American Journal of Numerical Analysis, 2014, Vol. 2, No. 4, 128-135
Available online at http://pubs.sciepub.com/ajna/2/4/5
© Science and Education Publishing
DOI:10.12691/ajna-2-4-5

A Domain Decomposition Method for Solving Singularly Perturbed Two Point Boundary Value Problems via Exponential Splines

P. Padmaja[1], Y.N. Reddy[2,*]

[1]Department of Mathematics, Prasad V Potluri Siddhartha Institute of Technology, Vijayawada, Andhra Pradesh, INDIA
[2]Department of Mathematics, National Institute of Technology, Warangal, INDIA
*Corresponding author: ynreddy_nitw@yahoo.com

Received August 07, 2014; Revised August 14, 2014; Accepted August 18, 2014

Abstract In this paper, we presented a domain decomposition method via exponential splines for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The method is distinguished by the following fact: The original singularly perturbed two-point boundary value problem is divided into two problems, namely inner and outer region problems. The terminal boundary condition is obtained from the solution of the reduced problem. Using stretching transformation, a modified inner region problem is constructed. Then, the inner region problem is solved as two-point boundary value problems by employing exponential splines. Several linear and nonlinear problems are solved to demonstrate the applicability of the method.

Keywords: *singular perturbation problems, inner region problem, outer region problem, boundary layer, exponential splines*

Cite This Article: P. Padmaja, and Y.N. Reddy, "A Domain Decomposition Method for Solving Singularly Perturbed Two Point Boundary Value Problems via Exponential Splines." *American Journal of Numerical Analysis*, vol. 2, no. 4 (2014): 128-135. doi: 10.12691/ajna-2-4-5.

1. Introduction

Singular perturbation problems occur very frequently in fluid mechanics and other branches of applied Mathematics. The solution of the singularly perturbed boundary value problems has a multi scale character. The solution varies rapidly in some parts and varies slowly in some other parts. The numerical treatment of singular perturbation problems is far from trivial, because of the boundary layer behaviour of the solution. There are many physical situations in which the sharp changes occur inside the domain of interest, and the narrow regions across which these changes take place are usually referred as shock layers in fluid and solid mechanics, transition points in quantum mechanics, Strokes lines and surfaces in Mathematics. These rapid changes cannot be handled slow scales, but they can be handled by fast or magnified or stretched scales. A common strategy for dealing with this type of problems consists of dividing the domain of integration into two sub domains and then to apply a different scheme on each sub domain [1,2]. In recent years a large number of analytical methods have been proposed [cf. Bender and Orszag [3], Kevorkian and Cole [4], O' Malley [5], Nayfeh [6], Smith [7], Hu et. al [8]. Numerical methods based on initial value techniques and boundary value techniques are given in [9,10,11,12]. Non linear single step methods for initial value problems were discussed by Van Niekerk [12]. A non standard explicit method for initial value problems is proposed by Ramos [13]. The more efficient, simpler computational techniques are required to solve singular perturbation problems.

In order to know the behavior of the solution of the singular perturbation problems in the boundary layer region, it is always suggestive to divide the original problem into two problems namely the inner region problem and the outer region problem and solve them separately. The general idea of domain decomposition process was originally introduced by Prandtl, which was later named as the method of matched asymptotic expansions. For many singular perturbation problems, a reduced problem is well defined and solution is known a priori. Based on this concept we presented a domain decomposition method via exponential splines for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The method is distinguished by the following fact: The original singularly perturbed two-point boundary value problem is divided into two problems, namely inner and outer region problems. The terminal boundary condition is obtained from the solution of the reduced problem. Using stretching transformation, a modified inner region problem is constructed. Then, the inner region problem is solved as two-point boundary value problems by employing exponential splines. Several linear and nonlinear problems are solved to demonstrate the applicability of the method.

2. Methods

2.1. Domain Decomposition Method: Linear Problems

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \mathrm{x} \in [0,\ 1] \quad (1)$$

with

$$\mathrm{y}\ (0) = \alpha \quad (2a)$$

and

$$\mathrm{y}\ (1) = \beta \quad (2b)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$ and α, β are known constants. We assume that $\mathrm{a}(\mathrm{x})$, $\mathrm{b}(\mathrm{x})$ and f(x) are sufficiently continuously differentiable functions in [0, 1]. Furthermore, we assume that $\mathrm{a}(\mathrm{x}) \geq \mathrm{M} > 0$ throughout the interval [0, 1], where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of x=0. Consider $\mathrm{x_p} = \mathrm{O}(\varepsilon)$ be the thickness of the boundary layer.

Now we divide the original problem into two problems, an inner region problem and an outer region problem. The inner region problem is defined in the interval $0 \leq \mathrm{x} \leq \mathrm{x_p}$ and the outer region problem is defined in the interval $\mathrm{x_p} \leq \mathrm{x} \leq 1$.

2.2. Terminal Boundary Condition

To obtain the boundary condition at terminal point $\mathrm{x_p}$, we solve the reduced problem with an appropriate boundary condition.

The reduced problem is

$$a(x)y'(x) + b(x)y(x) = f(x) \quad (3)$$

with

$$\mathrm{y}\ (1) = \beta \quad (4)$$

Let $\mathrm{y_R}\ (\mathrm{x})$ be the (analytical or numerical) solution of the reduced problem. At the terminal point $\mathrm{x_p}$ we get $\mathrm{y}(\mathrm{x_p}) = \mathrm{y_R}\ (\mathrm{x_p})$.

Let the terminal boundary condition : $\mathrm{y}(\mathrm{x_p}) = \gamma(\mathrm{say})$.

2.3. Solution of Inner Region Problem

Since the terminal boundary condition is common to both the inner and outer regions, we define the inner region problem as a two-point boundary value problem:

$$\begin{aligned} &\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \\ &0 \leq x \leq x_p \end{aligned} \quad (5)$$

with

$$\mathrm{y}(0) = \alpha \quad (6a)$$

and

$$\mathrm{y}(\mathrm{x_p}) = \gamma \quad (6b)$$

Now we introduce a stretched variable 't' to magnify the boundary layer region and there by eliminate any rapid variation that might be exhibited by the solution when the solution is considered as a function of the stretched variable.

We can think of two stretching transformations:

(i) $t = \dfrac{x}{\varepsilon}$ and

(ii) $t = \dfrac{\int_0^x a(s)ds}{\varepsilon}$

By using the stretching transformation $t = \dfrac{x}{\varepsilon}$, we have

$$y(x) = y(t\varepsilon) = Y(t), y'(x) = \frac{1}{\varepsilon}y'(t\varepsilon) = \frac{1}{\varepsilon}Y'(t),$$

$$y''(x) = \left(\frac{1}{\varepsilon^2}\right)y''(t\varepsilon) = \left(\frac{1}{\varepsilon^2}\right)Y''(t),$$

$$a(x) = a(t\varepsilon) = A(t), b(x) = b(t\varepsilon) = B(t),$$

$$f(x) = f(t\varepsilon) = F(t).$$

and $\mathrm{y}(0) = \alpha = \mathrm{Y}(0)$.

We get the inner region problem as:

$$Y''(t) + [A(t)Y'(t)]' + \varepsilon B(t)Y(t) = \varepsilon F(t) \quad (7)$$

and the boundary conditions becomes

$$Y(0) = \alpha \quad (8a)$$

and

$$Y(t_p) = \gamma \quad (8b)$$

where $t_p = \dfrac{x_p}{\varepsilon}$.

We solve this modified inner region problem (7)- (8) to obtain the solution over the interval $0 \leq t \leq t_p$. In order to solve this modified inner region problem, we derived a finite difference scheme based on exponential splines.

2.4. Solution of the Outer Region Problem

The solution of the reduced problem is considered as outer solution.

2.5. Exponential Spline Approximation

The spline proves to be an effective tool in the elementary process of interpolation and approximate integration. The outstanding characteristic, however, is its effectiveness in numerical differentiation. Splines are frequently used to find the solution of two-point boundary value problems.

To describe exponential spline approximation, we consider the two-point boundary value problem

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x); a \leq x \leq b \quad (9)$$

with

$$y(a) = \lambda_1 \tag{10a}$$

and

$$y(b) = \lambda_2 \tag{10b}$$

We divide the interval [a, b] into n equal parts with mesh size h and having nodes at a= x_0, x_1, x_2, x_3, …x_n=b.

By definition: A spline function of degree m with nodes at the points x_i ; i =0, 1, 2, ……,n is a function S(x) with the properties:

1. on each interval [x_{i-1}, x_i] ; i =0, 1, 2, …,n ; S(x) is a polynomial of degree m.
2. S(x) and its first m-1 derivatives are continuous on [a, b].

Let $y(x)$ be the exact solution of the problem (9)-(10) and S_i be the approximate solution to $y_i = y(x_i)$ obtained by the segment $Q_i(x)$ passing thorough the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Each mixed spline $Q_i(x)$ has the following form [c.f. 14-16]:

$$Q_i(x) = a_i e^{\tau(x-x_i)} + b_i e^{-\tau(x-x_i)} + c_i(x-x_i) + d_i \quad i = 0,1,2,.....,n \tag{11}$$

where a_i, b_i, c_i, d_i are constants and τ is a free parameter. To obtain the necessary conditions for the coefficients introduced in equation (11), the segment values of $Q_i(x_i), Q_i(x_{i+1}), Q_i'(x_i), Q_i'(x_{i+1})$ should be considered at the common node. We define $Q_i(x_i) = S_i, Q_i(x_{i+1}) = S_{i+1}, Q_i''(x_i) = M_i, Q_i''(x_{i+1}) = M_{i+1}$.

Using the above four conditions in (11) via a straight forward calculation, we obtain the following expressions:

$$a_i = \frac{h^2(M_{i+1} - e^{-\theta}M_i)}{2\theta^2 \sinh\theta}, b_i = \frac{h^2(e^{\theta}M_i - M_{i+1})}{2\theta^2 \sinh\theta}$$

$$c_i = \frac{(S_{i+1} - S_i)}{h} - \frac{h(M_{i+1} - M_i)}{\theta^2}, d_i = S_i - \frac{h^2 M_i}{\theta^2},$$

where

$\theta = \tau h$ and $i = 0,1,2,.....,n$.

Using the continuity of the first derivative at the point (x_i, S_i) from $Q_{i-1}'(x), Q_i'(x)$ we get the following relation:

$$S_{i-1} - 2S_i + S_{i+1} = h^2(\mu_1 M_{i+1} + \mu_2 M_i + \mu_1 M_{i-1}); \quad i = 1, 2, \ldots, n-1 \tag{12}$$

where

$$\mu_1 = \frac{(\sinh\theta - \theta)}{\theta^2 \sinh\theta} \text{ and } \mu_2 = \frac{(2\theta\cosh\theta - 2\sinh\theta)}{\theta^2 \sinh\theta}.$$

Remark: When the free parameter $\tau \to 0$, we have $\theta \to 0$. In this case $\mu_1 = \frac{1}{6}, \mu_2 = \frac{4}{6}$ and the relation defined in equation (12) reduces to the ordinary cubic spline relation

$$S_{i-1} - 2S_i + S_{i+1} = \frac{h^2}{6}(M_{i+1} + 4M_i + M_{i-1}). \tag{13}$$

At the point x_i the two point boundary value problem can be written as

$$M_i = -p_i S_i' - q_i S_i + r_i \tag{14}$$

where

$$p_i = p(x_i), q_i = q(x_i), r_i = r(x_i).$$

We have

$$S_i' = \frac{S_{i+1} - S_{i-1}}{2h}, S_{i+1}' = \frac{3S_{i+1} - 4S_i + S_{i-1}}{2h}, \quad S_{i-1}' = \frac{-S_{i+1} + 4S_i - 3S_{i-1}}{2h} \tag{15}$$

Substituting (14), (15) in (13) we obtain a tridiagonal scheme as

$$E_i S_{i-1} - F_i S_i + G_i S_{i+1} = H_i; \ i = 1,2,3, \ \ldots\ldots, N-1 \tag{16}$$

where

$$E_i = 1 + \frac{h}{2}(\mu_1 p_{i+1} - \mu_2 p_i - 3\mu_1 p_{i-1}) + \mu_1 h^2 q_{i-1}$$

$$F_i = 2 + \frac{h}{2}(4\mu_1 p_{i+1} - 4\mu_1 p_{i-1}) - \mu_2 h^2 q_i$$

$$G_i = 1 - \frac{h}{2}(-3\mu_1 p_{i+1} - \mu_2 p_i + \mu_1 p_{i-1}) + \mu_1 h^2 q_{i+1}$$

$$H_i = h^2(\mu_1 r_{i+1} + \mu_2 r_i + \mu_1 r_{i-1})$$

The solution of the above tridiagonal system can be obtained by using Thomas algorithm also called discrete invariant imbedding.

2.6. Solution of the original problem

After getting the solution of the inner region problem and outer region problem, we combine both to obtain the approximate solution of the original problem (1)- (2) over the interval $0 \le x \le 1$. We repeat the process for various choices of x_p, until the solution profiles do not differ materially from iteration to iteration. For computational purposes we use an absolute error criterion, namely

$$\left| Y(t)^{m+1} - Y(t)^m \right| \le \sigma; 0 \le t \le t_p$$

where $Y(t)^m$ is the mth iterate of the inner region solution and σ is the prescribed tolerance bound.

3. Analysis

3.1. Linear Examples

To demonstrate the applicability of the method we have applied it on three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example 3.1.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [[3], page 480; problem 9.17 with $\alpha = 0$]

$$\varepsilon y''(x) + y'(x) - y(x) = 0; x \in [0,1]$$

with y(0)=1 and y(1)=1.

The reduced problem is

$$y_0'(x) - y_o(x) = 0; y_0(1) = 1.$$

The solution of this problem is

$$y_0(x) = e^{x-1}.$$

The inner region problem is given by

$$Y''(t) + Y'(t) - \varepsilon Y(t) = 0; 0 \le t \le t_p$$

with $Y(0) = 1$, $Y(t_p) = e^{x_p - 1}$.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$$

Where $m_1 = (-1 + \sqrt{1 + 4\varepsilon}) / (2\varepsilon)$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon}) / (2\varepsilon)$.

The maximum absolute errors for different values of t_p in the inner region is presented in Tables 1(a), 1(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Example 3.1.2. Now consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x;\ \mathrm{x} \in [0,1]$$

with y(0)=0 and y(1)=1.

The reduced problem is

$$y_0'(x) = 1 + 2x; y(1) = 1.$$

The solution of this problem is

$$y_0(x) = x^2 + x - 1.$$

The inner region problem is given by

$$Y''(t) + Y'(t) = \varepsilon(1 + 2\varepsilon t); 0 \le t \le t_p$$

with $Y(0) = 0$, $Y(t_p) = x_p^2 + x_p - 1$.

The exact solution is given by

$$\mathrm{y(x)} = \mathrm{x(x+1-2}\varepsilon) + \frac{(2\varepsilon - 1)\left(1 - e^{-x/\varepsilon}\right)}{\left(1 - e^{-1/\varepsilon}\right)}.$$

The maximum absolute errors for different values of t_p in the inner region is presented in Tables 2(a), 2(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Example 3.1.3. Now we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole[[4], page 33; equations 2.3.26 and 2.3.27 with α=-1/2]

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0; \mathrm{x} \in [0,1]$$

with y(0)=0 and y(1)=1.

The reduced problem is

$$(1 - \frac{x}{2})y_0'(x) - \frac{1}{2}y_o(x) = 0; y_0(1) = 1.$$

The solution of this problem is

$$y_0(x) = \frac{1}{2 - x}.$$

The inner region problem is given by

$$Y''(t) + \left(1 - \frac{t\varepsilon}{2}\right)Y'(t) - \frac{\varepsilon}{2}Y(t) = 0; 0 \le t \le t_p$$

with $Y(0) = 0$, $Y(t_p) = \dfrac{1}{2 - x_p}$.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [[6], page 148; equation 4.2.32] as our 'exact' solution:

$$y(x) = \frac{1}{2 - x} - \frac{1}{2}e^{-(x - x^2/4)/\varepsilon}$$

The maximum absolute errors for different values of t_p in the inner region is presented in Tables 3(a), 3(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

3.2. Nonlinear Examples

We have applied the present method on three nonlinear singular perturbation problems with left-end boundary layer. Nonlinear singular perturbation problems are first converted as a sequence of linear singular perturbation problems by using Quasilinearization method. The solution of the reduced problem is taken as initial approximation.

Example 3.2.1. Consider the following singular perturbation problem from Bender and Orszag [[3], page 463; equations: 9.7.1]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \mathrm{x} \in [0,1]$$

with $\mathrm{y}(0) = 0$ and $\mathrm{y}(1) = 0$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

The reduced problem is

$$2y_0'(x) + e^{y_0(x)} = 0; y_0(1) = 0$$

and the solution of reduced problem is given by

$$y_0(x) = \log_e\left(\frac{2}{x+1}\right).$$

The inner region problem is given by

$$Y''(t) + 2Y'(t) + \frac{2\varepsilon}{1 + t\varepsilon}Y(t) = \left(\frac{2\varepsilon}{1 + t\varepsilon}\right)\left[\log_e\left(\frac{2}{1 + t\varepsilon}\right) - 1\right]$$

$0 \le t \le t_p$

with $Y(0) = 0$, $Y(t_p) = \log_e\left(\dfrac{2}{x_p + 1}\right)$

We have chosen to use Bender and Orszag's uniformly valid approximation [[3], page 463; equation: 9.7.6] for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)e^{-2x/\varepsilon}$$

The maximum absolute errors for different values of t_p in the inner region is presented in Tables 4(a), 4(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Example 3.2.2. Now consider the following singular perturbation problem from Kevorkian and Cole [[4], page 56; equation 2.5.1]

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0; x \in [0,1]$$

with $y(0) = -1$ and $y(1) = 3.9995$

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995.$$

The reduced problem is

$$[y_0(x)^2 / 2]' - y_0(x) = 0; y_0(1) = 3.9995;$$

whose solution is

$$y_0(x) = x + 2.9995.$$

The inner region problem is given by

$$Y''(t) + (t\varepsilon + 2.9995)Y'(t) = \varepsilon(t\varepsilon + 2.9995)$$
$$0 \le t \le t_p$$

with $Y(0) = -1$, $Y(t_p) = x_p + 2.9995$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation [[4], pages 57 and 58; equations (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right)$$

Where c_1=2.9995 and c_2=$(1/c_1)\log_e[(c_1-1)/(c_1+1)]$

For this example also we have a boundary layer of width O(ε) at x=0 [cf. Kevorkian and Cole [[4], pages 56-66].

The maximum absolute errors for different values of t_p in the inner region is presented in Table 5(a), Table 5(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Example 3.2.3. Finally we consider the following singular perturbation problem from O' Malley [[5], page 9; equation (1.10) case 2]:

$$\varepsilon y''(x) - y(x)y'(x) = 0;\ x \in [-1,1]$$

with y(-1)=0 and y(1)= -1.

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0$$

The reduced problem is

$$[y_0(x)^2 / 2]' = 0; y_0(1) = -1$$

whose solution is

$$y_0(x) = -1.$$

For this example the stretching transformation is :

$$t = \frac{x+1}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) = 0;\ with\ Y(0) = 0, Y(t_p) = -1.$$

We have chosen to use O' Malley's approximate solution [[5], pages 9 and 10; equations 1.13 and 1.14] for comparison,

$$y(x) = -\frac{\left(1 - e^{-(x+1)/\varepsilon}\right)}{\left(1 + e^{-(x+1)/\varepsilon}\right)}$$

For this example, we have a boundary layer of width O(ε) at x=-1 [[cf. O' Malley [5], pages 9 and 10, eqs. (1.10), (1.13), (1.14), case 2].

The maximum absolute errors for different values of t_p in the inner region is presented in Tables 6(a), 6(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

4. Results

4.1. Right-End Boundary Layer Problems

Now we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), x \in [0,\ 1] \quad (17)$$

with

$$y(0) = \alpha \quad (18a)$$

and

$$y(1) = \beta \quad (18b)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$ and α, β are known constants. We assume that a(x), b(x) and f(x) are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

Consider x_p = O(ε) be the thickness of the boundary layer (inner region). Now we divide the original problem into two problems, an inner region problem and an outer region problem. The outer region problem is defined in the interval $0 \le x \le 1 - x_p$ and the inner region problem is defined in the interval $1 - x_p \le x \le 1$.

4.2. Terminal Boundary Condition

To obtain the boundary condition at terminal point x_p, we solve the reduced problem with an appropriate boundary condition.

The reduced problem is

$$a(x)y'(x) + b(x)y(x) = f(x)\ with\ y(0) = a$$

Let the terminal boundary condition : $y(x_p) = \gamma(\text{say})$.

4.3. Solution of Inner Region Problem

Since the terminal boundary condition is common to both the inner and outer regions, we can formulate the inner region problem as:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); 1 - x_p \le x \le 1 \quad (19)$$

with

$$y(1 - x_p) = \gamma \quad (20a)$$

and

$$y(1) = \beta \quad (20b)$$

By using the stretching transformation $t = \frac{1-x}{\varepsilon}$, we have

$$y(x) = y(1 - t\varepsilon) = Y(t),$$
$$y'(x) = \frac{-1}{\varepsilon} y'(1 - t\varepsilon) = \frac{-1}{\varepsilon} Y'(t),$$
$$y''(x) = \left(\frac{1}{\varepsilon^2}\right) y''(1 - t\varepsilon) = \left(\frac{1}{\varepsilon^2}\right) Y''(t),$$
$$a(x) = a(1 - t\varepsilon) = A(t), b(x) = b(1 - t\varepsilon) = B(t),$$
$$f(x) = f(1 - t\varepsilon) = F(t),$$

and $y(1) = \beta = Y(0)$.

We get the modified inner region problem as:

$$Y''(t) - A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \quad (21)$$

and the boundary conditions becomes

$$Y(0) = \beta \quad (22a)$$

and

$$Y(t_p) = \gamma \quad (22b)$$

where $t_p = \frac{1 - x_p}{\varepsilon}$.

We solve this new inner region problem (21)- (22) to obtain the solution over the interval $0 \le t \le t_p$.

4.4. Solution of the Outer Region Problem

The solution of the reduced problem is considered as outer solution.

4.5. Solution of the Original Problem

In order to solve the two-point boundary value problem given by the equations (21) –(22) (inner region problem), we used finite difference scheme based on exponential splines which is discussed in section 2.5. We repeat the process for various choices of x_p, until the solution profiles do not differ materially from iteration to iteration. For computational purposes we use an absolute error criterion, namely

$$\left| Y(t)^{m+1} - Y(t)^m \right| \le \sigma; 0 \le t \le t_p$$

where $Y(t)^m$ is the mth iterate of the inner region solution and σ is the prescribed tolerance bound.

4.6. Examples with Right -End Boundary Layer

To illustrate the method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval we have implemented on three examples.

Example 4.6.1. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; \ x \in [0,1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at x=1. i.e., at the right end of the underlying interval. The reduced problem is

$$y_0'(x) = 0; y_0(0) = 1$$

whose solution is $y_0(x) = 1$.

For this example the stretching transformation is

$$t = \frac{1-x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) = 0; with\ Y(0) = 0, Y(t_p) = 1.$$

The exact solution is given by

$$y(x) = \frac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$$

The maximum absolute errors for different values of t_p in the inner region is presented in Table 7(a), Table 7(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Example 4.6.2. Now we consider the following singular perturbation problem $\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0$; $x \in [0,1]$.

with $y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$; and $y(1) = 1 + 1/e$.

Clearly this problem has a boundary layer at $x = 1$.

The reduced problem is

$$y_0'(x) + y_0(x) = 0; y_0(0) = 1.$$

The solution of this problem is

$$y_0(x) = e^{-x}.$$

For this example the stretching transformation

$$t = \frac{1-x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) - \varepsilon(1+\varepsilon)Y(t) = 0;$$

with $Y(0) = 1 + e^{-1}, Y(t_p) = e^{-x_p}$.

The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$$

The maximum absolute errors for different values of t_p in the inner region is presented in Table 8(a), Table 8(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

5. Conclusions

We have described the a domain decomposition method for solving a class of singularly perturbed two point boundary value problems with a boundary layer at one end point. It provides an alternate and supplementary method to the conventional ways of solving certain class of singular perturbation problems. It is a practical method, can be implemented on a computer with a modest amount of problem preparation. We have implemented the present method on three linear examples, three non-linear examples with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. The approximate solution is compared with exact solution. It can be observed from the results that the present method agrees with exact solution very well, which shows the efficiency of the method.

Table 1(a). Maximum absolute error for Example 1, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.4714429E-03	.4955828E-03	.4955530E-03	.4969537E-03
200	.4668534E-03	.5079210E-03	.4955530E-03	.4955828E-03
400	.4771948E-03	.4955828E-03	.5682111E-03	.4955828E-03

Table 1(b). Maximum absolute error for Example 1, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.4113534E-01	.4110917E-01	.4110852E-01	.4110852E-01
200	.4113534E-01	.4111528E-01	.4110852E-01	.4110852E-01
400	.4113534E-01	.4110852E-01	.4118821E-01	.4110852E-01

Table 2(a). Maximum absolute error for Example 2, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.6556511E-04	.2002716E-04	.2276897E-04	.2092123E-04
200	.2101064E-03	.2002716E-04	.2992153E-04	.2270937E-04
400	.6598234E-04	.1991987E-03	.2185702E-03	.2986193E-04

Table 2(b). Maximum absolute error for Example 2, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.4553795E-04	.3993511E-05	.3755093E-05	.2741814E-05
200	.1917481E-03	.2038479E-04	.6854534E-05	.4589558E-05
400	.4619360E-04	.1847148E-03	.2424121E-03	.8106232E-05

Table 3(a). Maximum absolute error for Example 3, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.2691150E-04	.3564358E-04	.1125038E-03	.2344549E-03
200	.3001094E-04	.1159310E-04	.1207590E-03	.2347231E-03
400	.2264977E-04	.2235174E-04	.5236268E-04	.2672672E-03

Table 3(b). Maximum absolute error for Example 3, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.2270937E-04	.4976988E-05	1519918E-05	.4768372E-06
200	.5289912E-04	.9417534E-05	.6943941E-05	.3337860E-05
400	.2303720E-04	.2334416E-03	.2917647E-04	.1424551E-04

Table 4(a). Maximum absolute error for Example 4, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.5860925E-03	.2459526E-02	.1741779E-01	.6585389E-01
200	.1133084E-03	.5230904E-03	.3857434E-02	.1719117E-01
400	.3746748E-03	.1668334E-03	.7692873E-03	.3630042E-02

Table 4(b). Maximum absolute error for Example 4, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.6262064E-03	.2542049E-02	.1763815E-01	.6651086E-01
200	.1523495E-03	.6067157E-03	.4071295E-02	.1763478E-01
400	.3208518E-03	.1300573E-03	.9858012E-03	.4060477E-02

Table 5(a). Maximum absolute error for Example 5, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.1074346E+01	.1090720E+01	.1239683E+01	.1190751E+01
200	.1073008E+01	.1074381E+01	.1088165E+01	.1239672E+01
400	.1071763E+01	.1073007E+01	.1079107E+01	.1088137E+01

Table 5(b). Maximum absolute error for Example 5, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.1074283E+01	.1090712E+01	.1239696E+01	.1190773E+01
200	.1072958E+01	.1074284E+01	.1088189E+01	.1239700E+01
400	.1071620E+01	.1072897E+01	.1078871E+01	.1088186E+01

Table 6(a). Maximum absolute error for Example 6, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.2145767E-05	.4053116E-05	.3993511E-05	.5602837E-05
200	.1523495E-03	.4887581E-04	.5912781E-04	.1019239E-04
400	.1555681E-04	.4479885E-03	.2405643E-03	.1001358E-04

Table 6(b). Maximum absolute error for Example 6, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.2145767E-05	.4053116E-05	.3993511E-05	.5602837E-05
200	.1523495E-03	.4887581E-04	.5912781E-04	.1019239E-04
400	.1555681E-04	.4479885E-03	.2405643E-03	.1001358E-04

Table 7(a). Maximum absolute error for Example 7, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.2163306E-02	.2171993E-02	.4971623E-03	.4985929E-03
200	.2217211E-02	.2175003E-02	.4915595E-03	.4971623E-03
400	.2301179E-02	.2216917E-02	.6645322E-03	.4915595E-03

Table 7(b). Maximum absolute error for Example 7, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.1898208E+00	.1812688E+00	.6429213E-01	.6429356E-01
200	.2212506E+00	.1898375E+00	2212020E+00	.6429213E-01
400	.2404358E+00	.2212492E+00	.2211437E+00	.2212020E+00

Table 8(a). Maximum absolute error for Example 8, for $\varepsilon=10^{-5}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.2170801E-02	.2185464E-02	.6164908E-03	.7321835E-03
200	.2201080E-02	2184510E-02	.6197691E-03	.7388592E-03
400	.2303123E-02	.2206206E-02	.5548000E-03	.7510185E-03

Table 8(b). Maximum absolute error for Example 8, for $\varepsilon=10^{-7}$

N	$t_p = 10$	$t_p = 20$	$t_p = 50$	$t_p = 100$
Inner region 100	.1898209E+00	.1812693E+00	.6429631E-01	.6429672E-01
200	.2211875E+00	1898309E+00	.2211976E+00	.6429744E-01
400	.2404387E+00	.2211742E+00	.2212206E+00	.2211971E+00

References

[1] Natesan, S., Vigo-Aguiar, J. & Ramanujam, N. (2003). A numerical algorithm for singular perturbation problems exhibiting weak boundary layers, Comput. Math. Appl. 45, 469-479.

[2] Robert, S. M. (1982). A Boundary-Value Technique for Singular Perturbation Problems, Journal of Mathematical Analysis and Applications, 87, 489-508.

[3] Bender, C.M. & Orszag, S. A. (1978). Advanced Mathematical Methods for Scientists and Engineers, Mc. Graw-Hill, New York.

[4] Kevorkian, J. & Cole, J. D. (1981). Perturbation Methods in Applied Mathematics, Springer-Verlag, New York.

[5] O' Malley, R. E. (1974). Introduction to Singular Perturbations, Academic Press, New York.

[6] Nayfeh, A. H. (1973). Perturbation Methods, Wiley, New York.

[7] Smith, D. R. (1985). Singular-Perturbation Theory an Introduction with Applications, Cambridge University Press, Cambridge.

[8] Hu, X.C., Manteuffel, T.A., Mccormick, S. & Russell, T.F. (1995). Accurate discretization for singular perturbations: the one-dimensional case, SIAM Journal off Numerical Analysis, 32 (1), 83-109.

[9] Kadalbajoo, M. K. & Reddy, Y. N. (1987). Initial-Value Technique for a Class of Nonlinear Singular Perturbation Problems, Journal of Optimization Theory and Applications, 53, 395-406.

[10] Kadalbajoo M.K. & Devendra Kumar (2008). A non –linear single step explicit scheme for non-linear two point singularly perturbed boundary value problems via initial value technique, Applied Mathematics and Computation, 202, 738-746.

[11] Reddy, Y.N. & Pramod Chakravarthy, P. (2003), Method of Reduction of Order for Solving Singularly Perturbed Two-Point Boundary Value Problems, Applied Mathematics and Computation, 136, 27-45.

[12] Van Niekerk, F.D. (1987), Non linear one step methods for initial value problems, Comput. Math. Appl., 13, 367-371.

[13] Higinio Ramos (2007). A non standard explicit integration scheme for initial value problems, Applied Mathematics and Computation, 189, 710-718.

[14] Zahra, W.K., Exponential spline solutions for a class of two point boundary value problems over a semu infinite range. Numer. Algor. 53, 561-573, 2009.

[15] Zahra, W.K., Finite difference technique based on exponential splines for solution of obstacle problems. Int. J. Computer Math. 88 (14), 3046-3060, 2011.

[16] Zahra, W.K., A smooth approximation based on exponential spline solutions for non linear fourth order two point boundary value problems, Appl. Math. Comput. 217, 8447-8457, 2011.