American Journal of Numerical Analysis, 2014, Vol. 2, No. 6, 184-189
Available online at http://pubs.sciepub.com/ajna/2/6/3

DOI:10.12691/ajna-2-6-3

A Fitted Second Order Finite Difference Method for Singular Perturbation Problems Exhibiting Dual Layers

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Received December 19, 2014; Revised December 25, 2014; Accepted December 29, 2014

Abstract In this paper a fitted second-order finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at both end (left and right) points. We have introduced a fitting factor in second-order tri-diagonal finite difference scheme and it is obtained from the theory of singular perturbations. The efficient Thomas algorithm is used to solve the tri-diagonal system. Maximum absolute errors are presented in tables to show the efficiency of the method.

Keywords: *singular perturbation problems, Boundary layer, dual layer, Finite differences, fitted method*

Cite This Article: H.S. Prasad, and Y.N. Reddy, "A Fitted Second Order Finite Difference Method for Singular Perturbation Problems Exhibiting Dual Layers." *American Journal of Numerical Analysis*, vol. 2, no. 6 (2014): 184-189. doi: 10.12691/ajna-2-6-3.

1. Introduction

Singularly perturbed differential equations (differential equations with a small parameter ε multiplying the highest order derivatives) are certainly of interest in many scientific and engineering applications. Among these are the fluid flow problems involving high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, control theory, electrical networks [Bender and Orszag (1978); Doolan *et al.* (1980)]. For small values of ε, it is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε, i.e., methods that are convergence ε-uniformly. The survey paper [Kadalbazoo and Reddy (1989)], gives an erudite outline of the singular perturbation problems and their treatment starting from Prandtl's paper on fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on singular perturbation problems. A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention [Hemker and Miller (1979); Kevorkian and Cole (1981); Reddy (1986); Reddy and Pramod Chakravarthy (2004)].

We shall be interested in defining numerical methods for the problem

$$-\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)=f(x)\ , \quad x\in[0,1]$$

with boundary conditions $y(0)=\alpha$ and $y(1)=\beta$ where ε is positive and very small. Moreover, we shall assume that in [0, 1], *b(x)* and *f(x)* are continuous and, for simplicity, *a(x)* is differentiable. This problem has been treated by several authors in the last years. The behaviour of the solution depends, of course, on the properties of the functions *a(x)* and *b(x)*. There are intervals of [0, 1] where the solution vary rapidly (layers). They may be localized either at the extreme points of the interval [0, 1] (boundary layers) or near the roots x_i of *a(x)*, which are called turning points (interior layers). The following table essentially taken from the above differential equation summarizes these facts.

$a(x)\neq 0$; $0\leq x\leq 1$	$a(x)<0$ boundary layer at $x=0$ $a(x)>0$ boundary layer at $x=1$
$a(x)=0$	$b(x)>0$ boundary layers at $x=0$ & $x=1$ $b(x)<0$ rapidly oscillatory solution $b(x)$ changes sign (turning points)
$a'(x_i)\neq 0$, $a(x_i)=0$	$a'(x_i)>0$ no boundary layers, but interior layer at x_i $a'(x_i)<0$ possible boundary layers, No interior layer at x_i

In this paper a fitted second-order finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at both end (left and right) points. We have introduced a fitting factor in second-order tridiagonal finite difference scheme and it is obtained from the theory of singular perturbations. The efficient Thomas algorithm is used to solve the tridiagonal system. Maximum absolute errors are presented in tables to show the efficiency of the method.

2. Description of the Differential Quadrature Method

Consider the singular perturbed two point boundary value problem of the form

$$-\varepsilon y'' + b(x)y(x) = f(x)\ ;x \in \Omega := (0,\ 1), \quad (1.1a)$$

with boundary conditions

$$y(0) = \alpha \text{ and } y(1) = \beta \quad (1.1b)$$

Consider the asymptotic expansion solution of for "Eq. (1.1a)" and "Eq. (1.1b)"

$$y(x,\varepsilon) = \sum_{i=0}^{\infty}\left[y_i(x) + v_i(\tau) + w_i(\eta)\right]\varepsilon^i \quad (2)$$

where $\tau = x/\sqrt{\varepsilon}$ and $\eta = (1-x)/\sqrt{\varepsilon}$

The zeroth order of the above asymptotic expansion is given by

$$y(x) = y_0(x) + v_0(\tau) + w_0(\eta) \quad (3)$$

where

$$y_0(x) = \frac{f(x)}{b(x)} \quad (4)$$

is the solution of the reduced problem of "Eq. (1.1a)" and "Eq. (1.1b)", which does not satisfy both the boundary conditions and v_0 is the left boundary layer correction (or solution) and w_0 is the right boundary layer correction (or solution). v_0, w_0 satisfy the differential equations

$$\frac{-d^2 v_0(\tau)}{d\tau^2} + b(0)v_0(\tau) = 0\ ;\ \tau \in (0,\ \infty) \quad (5)$$

$$\frac{-d^2 w_0(\eta)}{d\eta^2} + b(1)w_0(\eta) = 0\ ;\ \eta \in (0,\ \infty) \quad (6)$$

$$v_0(\tau = 0) + w_0(\eta = 1/\sqrt{\varepsilon}) = \alpha - y_0(0)$$

with $v_0(\tau = 1/\sqrt{\varepsilon}) + w_0(\eta = 0) = \beta - y_0(1)$

$$v_0(\tau = \infty) = w_0(\eta = \infty) = 0$$

Solutions of "Eq. (5)" and "Eq. (6)" are given by

$$v_0(\tau) = Ae^{-\sqrt{b(0)}\tau} \quad (7)$$

$$w_0(\eta) = Be^{-\sqrt{b(1)}\eta} \quad (8)$$

Therefore, solution of "Eq. (1.1a)" and "Eq. (1.1b)" becomes

$$y(x) = y_0(x) + Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x} + Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x)} \quad (9)$$

where A and B are given by

$$A = \frac{(\beta - y_0(1)) - (\alpha - y_0(0))e^{-\sqrt{\frac{b(0)}{\varepsilon}}}}{1 - e^{-\frac{\left(\sqrt{b(0)}+\sqrt{b(1)}\right)}{\sqrt{\varepsilon}}}} \quad (10)$$

$$B = \frac{(\alpha - y_0(0)) - (\beta - y_0(1))e^{-\sqrt{\frac{b(1)}{\varepsilon}}}}{1 - e^{-\frac{\left(\sqrt{b(0)}+\sqrt{b(1)}\right)}{\sqrt{\varepsilon}}}} \quad (11)$$

We rearrange the differential equation $-\varepsilon y'' + b(x)y(x) = f(x)$ as $\varepsilon y''(x) = g(x,y)$ where $g(x,y) = b(x)y(x) - f(x)$.

Now we divide the interval [0,1] into N equal parts with constant mesh length h. let $0 = x_0, x_1, x_2, \ldots\ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih\ ;\ i = 0,1,\ldots..N$. We choose n such that $x_n = \frac{1}{2}$. In the interval $\left[0,\frac{1}{2}\right]$ the boundary layer will be in the left hand side i.e., at $x = 0$ and in the interval $\left[\frac{1}{2},1\right]$ the boundary layer will be in the right hand side i.e., at $x = 1$. At $x = x_i$ the above differential equation can be written as $\varepsilon y_i''(x) = g(x_i, y_i)$ where $g(x_i, y_i) = b(x_i)y(x_i) - f(x_i)$

By considering the second order finite difference scheme "according to Jain [1984]" for the differential equation, we have

$$\begin{aligned}
&\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) = \frac{1}{6}\left(g_{i-1} + 4g_i + g_{i+1}\right)\\
&\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) =\\
&\frac{1}{6}\left(b_{i-1}y_{i-1} - f_{i-1} + 4b_i y_i - 4f_i + b_{i+1}y_{i+1} - f_{i+1}\right)\\
&\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right)\\
&-\frac{1}{6}\left(b_{i-1}y_{i-1} + 4b_i y_i + b_{i+1}y_{i+1}\right)\\
&= \frac{-1}{6}\left(f_{i-1} + 4f_i + f_{i+1}\right)
\end{aligned} \quad (12)$$

In the interval $\left[0,\frac{1}{2}\right]$, we introduce a fitting factor σ_1 in the above difference scheme as

$$\begin{aligned}
&\varepsilon\sigma_1\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right)\\
&-\frac{1}{6}\left(b_{i-1}y_{i-1} + 4b_i y_i + b_{i+1}y_{i+1}\right)\\
&= \frac{-1}{6}\left(f_{i-1} + 4f_i + f_{i+1}\right)
\end{aligned} \quad (13)$$

for i = 1,2,…,n-1

To find σ_1 on the left boundary layer we use the asymptotic solution

$$v_0(x_i) = y_i = Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x_i} \quad (14)$$

and A is given by "Eq. (9)". We assume that solution converges uniformly to the solution of "Eq. (1)", then $f_{i-1} + 4f_i + f_{i+1}$ is bounded.

As $h \to 0$ equation (13) becomes

$$\begin{aligned}
&\lim_{h\to 0}\frac{\sigma_1}{\rho^2}\left(y_{i-1} - 2y_i + y_{i+1}\right)\\
&= \frac{b(0)}{6}\lim_{h\to 0}\left(y_{i-1} + 4y_i + y_{i+1}\right)
\end{aligned} \quad (15)$$

where $\rho = \frac{h}{\sqrt{\varepsilon}}$

Substituting "Eq. (14)" in "Eq. (15)" and simplifying, we get the fitting factor as

$$\sigma_1 = \frac{\rho^2 b(0)\left(e^{\sqrt{b(0)}\rho} + e^{-\sqrt{b(0)}\rho} + 4\right)}{24 Sinh^2\left(\frac{\sqrt{b(0)}\rho}{2}\right)} \tag{16}$$

which is a constant fitting factor. This will be the fitting factor in the interval $\left[0, \frac{1}{2}\right]$.

Substituting the fitting factor "Eq. (16)" in "Eq. (13)", we have the three term recurrence relation as

$$\begin{aligned}&\left(\frac{\varepsilon\sigma_1}{h^2} - \frac{b_{i-1}}{6}\right) y_{i-1} - \left(\frac{2\varepsilon\sigma_1}{h^2} + \frac{4}{6} b_i\right) y_i \\ &+ \left(\frac{\varepsilon\sigma_1}{h^2} - \frac{b_{i+1}}{6}\right) y_{i+1} = \frac{-1}{6}\left(f_{i-1} + 4 f_i + f_{i+1}\right)\end{aligned} \tag{17}$$

for i = 1,2,…,n-1.

We solve the above tridiagonal system by Thomas algorithm. The value of $y_n = y\left(x = \frac{1}{2}\right)$ is obtained by the solution of the reduced problem i.e., $y_0(x)$.

In the interval $\left[\frac{1}{2}, 1\right]$, the boundary layer will be in the right hand side, i.e., at *x=1*. We introduce a fitting factor σ_2 in the difference scheme "Eq. (13)" as

$$\begin{aligned}&\varepsilon\sigma_2\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) - \frac{1}{6}\left(b_{i-1} y_{i-1} + 4 b_i y_i + b_{i+1} y_{i+1}\right) \\ &= \frac{-1}{6}\left(f_{i-1} + 4 f_i + f_{i+1}\right)\end{aligned} \tag{18}$$

for i = n+1, n+2,…..N-1.

To find σ_2 on the right boundary layer we use the asymptotic solution

$$w_0(x_i) = y_i = B e^{-\sqrt{\frac{b(1)}{\varepsilon}}(1 - x_i)} \tag{19}$$

where B is given by (11). Assume that solution converges uniformly to the solution of "Eq. (1.1a)" and "Eq. (1.1b)", then $f_{i-1} + 4 f_i + f_{i+1}$ is bounded.

As $h \to 0$ equation "Eq. (18)" becomes

$$\lim_{h\to 0} \frac{\sigma_2}{\rho^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) = \frac{b(1)}{6} \lim_{h\to 0}\left(y_{i-1} + 4y_i + y_{i+1}\right) \tag{20}$$

where $\rho = \frac{h}{\sqrt{\varepsilon}}$.

Substituting "Eq. (19)" in "Eq. (20)" and simplifying, we get the fitting factor as

$$\sigma_2 = \frac{\rho^2 b(1)\left(e^{\sqrt{b(1)}\rho} + e^{-\sqrt{b(1)}\rho} + 4\right)}{24 Sinh^2\left(\frac{\sqrt{b(1)}\rho}{2}\right)} \tag{21}$$

which is a constant fitting factor. This will be the fitting factor in the interval $\left[\frac{1}{2}, 1\right]$.

From "Eq. (18)", we have the three term recurrence relation

$$\begin{aligned}&\left(\frac{\varepsilon\sigma_2}{h^2} - \frac{b_{i-1}}{6}\right) y_{i-1} - \left(\frac{2\varepsilon\sigma_2}{h^2} + \frac{4}{6} b_i\right) y_i \\ &+ \left(\frac{\varepsilon\sigma_2}{h^2} - \frac{b_{i+1}}{6}\right) y_{i+1} = \frac{-1}{6}\left(f_{i-1} + 4 f_i + f_{i+1}\right)\end{aligned} \tag{22}$$

for i = n+1, n+2,…N-1.

We solve the above tri-diagonal system by Thomas algorithm. The value of $y_n = y\left(x = \frac{1}{2}\right)$ is obtained by the solution of the reduced problem i.e., $y_0(x)$.

Remark: When *b(0) = b(1)* , both the fitting factors become equal and the constant fitting factor is

$$\sigma = \frac{\rho^2 b(0)\left(e^{\sqrt{b(0)}\rho} + e^{-\sqrt{b(0)}\rho} + 4\right)}{24 Sinh^2\left(\frac{\sqrt{b(0)}\rho}{2}\right)}$$

3. Stability and Convergence Analysis

Theorem 3.1. Under the assumptions $\varepsilon > 0$ and $b(x) \geq M > 0$, $\forall x \in [0, 1]$, the solution to the system of the difference equations "Eq. (17)" and "Eq. (22)", together with the given boundary conditions exists, is unique and satisfies

$$\|y\|_{h,\infty} \leq 6M^{-1}\|f\|_{h,\infty} + \left(|\alpha| + |\beta|\right)$$

where $\|\cdot\|_{h,\infty}$ is the discrete $l_\infty - norm$, given by $\|x\|_{h,\infty} = \max_{0 \leq i \leq N}\{|x_i|\}$.

Proof. Let $L_h(.)$ denote the difference operator on left hand side of "Eq. (12)" and w_i be any mesh function satisfying $L_h(w_i) = f_i$.

By rearranging the difference scheme "Eq. (17)" and using non-negativity of the coefficients E_i, F_i and G_i, we obtain

$$F_i\ |w_i| \leq |H_i| + E_i\ |w_{i-1}| + G_i\ |w_{i+1}|$$

$$\left(\frac{2\varepsilon\sigma}{h^2} + \frac{4b_i}{6}\right)|w_i| \leq$$

$$|H_i| + \left(\frac{\varepsilon\sigma}{h^2} - \frac{b_{i-1}}{6}\right)|w_{i-1}| + \left(\frac{\varepsilon\sigma}{h^2} - \frac{b_{i+1}}{6}\right)|w_{i+1}|$$

for i =1, 2, …, N-1 Now using the assumption $b(x) \geq M$, the definition of l_∞ -norm and manipulating, we obtain

$$\begin{aligned}&\sigma\ \varepsilon\frac{\left(|w_{i+1}| - 2|w_i| + |w_{i-1}|\right)}{h^2} - \\ &\frac{\|b\|_{\infty,h}}{6}\left(|w_{i+1}| + 4|w_i| + |w_{i-1}|\right) + |H_i| \geq 0\end{aligned} \tag{23}$$

To prove the uniqueness and existence, let $\{u_i\}$, $\{v_i\}$ be two sets of solution of the difference equation "Eq. (17)" satisfying boundary conditions. Then $w_i = u_i - v_i$ satisfies $L_h(w_i) = f_i$ where $f_i = 0$ and $w_0 = w_N = 0$.

Summing "Eq. (23)" over i = 1, 2, ……, N-1, we obtain

$$-\frac{\sigma\varepsilon}{h^2}\left(|w_1|+|w_{N-1}|\right)-\frac{5}{6}\|b\|_{h,\infty}\sum_{i=1}^{N-1}|w_i| \ge 0 \tag{24}$$

Since

$\varepsilon > 0,\ \sigma > 0,\ b_i > 0$ and $|w_i| \ge 0\ \ \forall i,\ i = 1,2,......, N-1,$

therefore for inequality "Eq. (24)" to hold, we must have $w_i = 0\ \ \forall i,\ \ i = 1,2,.....N-1$.

This implies the uniqueness of the solution of the tridiagonal system of difference "Eq. (17)". For linear equations, the existence is implied by uniqueness. Now to establish the estimate, let $w_i = y_i - l_i$,

Where y_i satisfies difference "Eq. (12)", the boundary conditions and

$$l_i = (1-ih)\alpha + (ih)\beta,$$

then $w_0 = w_N = 0$, and $w_i,\ \ i = 1,2,...N-1$

$$L_h(w_i) = f_i$$

Now let $|w_n| = \|w\|_{h,\infty} \ge |w_i|,\ i = 0,1,......., N.$

Then summing "Eq. (23)" from i = n to N-1 and using the assumption on b(x), which gives

$$\begin{aligned}&-\frac{\sigma\varepsilon}{h^2}\left(|w_n|-|w_{n-1}|\right)-\frac{\sigma\varepsilon}{h^2}|w_{N-1}|-\\&\frac{\|b\|}{6}\left(|w_n|+|w_{n-1}|\right)-\frac{5}{6}\|b\|\sum_{i=n}^{N-1}|w_i|+\sum_{i=n}^{N-1}|H_i| \ge 0\end{aligned} \tag{25}$$

Inequality "Eq. (25) ", together with the condition on b(x) implies that

$$\frac{M}{6}|w_n| \le \sum_{i=n}^{N-1}|H_i| \le \sum_{i=0}^{N}|H_i| \le \|H\|_{h,\infty},$$

i.e., we have

$$|w_n| \le 6\,M^{-1}\|H\|_{h,\infty} \tag{26}$$

Also, we have

$$y_i = w_i + l_i$$

$$\begin{aligned}\|y\|_{h,\infty} &= \max_{0\ \le i\ \le N}\{\ |y_i|\ \}\\ &\le\ \|w\|_{h,\infty} + \|l\|_{h,\infty}\\ &\le\ |w_n| + \|l\|_{h,\infty}.\end{aligned} \tag{27}$$

Now to complete the estimate, we have to find out the bound on l_i

$$\begin{aligned}\|l\|_{h,\infty} &= \max_{0\ \le i\ \le N}\{\ |l_i|\ \}\\ &\le \max_{0\ \le i\ \le N}\{\ |(1-ih)|\ \ |\alpha| + |ih|\ \ |\beta|\}\\ &\le \max_{0\ \le i\ \le N}\{\ (1-ih)\ \ |\alpha| + (ih)\ \ |\beta|\},\end{aligned}$$

i.e., we have

$$\|\ l\ \|_{h,\infty} \le |\alpha| + |\beta|. \tag{28}$$

From "Eq. (26)" – "Eq. (28)", we obtain the estimate

$$\|y\|_{h,\infty} \le 6M^{-1}\|\ f\ \|_{h,\infty} + \left(|\alpha| + |\beta|\right).$$

This theorem implies that the solution to the system of the difference equations "Eq. (17)"and "Eq. (22)" are uniformly bounded independent of mesh size h and the perturbation parameter ε. Thus the scheme is stable for all step sizes.

Corollary 3.2. Under the conditions for theorem 1, the error $e_i = y(x_i) - y_i$ between the solution y(x) of the continues problem and the solution y_i of the discretized problem, with boundary conditions, satisfies the estimate

$$\|e\|_{h,\infty} \le 6M^{-1}\|\tau\|_{h,\infty}, \text{ where} |\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{\sigma h^2}{6}\left|y^{(4)}(x)\right|\right\}$$

Proof. From the finite differences, we have

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} = y_i'' + \frac{h^2}{12}y_i^{(4)} \tag{29}$$

and

$$y_{i+1}'' - 2y_i'' + y_{i-1}'' = h^2 y_i^{(4)} \tag{30}$$

dividing the "Eq. (30)" by 6, adding and subtracting y_i'', we have

$$\begin{aligned}y_i'' + \frac{y_{i+1}''-2y_i''+y_{i-1}''}{6} &= y_i'' + \frac{h^2}{6}y_i^{(4)}\\ \frac{y_{i+1}''+4y_i''+y_{i-1}''}{6} &= y_i'' + \frac{h^2}{6}y_i^{(4)}\end{aligned} \tag{31}$$

From "Eq. (29)" and "Eq. (31)", we get

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} - \frac{h^2}{12}y_i^{(4)} = \frac{y_{i+1}''+4y_i''+y_{i-1}''}{6} - \frac{h^2}{6}y_i^{(4)}$$

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} = \frac{1}{6}\left(f_{i+1}+4f_i+f_{i-1}\right) - \frac{h^2}{12}y_i^{(4)}$$

which is a scheme of second order.

Hence the second order fitted scheme for the singularly perturbed problem $\varepsilon y'' = g(x,y)$ is

$$\sigma\varepsilon\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right) = \frac{1}{6}\left(f_{i+1}+4f_i+f_{i-1}\right) - \frac{h^2}{12}y_i^{(4)}$$

Truncation error τ_i is given by

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{\sigma h^2}{12}\left|y^{(4)}(x)\right|\right\}$$

One can easily show that the error e_i, satisfies

$$L_h\left(e(x_i)\right) = L_h\left(y(x_i)\right) - L_h\left(y_i\right) = \tau_i,\ \ i = 1,2,..., N-1$$

and $e_0 = e_N = 0$.

Then Theorem 3.1 implies that

$$\|e\|_{h,\infty} \le 6M^{-1}\|\tau\|_{h,\infty} \tag{32}$$

The estimate "Eq. (32)" establishes the convergence of the difference scheme for the fixed values of the parameter ε .

4. Numerical Illustrations

To demonstrate the applicability of the method we have applied it to four linear singular perturbation problems with dual boundary layers. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. Maximum absolute errors are presented in tables.

Example .1. Consider the following non-homogeneous singular perturbation problem:

$\varepsilon y''(x) - y(x) = \cos^2 \pi x + 2\varepsilon\pi^2 \cos 2\pi x$; $x \in [0,1]$

with $y(0) = 0$ and $y(1) = 0$. The exact solution is given by

$$y(x) = \frac{\left(e^{\left(-(1-x)/\sqrt{\varepsilon}\right)} + e^{\left(-x/\sqrt{\varepsilon}\right)}\right)}{\left(1 + e^{-1/\sqrt{\varepsilon}}\right)} - \cos^2 \pi x$$

The maximum absolute errors are given in tables 1 for different values of h and ε respectively.

Example 4.2. Consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) + y(x) = 1 + 2\sqrt{\varepsilon}\left[e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}}\right];\ \mathrm{x} \in [0,1]$$

with y(0)=0 and y(1)=-0. The exact solution is given by

$$y(x) = 1 - (1-x)e^{-x/\sqrt{\varepsilon}} - xe^{(x-1)/\sqrt{\varepsilon}}$$

The maximum absolute errors are given in tables 2 for different values of h and ε respectively.

Example 4.3. Consider the following variable coefficient singular perturbation problem

$$\varepsilon^2 y''(x) - (2 - x^2) y(x) = -1; x \in [-1,\ 1]$$

with $y(-1) = 0$ and $y(1) = 0$.

The exact solution is given by

$$y(x) = \frac{1}{2 - x^2} - e^{\frac{-(1+x)}{\varepsilon}} - e^{\frac{-(1-x)}{\varepsilon}}$$

The maximum absolute errors are given in tables 3 for different values of h and ε respectively.

5. Discussion and Conclusions

We have presented a fitted second order finite difference method for solving singularly perturbed two-point boundary value problems with boundary layer at both (left and right) end points. We have introduced a fitting factor the difference scheme and obtained its value from the theory of singular perturbations. We have implemented the present method on standard test problems. Maximum absolute errors are presented in tables. It is observed from the results that the present method approximate the exact solution very well.

Table 1. The maximum errors in solution of Problem 4.1

ε / h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	3.65(-2)	9.30(-3)	2.30(-3)	5.85(-4)	1.46(-4)	3.66(-5)	9.15(-6)	2.28(-6)
2^{-4}	3.19(-2)	8.10(-3)	2.00(-3)	5.08(-4)	1.27(-4)	3.17(-5)	7.94(-6)	1.98(-6)
2^{-5}	2.85(-2)	7.20(-3)	1.80(-3)	4.48(-4)	1.12(-4)	2.80(-5)	7.01(-6)	1.75(-6)
2^{-6}	2.70(-2)	6.70(-3)	1.70(-3)	4.16(-4)	1.04(-4)	2.60(-5)	6.50(-6)	1.62(-6)
2^{-10}	3.66(-2)	7.50(-3)	1.70(-3)	4.06(-4)	1.00(-4)	2.51(-5)	6.27(-6)	1.56(-6)

Table 2. The maximum absolute errors in solution of Problem 4.2

ε / h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	2.30(-3)	5.64(-4)	1.40(-4)	3.51(-5)	8.78(-6)	2.93(-6)	5.49(-7)	1.37(-7)
2^{-4}	2.80(-3)	6.83(-4)	1.70(-4)	4.24(-5)	1.06(-5)	2.65(-6)	6.63(-7)	1.65(-7)
2^{-5}	3.10(-3)	7.58(-4)	1.87(-4)	4.69(-5)	1.17(-5)	2.93(-6)	7.32(-7)	1.83(-7)
2^{-6}	4.30(-3)	9.96(-4)	2.44(-4)	6.08(-5)	1.51(-5)	3.79(-6)	9.49(-7)	2.37(-7)
2^{-10}	8.50(-3)	3.90(-3)	1.10(-4)	2.45(-4)	6.02(-5)	1.49(-5)	3.74(-6)	9.35(-7)

Table 3. The Maximum absolute errors in solution of Problem 4.3

ε / h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	6.74(-2)	6.92(-2)	7.04(-2)	7.03(-2)	7.03(-2)	7.03(-2)	7.03(-2)	7.03(-2)
2^{-4}	3.34(-2)	3.36(-2)	3.55(-2)	3.55(-2)	3.55(-2)	3.55(-2)	3.55(-2)	3.55(-2)
2^{-5}	1.39(-2)	1.53(-2)	1.62(-2)	1.67(-2)	1.67(-2)	1.68(-2)	1.68(-2)	1.68(-2)
2^{-6}	1.06(-2)	5.60(-3)	6.70(-3)	7.50(-3)	7.70(-3)	7.70(-3)	7.80(-3)	7.80(-3)
2^{-10}	1.06(-2)	3.90(-3)	1.20(-3)	3.54(-4)	9.84(-4)	1.19(-4)	3.11(-4)	3.89(-4)

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