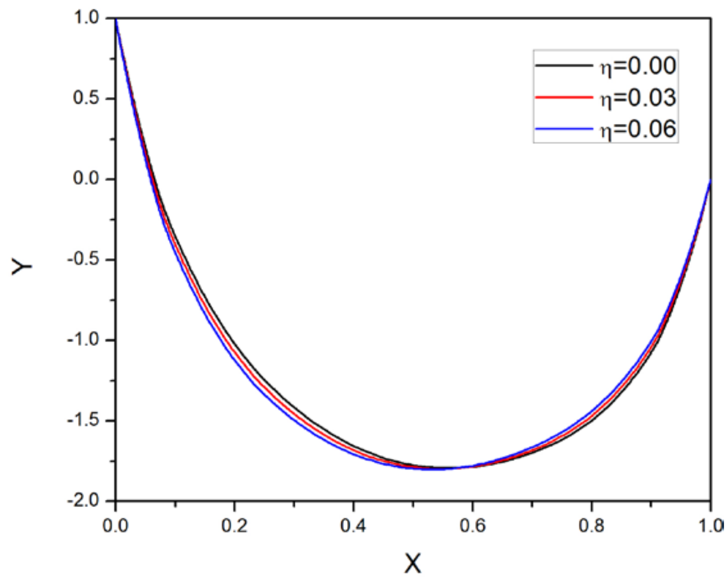

Figure 4. Numerical solution of Example 2 for $\varepsilon = 0.1$ and $\delta = 0.07$

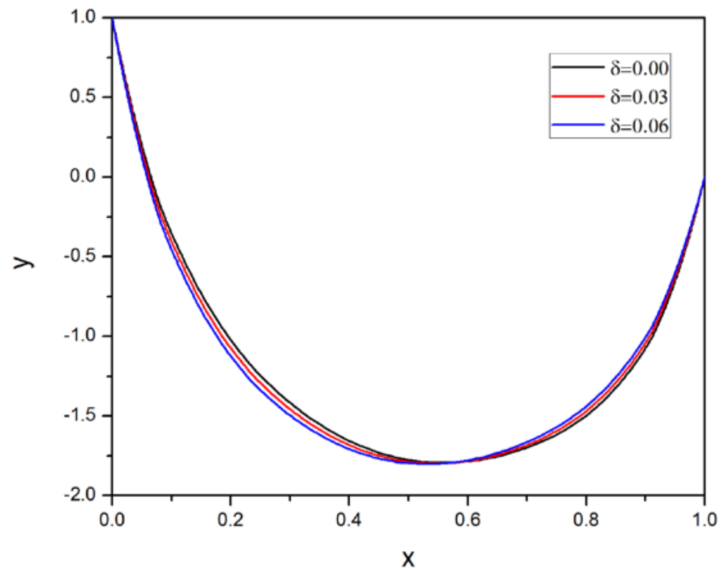

Figure 5. Numerical solution of Example 2 for $\varepsilon = 0.1$ and $\eta = 0.05$

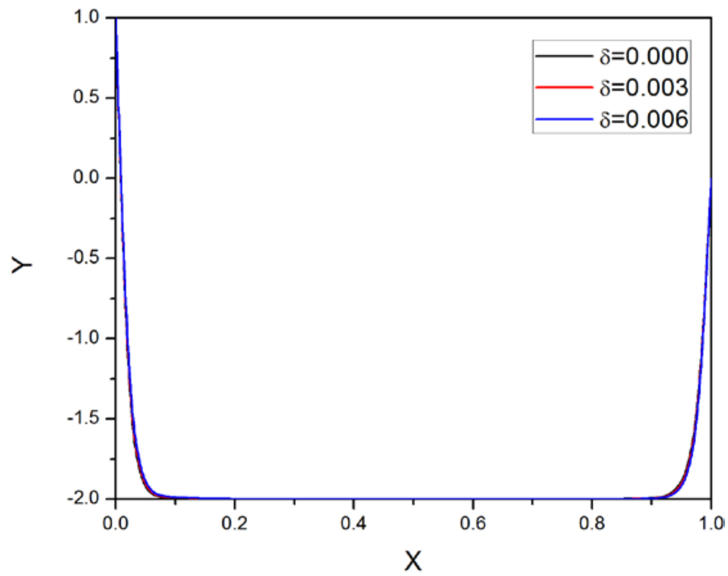

Figure 6. Numerical solution of Example 2 for $\varepsilon = 0.01$ and $\eta = 0.007$

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