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Initial Value Approach for a Class of Singular Perturbation Problems

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Abstract In this paper, we present an initial value approach for a class of singularly perturbed two point boundary value problems with a boundary layer at one end point. The idea is to replace the original two point boundary value problem by set of suitable initial value problems. This replacement is significant from the computational point of view. This method does not depend on asymptotic expansions. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory.

Keywords: *singular perturbations, boundary value problems, boundary layer, initial value approach*

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1. Introduction

Singular perturbation problem now is a maturing mathematical subject with fairly long history and a strong promise for continued important applications throughout science and engineering. A singular perturbation problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the interval, as the perturbation parameter $\varepsilon \to 0$. Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior. Detailed theory and analytical discussion on singular perturbation problems is given in the books and high level monographs: Ref: [1-12].

In this paper, we present an initial value approach for a class of singularly perturbed two point boundary value problems with a boundary layer at one end point. The idea is to replace the original two point boundary value problem by set of suitable initial value problems. This replacement is significant from the computational point of view. This method does not depend on asymptotic expansions. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory.

2. Initial Value Approach

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), x \in [0, 1] \quad (1)$$

with

$$y(0) = \alpha, y(1) = \beta \quad (2)$$

where ε is a small positive parameter ($0<\varepsilon<<1$) and α, β are known constants. We assume that a(x), b(x) and f(x) are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \geq M > 0$ throughout the interval [0, 1], where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of x=1.

The method consist the following steps:

Step 1. Setup the two first order equations equivalent to the equation (1) as follows:

$$z'(x) + [b(x) - a'(x)]y(x) = f(x) \quad (3)$$

and

$$\varepsilon y'(x) + a(x)y(x) = z(x) \quad (4)$$

Step 2. Obtain the reduced problem by setting ε=0 in equation (1) with appropriate boundary condition

$$a(x)y_0'(x) + b(x)y_0(x) = f(x) \quad (5)$$

with

$$y_0(1) = \beta \tag{6}$$

Step 3. Setup the initial conditions as follows:

Using $y_0(x)$, the solution of the reduced problem, in equation (4) we have

$$z(1) = \varepsilon y_0'(1) + a(1)y_0(1) \tag{7}$$

From (5), we have

$$a(1)y_0'(1) + b(1)y_0(1) = f(1).$$

Using this in (7), we get

$$z(1) = \varepsilon\left(\frac{f(1) - b(1)\beta}{a(1)}\right) + a(1)\beta \tag{8}$$

Step 4. Get the set of initial value problem as follows: Replacing y(x) by $y_0(x)$ in (3), we get

$$z'(x) + [b(x) - a'(x)]y_0(x) = f(x) \tag{9}$$

Now the differential equation (9) with the condition (8) and the differential equation (5) with (6) constitute an initial value problem and the differential equation (4) with the condition y(0)=α constitute another initial value problem.

Therefore the set of initial value problems corresponding to (1)-(2) are given by

$$a(x)y_0'(x) + b(x)y_0(x) = f(x) \text{ with } y_0(1) = \beta$$
$$z'(x) + [b(x) - a'(x)]y_0(x) = f(x)$$

with

$$z(1) = \varepsilon\left(\frac{f(1) - b(1)\beta}{a(1)}\right) + a(1)\beta \tag{10}$$

$$\varepsilon y'(x) + a(x)y(x) = z(x) \text{ with } y(0) = a \tag{11}$$

Thus in a manner of speaking, we have replaced the original boundary value problem (1)-(2) by a set of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. The novelty of this method is that it does not require the analytical solution of the reduced problem. The initial value problem (10) does not contain the perturbation parameter. Hence we solve (10) to get $z(x)$ using classical Runge-Kutta method. In fact any standard method can be used.

To solve the initial value problem (11) we use the trapezoidal formula for the numerical integration to obtain a two term relationship as follows: Consider the initial value problem (11) i.e., $\varepsilon y'(x) + a(x)y(x) = z(x)$ with y(0)=α.

We now divide the interval [0, 1] into N equal parts with mesh size h. i.e., $h = \frac{1}{N}$ and $x_i = ih$ for $i = 0,1,2,......., N$. Integrating by parts the equation (11) in $[x_i, x_{i+1}], i = 0,1,2,3,......, N-1$, we get

$$y(x_{i+1}) - y(x_i) = \frac{1}{\varepsilon}\int_{x_i}^{x_{i+1}} [z(x) - a(x)y(x)]dx.$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$y(x_{i+1}) - y(x_i) = \frac{h}{2\varepsilon}[z(x_{i+1}) + z(x_i)] - \frac{h}{2\varepsilon}[a(x_{i+1})y(x_{i+1}) + a(x_i)y(x_i)].$$

We consider $\rho = \frac{h}{\varepsilon}$.

The above relation becomes,

$$y(x_{i+1}) - y(x_i) = \frac{\rho}{2}[z(x_{i+1}) + z(x_i)] - \frac{\rho}{2}[a(x_{i+1})y(x_{i+1}) + a(x_i)y(x_i)].$$

After simple manipulation, we obtain a two term recurrence relationship,

$$y(x_{i+1}) = \frac{\left(1 - \frac{\rho a(x_i)}{2}\right)}{\left(1 + \frac{\rho a(x_{i+1})}{2}\right)} y(x_i) + \frac{\frac{\rho}{2}\left(z(x_i) + z(x_{i+1})\right)}{\left(1 + \frac{\rho a(x_{i+1})}{2}\right)}. \tag{12}$$

The initial condition $y(0) = \alpha$ is used in (12) to obtain the numerical solution, in the interval [0, 1].

3. Numerical Examples

To demonstrate the applicability of the method, we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example. 1.: Consider the following homogeneous singular perturbation problem from Bender and Orszag [2], page: 480; problem 9.17 with α=0; $\varepsilon y''(x) + y'(x) - y(x) = 0$; x∈[0,1] with y(0)=1 and y(1)=1. The exact solution is given by $y(x) = [(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]/[e^{m_2} - e^{m_1}]$ where $m_1 = (-1 + \sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1+4\varepsilon})/(2\varepsilon)$.

The set of initial value problems corresponding to this example are

$$y_0'(x) - y_0(x) = 0, z'(x) - y_0(x) = 0$$

with $y_0(1) = 1, z(1) = \varepsilon + 1$

$$\varepsilon y'(x) + y(x) = z(x) \text{ with } y(0) = 1.$$

Table 1(a). Numerical Results of Example 1 with ε=10^{-3}, h=10^{-3}

x	y(x)	Exact solution
.0000000	1.0000000	1.0000000
.0010000	.5791308	.6007948
.0020000	.4390866	.4543152
.0030000	.3926510	.4007156
.0040000	.3774184	.3812507
.0050000	.3725871	.3743310
.1000000	.4067981	.4069397
.2000000	.4495210	.4496925
.3000000	.4967420	.4969368
.4000000	.5489343	.5491446
.5000000	.6066216	.6068373
.6000000	.6703823	.6705912
.7000000	.7408558	.7410430
.8000000	.8187488	.8188963
.9000000	.9048424	.9049289
1.0000000	1.0000000	1.0000000

Table 1(b). Numerical Results of Example 1 with ε=10^{-4} , h=10^{-4}

x	y(x)	Exact solution
.0000000	1.0000000	1.0000000
.0001000	.5786408	.6004911
.0002000	.4382122	.4535580
.0003000	.3914272	.3995329
.0004000	.3758567	.3796835
.0005000	.3706911	.3724053
.1000000	.4065924	.4066546
.2000000	.4493482	.4494124
.3000000	.4966010	.4966660
.4000000	.5488238	.5488880
.5000000	.6065397	.6066011
.6000000	.6703262	.6703823
.7000000	.7408220	.7408698
.8000000	.8187330	.8187687
.9000000	.9048382	.9048584
1.0000000	1.0000000	1.0000000

The numerical results are given in tables 1(a), 1(b) for ε=10^{-3} and 10^{-4} respectively.

Example 2.: Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity.

$$\varepsilon y''(x) + y'(x) = 1 + 2x;\ x \in [0,1]$$

with y(0)=0 and y(1)=1.

Table 2(a). Numerical Results of Example 2 with ε=10^{-3}, h=10^{-3}

x	y(x)	Exact solution
.0000000	.0000000	.0000000
.0010000	-.6656657	-.6298574
.0020000	-.8868849	-.8609354
.0030000	-.9599540	-.9453095
.0040000	-.9836383	-.9757130
.0050000	-.9908597	-.9862605
.1000000	-.8900000	-.8882000
.2000000	-.7600000	-.7584000
.3000000	-.6100000	-.6086000
.4000000	-.4400000	-.4388000
.5000000	-.2500000	-.2490000
.6000000	-.0399999	-.0392000
.7000000	.1900001	.1906001
.8000000	.4400001	.4404000
.9000000	.7100000	.7102001
1.0000000	1.0000000	1.0000000

Table 2(b). Numerical Results of Example 2 with ε=10^{-4} , h=10^{-4}

x	y(x)	Exact solution
.0000000	.0000000	.0000000
.0001000	-.6665667	-.6318942
.0002000	-.8886889	-.8642918
.0003000	-.9626629	-.9497229
.0004000	-.9872542	-.9810880
.0005000	-.9953846	-.9925632
.1000000	-.8900000	-.8898200
.2000000	-.7599999	-.7598400
.3000000	-.6100003	-.6098601
.4000000	-.4400001	-.4398801
.5000000	-.2500001	-.2499000
.6000000	-.0400001	-.0399201
.7000000	.1899999	.1900600
.8000000	.4399999	.4400399
.9000000	.7099998	.7100199
1.0000000	1.0000000	1.0000000

The exact solution is given by

$$y(x) = x(x+1-2e) + (2e-1)(1-e^{-x/e}) / (1-e^{-1/e}).$$

The set of initial value problems corresponding to this example are

$$y_0'(x) = 1 + 2x$$
$$z'(x) = 1 + 2x \ with\ y_0(1) = 1, z(1) = 3\varepsilon + 1$$
$$\varepsilon y'(x) + y(x) = z(x)\ with\ y(0) = 0.$$

The numerical results are given in tables 2(a), 2(b) for ε=10^{-3} and 10^{-4} respectively.

Example 3.: Now we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [5], page: 33; equations 2.3.26 and 2.3.27 with α=-1/2;

$$\varepsilon y''(x) + (1 - \frac{x}{2}) y'(x) - \frac{1}{2} y(x) = 0;\ x \in [0,1]$$

with y(0)=0 and y(1)=1.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [7], page: 148; equation 4.2.32) as our 'exact' solution; $y(x) = \frac{1}{2-x} - \frac{1}{2} e^{-(x-x^2/4)/\varepsilon}$ The set of initial value problems related to this example are $(1 - \frac{x}{2}) y_0'(x) - \frac{1}{2} y_o(x) = 0$ $z'(x) = 0$ with $y_0(1)=1$, z(1)=ε+1/2 $\varepsilon y'(x) + (1 - \frac{x}{2}) y(x) = z(x)$ with y(0)=0. The numerical results are given in tables 3(a), 3(b) for ε=10^{-3} and 10^{-4} respectively.

Table 3(a). Numerical Results of Example 3 with ε=10^{-3}, h=10^{-3}

x	y(x)	Exact solution
.0000000	.0000000	.0000000
.0010000	.3340557	.3162644
.0020000	.4455561	.4327652
.0030000	.4829087	.4758015
.0040000	.4955414	.4918075
.0050000	.4999274	.4978630
.1000000	.5270767	.5263158
.2000000	.5563236	.5555556
.3000000	.5890047	.5882353
.4000000	.6257619	.6250000
.5000000	.6674078	.6666667
.6000000	.7149862	.7142857
.7000000	.7698603	.7692308
.8000000	.8338451	.8333333
.9000000	.9094108	.9090909
1.0000000	1.0000080	1.0000000

Table 3(b). Numerical Results of Example 3 with ε=10^{-4} , h=10^{-4}

x	y(x)	Exact solution
.0000000	.0000000	.0000000
.0001000	.3334056	.3160807
.0002000	.4445556	.4323756
.0003000	.4816241	.4751759
.0004000	.4939984	.4909385
.0005000	.4981406	.4967540
.1000000	.5263919	.5263158
.2000000	.5556324	.5555555
.3000000	.5883123	.5882353
.4000000	.6250762	.6250000
.5000000	.6667408	.6666667
.6000000	.7143557	.7142857
.7000000	.7692936	.7692308
.8000000	.8333843	.8333333
.9000000	.9091225	.9090909
1.0000000	1.0000000	1.0000000

4. Non-Linear Problems

We now extend this method for a class of non-linear singularly perturbed two point boundary value problems with left end boundary layer of the underlying interval. For this we consider a class of non-linear singularly perturbed two point boundary value problems of the form:

$$\varepsilon y''(x) + \left[a(x)y(x)\right]' + b(x, y(x)) = 0,\ x \in [0,1] \quad (13)$$

with

$$y(0) = a \text{ and } y(1) = b \quad (14)$$

where ε is a small positive parameter (0<ε<<1) and α, β are known constants. We assume that a(x), b(x, y) are sufficiently continuously differentiable functions in [0, 1]. Furthermore, we assume that (13)-(14) has a solution which displays a boundary layer of width O(ε) at x=1 for small values of ε.

Step 1. Setup the two first order equations equivalent to equation (12) as follows:

$$z'(x) + b(x, y(x)) = 0 \quad (15)$$

and

$$\varepsilon y'(x) + a(x)y(x) = z(x) \quad (16)$$

Step2. Obtain the reduced problem by setting ε=0 in equation (13) with appropriate boundary condition

$$[a(x)y_0(x)]' + b(x, y_0(x)) = 0 \quad (17)$$

with

$$y_0(1) = \beta \quad (18)$$

Step 3. Setup the initial conditions as follows:

Using $y_0(x)$, the solution of the reduced problem, in equation (15) we have

$$z(1) = \varepsilon y_0'(1) + a(1)y_0(1)). \quad (19)$$

This will be the initial condition for equation (15) and y(0)=α will be the initial condition for equation (16).

Step 4. Get the set of initial value problems as follows:

Replacing y(x) by $y_0(x)$ in (15), we get

$$z'(x) + b(x, y_0(x)) = 0 \quad (20)$$

Now the differential equation (20) with the condition (19) and the differential equation (17) with (18) constitute an initial value problem and the differential equation (16) with the condition y(0)=α constitute another initial value problem.

Therefore the set of initial value problems corresponding to equation (12)-(13) are given by

$$[a(x)y_0(x)]' + b(x, y_0(x)) = 0 \text{ with } y_0(1) = \beta$$

$$z'(x) + b(x, y_0(x)) = 0 \text{ with } z(1) = \varepsilon y_0'(1) + a(1)\beta \quad (21)$$

$$\varepsilon y'(x) + a(x)y(x) = z(x) \text{ with } y(0) = a \quad (22)$$

Thus in a manner of speaking, we have replaced the original boundary value problem (13)-(14) by a set of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. The present method does not require the analytical solution of the reduced problem. The initial value problem (21) does not contain the perturbation parameter. It is not a perturbation problem. Hence we solve (21) to get $z(x)$ using classical Runge-Kutta method. In fact any standard method can be used. The initial value problem (22) is a singular perturbation problem. To solve the initial value problem (22) we use the trapezoidal formula for the numerical integration of the first order differential equation to obtain a two term relationship which is described in section 2. By using the initial condition and the two term relationship, we obtain the numerical solution of the original boundary value problem.

5. Non Linear Examples

Again to demonstrate the applicability of the method, we have applied it to a non-linear singular perturbation problem with left-end boundary layer.

Example 4.: Consider the following non linear singular perturbation problem from Bender and Orszag [2], page: 463; equations: 9.7.1;

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0;\ x \in [0,1]$$

with y(0)=0 and y(1)=0.

We have chosen to use Bender and Orszag's uniformly valid approximation (ref. [2], page: 463; equation: 9.7.6) for comparison. $y(x)=\log_e(2/(1+x))-(\log_e 2)e^{-2x/\varepsilon}$. For this example, we have boundary layer of thickness O(ε) at x=0. (cf. Bender and Orszag [2]).

Table 4(a). Numerical Results of Example 4 with $\varepsilon=10^{-3}$, $h=10^{-3}$

x	y(x)	Exact solution
.0000000	.0000000	.0000000
.0010000	.6917068	.5983404
.0020000	.6907094	.6784537
.0030000	.6897131	.6884335
.0040000	.6887177	.6889226
.0050000	.6877234	.6881282
.1000000	.5974989	.5978370
.2000000	.5105671	.5108256
.3000000	.4305866	.4307829
.4000000	.3565277	.3566749
.5000000	.2875737	.2876821
.6000000	.2230667	.2231435
.7000000	.1624674	.1625189
.8000000	.1053297	.1053605
.9000000	.0512794	.0512933
1.0000000	.0000000	.0000000

Table 4(b). Numerical Results of Example 4 with $\varepsilon=10^{-4}$, $h=10^{-4}$

x	y(x)	Exact solution
.0000000	.0000000	.0000000
.0001000	.6930033	.5992399
.0002000	.6929034	.6802518
.0003000	.6928034	.6911291
.0004000	.6927035	.6925147
.0005000	.6926035	.6926158
.1000000	.5978034	.5978370
.2000000	.5108001	.5108256
.3000000	.4307635	.4307829
.4000000	.3566604	.3566750
.5000000	.2876713	.2876821
.6000000	.2231359	.2231436
.7000000	.1625138	.1625189
.8000000	.1053575	.1053605
.9000000	.0512919	.0512933
1.0000000	.0000000	.0000000

The set of initial value problems related to this example are $2y_0'(x)+e^{y_0(x)}=0$ $z'(x)+e^{y_0(x)}=0$ with $y_0(1)=0, z(1)=-\dfrac{\varepsilon}{2}$ and $\varepsilon y'(x)+2y(x)=z(x)$ with $y(0)=0$. The numerical results are given in tables 4(a), 4(b) for ε=10^{-3} and 10^{-4} respectively.

6. Right End Boundary Layer Problems

Finally, we extend this method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)=f(x),\ x\in[0,\ 1] \quad (23)$$

with

$$y(0)=a \text{ and } y(1)=b \quad (24)$$

where ε is a small positive parameter (0<ε<<1) and α, β are known constants. We assume that a(x), b(x) and f(x) are sufficiently continuously differentiable functions in [0, 1]. Furthermore, we assume that a(x)≤ M<0 throughout the interval [0,1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of x=1.

Step 1. Setup the two first order equations equivalent to the equation (23) as follows:

$$z'(x)+[b(x)-a'(x)]y(x)=f(x) \quad (25)$$

and

$$\varepsilon y'(x)+a(x)y(x)=z(x) \quad (26)$$

Step 2. Obtain the reduced problem by setting ε=0 in equation (23) with the appropriate boundary condition.

$$a(x)y_0'(x)+b(x)y_0(x)=f(x) \quad (27)$$

with

$$y_0(0)=\alpha \quad (28)$$

Step 3. Set up the initial conditions as follows:

Using $y_0(x)$, the solution of the reduced problem, in equation (26) we have

$$z(0)=\varepsilon y_0'(0)+a(0)y_0(0). \quad (29)$$

From (3.27) we have $y_0'(x)=\dfrac{f(0)-b(0)y_0(0)}{a(0)}$.

Substituting this in (3.29), we get

$$z(0)=\frac{\left(f(0)-b(0)\alpha\right)}{\varepsilon.a(0)}+a(0)\alpha. \quad (30)$$

This is the initial condition for equation (25). and y(1)=β will be the initial condition for equation (26).

Step 4. Get the set of initial value problems as follows:

Replacing y(x) by $y_0(x)$ in (25), we get

$$z'(x)+[b(x)-a'(x)]y_0(x)=f(x) \quad (31)$$

Now the differential equation (31) with the condition (30) and the differential equation (27) with (28) constitute an initial value problem and the differential equation (26) with the condition y(1)= β constitute another initial value problem.

Therefore the set of initial value problems corresponding to equation (25)-(26) are given by

$$a(x)y_0'(x)+b(x)y_0(x)=f(x)$$
$$z'(x)+[b(x)-a'(x)]y_0(x)=f(x)$$

with

$$y_0(0)=\alpha, z(0)=\frac{\left(f(0)-b(0)\alpha\right)}{\varepsilon.a(0)}+a(0)\alpha. \quad (32)$$

$$\varepsilon y'(x)+a(x)y(x)=z(x) \text{ with } y(1)=b \quad (33)$$

Thus in a manner of speaking, we have replaced the original boundary value problem (23)-(24) by a set of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. We solve these initial value problems to obtain the solution over the interval [0, 1]. The present method does not require the analytical solution of the reduced problem. The initial value problem (32) does not contain the perturbation parameter. It is not a perturbation problem. Hence we solve (32) to get $z(x)$ using classical Runge-Kutta method. In fact any standard method can be used.

The initial value problem (33) is a singular perturbation problem. To solve the initial value problem (33) we use the trapezoidal formula for the numerical integration of the first order differential equation to obtain a two term relationship. By using the initial condition and the two term relationship, we obtain the numerical solution of the original boundary value problem. The main feature of this method is that it does not require very fine mesh. Some numerical experiments have been included to demonstrate the applicability of the method.

Now we consider the initial value problem (33) i.e., $\varepsilon y'(x)+a(x)y(x)=z(x)$ with y(1)= β.

We now divide the interval [0, 1] into N equal parts with mesh size h. i.e., $h=\dfrac{1}{N}$ and $x_i=ih$ for $i=0,1,2,.......,N$.

Integrating by parts the equation (33) in $[x_{i-1},x_i], i=1,2,3,......,N$, we get

$$y(x_i)-y(x_{i-1})=\frac{1}{\varepsilon}\int_{x_{i-1}}^{x_i}\left[z(x)-a(x)y(x)\right]dx.$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$y(x_i)-y(x_{i-1})=\frac{h}{2\varepsilon}\left[z(x_i)+z(x_{i-1})\right]$$
$$-\frac{h}{2\varepsilon}\left[a(x_i)y(x_i)+a(x_{i-1})y(x_{i-1})\right].$$

We consider $\rho=\dfrac{h}{\varepsilon}$.

The above relation becomes,

$$y(x_i) - y(x_{i-1}) = \frac{\rho}{2}\left[z(x_i) + z(x_{i-1})\right]$$

$$-\frac{\rho}{2}\left[a(x_i)y(x_i) + a(x_{i-1})y(x_{i-1})\right].$$

After simple manipulation, we obtain a two term recurrence relationship,

$$y(x_{i-1}) = \frac{\left(1+\frac{\rho a(x_i)}{2}\right)}{\left(1-\frac{\rho a(x_{i-1})}{2}\right)} y(x_i) - \frac{\frac{\rho}{2}\left(z(x_i)+z(x_{i-1})\right)}{\left(1-\frac{\rho a(x_{i+1})}{2}\right)}. \quad (34)$$

The initial condition $y(1) = \beta$ is used in (34) to obtain the numerical solution, in the interval [0, 1].

7. Examples with Right-End Boundary Layer

To illustrate the method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval we considered two examples.

Table 5(a). Numerical Results of Example 5 with ε=10^{-3}, h=10^{-3}

x	y(x)	Exact solution
.0000000	1.0000000	1.0000000
.1000000	1.0000000	1.0000000
.2000000	1.0000000	1.0000000
.3000000	1.0000000	1.0000000
.4000000	1.0000000	1.0000000
.5000000	1.0000000	1.0000000
.6000000	1.0000000	1.0000000
.7000000	1.0000000	1.0000000
.8000000	1.0000000	1.0000000
.9000000	1.0000000	1.0000000
.9950001	.9958848	.9932616
.9960001	.9876543	.9816834
.9970000	.9629630	.9502110
.9980000	.8888889	.8646612
.9990001	.6666667	.6320939
1.0000000	.0000000	.0000000

Table 5(b). Numerical Results of Example 5 with ε=10^{-4} , h=10^{-4}

x	y(x)	Exact solution
.0000000	1.0000000	1.0000000
.1000000	1.0000000	1.0000000
.2000000	1.0000000	1.0000000
.3000000	1.0000000	1.0000000
.4000000	1.0000000	1.0000000
.5000000	1.0000000	1.0000000
.6000000	1.0000000	1.0000000
.7000000	1.0000000	1.0000000
.8000000	1.0000000	1.0000000
.9000000	1.0000000	1.0000000
.9995000	.9958848	.9932636
.9996000	.9876543	.9816856
.9997000	.9629630	.9502377
.9998000	.8888889	.8647096
.9999000	.6666667	.6321816
1.0000000	.0000000	.0000000

Example 5.: Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; \mathrm{x} \in [0,1]$$

with y(0)=1 and y(1)=0.

Clearly, this problem has a boundary layer at x=1. i.e.; at the right end of the underlying interval.

The exact solution is given by $y(x)=(e^{(x-1)/\varepsilon}-1)/(e^{-1/\varepsilon}-1)$.

The set of initial value problems for this are:

$$y_0'(x) = 0, z'(x) = 0 \text{ with } y_0(0) = 1,\ z(0) = -1$$

$$\varepsilon y'(x) - y(x) = z(x) \text{with } y(1) = 0$$

The numerical results are given in tables 5(a), 5(b) for ε=10^{-3} and 10^{-4} respectively.

Example 6.: Now we consider the following singular perturbation problem $\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0$; x∈[0, 1] with y(0)=1+exp(-(1+ε)/ε); and y(1)=1+1/e.

Clearly this problem has a boundary layer at x=1.

The exact solution is given by $y(x)=e^{(1+\varepsilon)(x-1)/\varepsilon}+e^{-x}$

The set of initial value problems related to this example are $y_0'(x) + y_0(x) = 0$ $z'(x) = (1+\varepsilon)e^{-x}$ with $y_0(0)=1$, z(0)=-ε-1 and $\varepsilon y'(x) - y(x) = z(x)$ with y(1)=1+1/e.

The numerical results are given in tables 6(a), 6(b) for ε=10^{-3} and 10^{-4} respectively.

Table 6(a). Numerical Results of Example 6 with ε=10^{-3}, h=10^{-3}

x	y(x)	Exact solution
.0000000	.9990000	1.0000000
.1000000	.9038373	.9048374
.2000000	.8177306	.8187308
.3000000	.7398182	.7408182
.4000000	.6693200	.6703200
.5000000	.6055306	.6065307
.6000000	.5478117	.5488116
.7000000	.4955852	.4965853
.8000000	.4483290	.4493290
.9000000	.4055696	.4065697
.9950001	.3728428	.3764282
.9960001	.3807120	.3875974
.9970000	.4050588	.4186246
.9980000	.4788382	.5036843
.9990001	.7009141	.7357859
1.0000000	1.3678790	1.3678790

Table 6(b). Numerical Results of Example 6 with ε=10^{-4} , h=10^{-4}

x	y(x)	Exact solution
.0000000	.9999000	1.0000000
.1000000	.9047377	.9048374
.2000000	.8186311	.8187308
.3000000	.7407186	.7408183
.4000000	.6702206	.6703200
.5000000	.6064312	.6065307
.6000000	.5487121	.5488117
.7000000	.4964858	.4965853
.8000000	.4492294	.4493290
.9000000	.4064699	.4065697
.9995000	.3720793	.3747965
.9996000	.3802738	.3863337
.9997000	.4049308	.4177372
.9998000	.4789754	.5032163
.9999000	.7011830	.7356979
1.0000000	1.3678790	1.3678790

8. Discussion and Conclusions

We have presented and illustrated Initial value approach for solving singularly perturbed two point boundary value problems. The solution of the given singularly perturbed boundary value problem is computed numerically by solving a set of initial value problems, which are deduced from the original problem. This method is very easy to implement on any computer with minimum problem preparation. We have implemented the present method on three linear examples, three non-linear examples with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. To solve the unperturbed initial value problem we used the classical fourth order Runge-Kutta method. In fact any standard analytical or numerical method can be used. To solve perturbed initial value problem, we use the trapezoidal formula for the numerical integration of the first order differential equation to obtain a two term relationship. By using the initial condition and the two term relationship, we obtain the numerical solution of the original boundary value problem. The main feature of this method is that it does not require very fine mesh. Several numerical experiments have been included to demonstrate the applicability of the present method. Computational results are presented in tables. Here we have given results for only few values, although the solutions are computed at all points with mesh size h. The approximate solution is compared with exact solution. It can be observed from the results that the present method agrees with exact solution very well, which shows the efficiency of the method.

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