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Interpolation Splines Minimizing Semi-Norm in $K_2(P_2)$ Space

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Abstract In the present paper using S.L. Sobolev's method interpolation splines minimizing the semi-norm in $K_2(P_2)$ space are constructed. Explicit formulas for coefficients of interpolation splines are obtained. The obtained interpolation spline is exact for the functions $e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$ and $e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$. Also we give some numerical results where we showed connection between optimal quadrature formula and obtained interpolation spline in the space $K_2(P_2)$.

Keywords: *interpolation spline, Hilbert space, the norm minimizing property, S.L. Sobolev's method, discrete argument function*

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1. Introduction

In order to find an approximate representation of a function φ by elements of a certain finite dimensional space, it is possible to use values of this function at some finite set of points x_β, $\beta = 0,1,...,N$. The corresponding problem is called *the interpolation problem*, and the points x_β *the interpolation nodes*.

There are polynomial and spline interpolations. Now the theory of spline interpolation is fast developing. Many books are devoted to the theory of splines, for example, Ahlberg et al [1], Arcangeli et al [2], Attea [3], Berlinet and Thomas-Agnan [4], Bojanov et al [5], de Boor [7], Eubank [9], Green and Silverman [12], Ignatov and Pevniy [17], Korneichuk et al [18], Laurent [19], Mastroianni and Milovanovic [20], Nürnberger [21], Schumaker [23], Stechkin and Subbotin [29], Vasilenko [30], Wahba [32] and others.

Suppose the functions φ belong to the Hilbert space (see [[1], Chapter 3])

$$K^m(a,b)=\{\varphi:[a,b]\to\mathbb{R} \mid \varphi^{(m-1)} \text{ is absolutely continuous and } \varphi^{(m)}\in L_2(a,b)\},$$

equipped with the norm

$$\|\varphi\| = \left(\int_a^b \left(L\varphi(x)\right)^2 dx\right)^{1/2} \quad (1.1)$$

and $\int_0^1 \left(L\varphi(x)\right)^2 dx < \infty$, where

$$L \equiv a_m(x)\cdot\frac{d^m}{dx^m} + a_{m-1}(x)\cdot\frac{d^{m-1}}{dx^{m-1}} + ... + a_0(x), \quad (1.2)$$

here each $a_j(x)$ ($j = 0,1,...,m$) is in $C^j[a,b]$ and $a_m(x)$ does not vanish on $[a,b]$. Let L^* be a formal adjoint of L and

$$L^* \equiv (-1)^m \frac{d^m}{dx^m}\{a_m(x)\cdot\} + (-1)^{m-1}\frac{d^{m-1}}{dx^{m-1}}\{a_{m-1}(x)\cdot\} + ... - \frac{d}{dx}\{a_1(x)\cdot\} + a_0(x).$$

The equality (1.1) is semi-norm and $\|\varphi\| = 0$ only for a solution of the equation $L\varphi = 0$. We give definition of generalized splines following [1, Chapter 6]. If $\Delta: a = x_0 < x_1 < ... < x_N = b$ is a mesh on $[a,b]$, then a *generalized spline* (or L *-spline*) of deficiency k ($0 \le k \le m$) with respect to Δ is a function $S_\Delta(x)$ which is in $K^{2m-k}(a,b)$ and satisfies the differential equation

$$L^* L S_\Delta = 0$$

on each open mesh interval of Δ. The ordinary spline (deficiency one) allows discontinuities in the $(2m-1)$ th derivative, but only at mesh points.

If the exact values $\varphi(x_\beta)$ of an unknown smooth function $\varphi(x)$ at the set of points $\{x_\beta, \beta = 0,1,...,N\}$ in an interval $[a,b]$ are known, it is usual to approximate φ by minimizing

$$\int_a^b (g^{(m)}(x))^2 dx \tag{1.3}$$

in the set of interpolating functions (i.e., $g(x_\beta)=\varphi(x_\beta)$, $\beta=0,1,...,N$) of the Sobolev space $L_2^{(m)}(a,b)$. Here $L_2^{(m)}(a,b)$ is the Sobolev space of functions with a square integrable m-th generalized derivative. It turns out that the solution is a natural polynomial spline of degree $2m-1$ with knots $x_0,x_1,...,x_N$ called the interpolating D^m spline for the points $(x_\beta,\varphi(x_\beta))$. In non periodic case first this problem was investigated by Holladay [16] for $m=2$ and the result of Holladay was generalized by de Boor [6] for any m. In the Sobolev space $\widetilde{L_2^{(m)}}$ of periodic functions the minimization problem of integrals of type (1.3) was investigated by I.J.Schoenberg [22], M.Golomb [13], W.Freeden [10,11] and others.

We consider the Hilbert space

$$\mathrm{K}_2(\mathrm{P}_2)=\{\varphi:[0,1]\to\mathbb{R} \mid \varphi' \text{ is absolutely continuous and } \varphi''\in \mathrm{L}_2(0,1)\},$$

equipped with the norm

$$\|\varphi|K_2(P_2)\|=\left\{\int_0^1\left(P_2\left(\frac{d}{dx}\right)\varphi(x)\right)^2 dx\right\}^{1/2}, \tag{1.4}$$

where $P_2\left(\frac{d}{dx}\right)=\frac{d^2}{dx^2}+\frac{d}{dx}+1$ and

$$\int_0^1\left(P_2\left(\frac{d}{dx}\right)\varphi(x)\right)^2 dx<\infty.$$

The equality (1.4) is semi-norm and $\|\varphi\|=0$ if and only if $\varphi(x)=c_1 e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x+c_2 e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$.

Consider the following interpolation problem:

Problem 1. Find the function $S(x)\in K_2(P_2)$ which gives minimum to the norm (1.4) and satisfies the interpolation condition

$$S(x_\beta)=\varphi(x_\beta),\ \ \beta=0,1,...,N \tag{1.5}$$

for any $\varphi\in K_2(P_2)$, where $x_\beta\in[0,1]$ are the nodes of interpolation.

Following [[30], p.45, Theorem 2.2] we get the analytic representation of the interpolation spline $S(x)$

$$S(x)=\sum_{\gamma=0}^N C_\gamma G(x-x_\gamma)+d_1 e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x+d_2 e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x, \tag{1.6}$$

where C_γ, $\gamma=0,1,...,N$, d_1, d_2 are real numbers,

$$G(x)=\frac{\mathrm{sign}(x)}{2}\left(\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\cdot\cosh\frac{x}{2}-\cos\frac{\sqrt{3}}{2}x\cdot\sinh\frac{x}{2}\right) \tag{1.7}$$

and $G(x)$ is a fundamental solution of the operator $\frac{d^4}{dx^4}+\frac{d^2}{dx^2}+1$, i.e., is a solution of the equation

$$G^{(4)}(x)+G^{(2)}(x)+G(x)=\delta(x),$$

here $\delta(x)$ is Dirac's delta function. It should be noted that the rule for finding a fundamental solution of a linear differential operator

$$L\equiv\frac{d^n}{dx^n}+a_1\frac{d^{n-1}}{dx^{n-1}}+...+a_n,$$

where a_j are real numbers, is given in [31, p.88]. Using this rule, it is found the function $G(x)$ which is a fundamental solution of the operator $\frac{d^4}{dx^4}+\frac{d^2}{dx^2}+1$ and has the form (1.7).

Furthermore from [30, p.45-47] it follows that the solution $S(x)$ of the form (1.6) of Problem 1 is exists, unique when $N=1,2,...$ and coefficients C_γ, $\gamma=0,1,2,...,N$, d_1, d_2 of $S(x)$ are defined by the following system of $N+3$ linear equations

$$\sum_{\gamma=0}^N C_\gamma G(x_\beta-x_\gamma)+d_1 e^{-\frac{1}{2}x_\beta}\sin\frac{\sqrt{3}}{2}x_\beta+d_2 e^{-\frac{1}{2}x_\beta}\cos\frac{\sqrt{3}}{2}x_\beta=\varphi(x_\beta),\qquad \beta=0,1,...,N, \tag{1.8}$$

$$\sum_{\gamma=0}^N C_\gamma e^{-\frac{1}{2}x_\gamma}\sin\frac{\sqrt{3}}{2}x_\gamma=0, \tag{1.9}$$

$$\sum_{\gamma=0}^N C_\gamma e^{-\frac{1}{2}x_\gamma}\cos\frac{\sqrt{3}}{2}x_\gamma=0, \tag{1.10}$$

where $\varphi\in K_2(P_2)$.

It should be noted that systems for coefficients of D^m splines similar to the system (1.8)-(1.10) were investigated, for example, in [2,8,17,19,30].

The main aim of the present paper is to solve Problem 1, i.e., to solve the system (8)-(10) for equal spaced nodes $x_\beta=h\beta$, $\beta=0,1,...,N$, $h=1/N$, $N=1,2,...$ and to find analytic formula for coefficients C_γ, γ=0,1,...,N, d_1 and d_2 of $S(x)$.

The rest of the paper is organized as follows: in section 2 we give the algorithm for solution of system (1.8)-(2.10) when the nodes x_β are equal spaced. Using this algorithm coefficients of the interpolation spline $S(x)$ are computed in section 3. In section 4 some numerical results are presented.

2. The Algorithm for Computation of Coefficients of Interpolation Splines

In the present section we give the algorithm for solution of system (1.8)-(1.10) when the nodes x_β are equal spaced. Here we use similar method suggested by S.L. Sobolev [26,28] for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}$. Below mainly is used the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [27,28]. For completeness we give some definitions about functions of discrete argument.

Assume that the nodes x_β are equal spaced, i.e., $x_\beta = h\beta,\ h=\frac{1}{N},\ N=1,2,\ldots$.

Definition 2.1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 2.2. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$\left[\varphi(h\beta),\psi(h\beta)\right]=\sum_{\beta=-\infty}^{\infty}\varphi(h\beta)\cdot\psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 2.3. *The convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta)*\psi(h\beta)=\sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\beta-h\gamma).$$

Now we turn to our problem.

Suppose that $C_\beta=0$ when $\beta<0$ and $\beta>N$. Using above mentioned definitions, we rewrite the system (1.8)-(1.10) in the convolution form

$$\begin{aligned}&G(h\beta)*C_\beta+d_1e^{-\frac{1}{2}(h\beta)}\sin\frac{\sqrt{3}}{2}(h\beta)+\\&+d_2e^{-\frac{1}{2}(h\beta)}\cos\frac{\sqrt{3}}{2}(h\beta)=\varphi(h\beta),\qquad \beta=0,1,\ldots,N,\end{aligned}\tag{2.1}$$

$$\sum_{\beta=0}^{N}C_\beta e^{-\frac{1}{2}(h\beta)}\sin\frac{\sqrt{3}}{2}(h\beta)=0,\tag{2.2}$$

$$\sum_{\beta=0}^{N}C_\beta e^{-\frac{1}{2}(h\beta)}\cos\frac{\sqrt{3}}{2}(h\beta)=0.\tag{2.3}$$

Thus we have the following problem.

Problem 2. Find the discrete function C_β, $\beta=0,1,\ldots,N$ and unknown constants d_1, d_2 which satisfy the system (2.1)-(2.3).

Further we investigate Problem 2 which is equivalent to Problem 1. Instead of C_β we introduce the following functions

$$v(h\beta)=G(h\beta)*C_\beta,\tag{2.4}$$

$$\begin{aligned}u(h\beta)=v(h\beta)&+d_1e^{-\frac{1}{2}(h\beta)}\sin\frac{\sqrt{3}}{2}(h\beta)+\\&+d_2e^{-\frac{1}{2}(h\beta)}\cos\frac{\sqrt{3}}{2}(h\beta).\end{aligned}\tag{2.5}$$

In such statement it is necessary to express the coefficients C_β by the function $u(h\beta)$. For this we have to construct such operator $D(h\beta)$ which satisfies the equality

$$D(h\beta)*G(h\beta)=\delta(h\beta),\tag{2.6}$$

where $\delta(h\beta)$ is equal to 0 when $\beta\neq 0$ and is equal to 1 when $\beta=0$, i.e., $\delta(h\beta)$ is the discrete delta-function. In connection with this the discrete analogue $D(h\beta)$ of the operator $\frac{d^4}{dx^4}+\frac{d^2}{dx^2}+1$, which satisfies equation (2.6) is constructed in [14] and its some properties were investigated. Following in [14] we have:

Theorem 2.1. *The discrete analogue of the differential operator* $\frac{d^4}{dx^4}+\frac{d^2}{dx^2}+1$ *satisfying the equation.(2.6) has the form*

$$D(h\beta)=K\begin{cases}A_1\lambda_1^{|\beta|-1}, & |\beta|\geq 2,\\ 1+A_1, & |\beta|=1,\\ C+\frac{A_1}{\lambda_1}, & \beta=0,\end{cases}\tag{2.7}$$

where $K=\dfrac{1}{\frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}h)\cdot\cosh(\frac{h}{2})-\cos(\frac{\sqrt{3}}{2}h)\cdot\sinh(\frac{h}{2})}$,

$$C=-p-4\cos(\frac{\sqrt{3}}{2}h)\cosh(\frac{h}{2}),$$

$$p=\frac{\sinh(h)-\frac{1}{\sqrt{3}}\sin(\sqrt{3}h)}{\frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}h)\cdot\cosh(\frac{h}{2})-\cos(\frac{\sqrt{3}}{2}h)\cdot\sinh(\frac{h}{2})},$$

$$A_1=\frac{P_4(\lambda_1)}{\lambda_1^2-1},$$

$$\begin{aligned}P_4(\lambda_1)=\lambda_1^4-4\lambda_1^3\cos(\frac{\sqrt{3}}{2}h)\cosh(\frac{h}{2})+2(1+\cos(\sqrt{3}h)\\+\cosh(h))\lambda_1^2-4\lambda_1\cos(\frac{\sqrt{3}}{2}h)\cosh(\frac{h}{2})+1\end{aligned}\tag{2.8}$$

and

$$\lambda_1=\frac{\sin(\sqrt{3}h)-\sqrt{3}\sinh(h)+2\sqrt{\left(\cos^2(\frac{\sqrt{3}}{2}h)-\cosh^2(\frac{h}{2})\right)\left(\sin^2(\frac{\sqrt{3}}{2}h)-3\sinh^2(\frac{h}{2})\right)}}{2(\sin(\frac{\sqrt{3}}{2}h)\cosh(\frac{h}{2})-\sqrt{3}\cos(\frac{\sqrt{3}}{2}h)\sinh(\frac{h}{2}))}\tag{2.9}$$

is a zero of the polynomial

$$Q_2(\lambda) = \lambda^2 - p\lambda + 1,$$

and $|\lambda_1| < 1$ *and* h *is a small parameter.*

Theorema 2.2. *The discrete analogue* $D(h\beta)$ *of the differential operator* $\frac{d^4}{dx^4} + \frac{d^2}{dx^2} + 1$ *satisfies the following equalities:*

1) $D(h\beta) * e^{-\frac{1}{2}(h\beta)} \sin\frac{\sqrt{3}}{2}(h\beta) = 0$,

2) $D(h\beta) * e^{-\frac{1}{2}(h\beta)} \cos\frac{\sqrt{3}}{2}(h\beta) = 0$,

3) $D(h\beta) * e^{\frac{1}{2}(h\beta)} \sin\frac{\sqrt{3}}{2}(h\beta) = 0$,

4) $D(h\beta) * e^{\frac{1}{2}(h\beta)} \cos\frac{\sqrt{3}}{2}(h\beta) = 0$,

5) $D(h\beta) * G(h\beta) = \delta(h\beta)$.

Here $G(h\beta)$ *is the function of discrete argument, corresponding to the function* $G(x)$ *defined by (1.7) and* $\delta(h\beta)$ *is the discrete delta function.*

Then taking into account (2.5), (2.6) and Theorems 2.1 and 2.2, for the coefficients we have

$$C_\beta = D(h\beta) * u(h\beta). \tag{2.10}$$

Thus if we find the function $u(h\beta)$ then the coefficients C_β can be obtained from equality (10). In order to calculate the convolution (2.10) we need a representation of the function $u(h\beta)$ for all integer values of β. From equality (2.1) we get that $u(h\beta) = \varphi(h\beta)$ when $h\beta \in [0,1]$. Now we need to find a representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$. Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D(h\beta) * u(h\beta) = 0$, $h\beta \notin [0,1]$. Now we calculate the convolution $v(h\beta) = G(h\beta) * C_\beta$ when $\beta \le 0$ and $\beta \ge N$.

Suppose $\beta \le 0$ then taking into account equalities (1.7), (2.2), (2.3), we have

$$v(h\beta) = \sum_{\gamma=-\infty}^{\infty} C_\gamma G(h\beta - h\gamma)$$

$$= \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sign}(h\beta - h\gamma)}{2}\left(\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(h\beta - h\gamma)\cdot\cosh\frac{h\beta - h\gamma}{2}\right.$$

$$\left. - \cos\frac{\sqrt{3}}{2}(h\beta - h\gamma)\cdot\sinh\frac{h\beta - h\gamma}{2}\right)$$

$$= -\frac{1}{4}\begin{bmatrix} \frac{1}{\sqrt{3}}\sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}h\gamma \\ + \sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \end{bmatrix} e^{-\frac{1}{2}h\beta}\sin\frac{\sqrt{3}}{2}h\beta$$

$$-\frac{1}{4}\begin{bmatrix} -\frac{1}{\sqrt{3}}\sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \\ + \sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}h\gamma \end{bmatrix} e^{-\frac{1}{2}h\beta}\cos\frac{\sqrt{3}}{2}h\beta.$$

Denoting

$$b_1 = \frac{1}{4}\begin{bmatrix} \frac{1}{\sqrt{3}}\sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}h\gamma \\ + \sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \end{bmatrix},$$

$$b_2 = \frac{1}{4}\begin{bmatrix} -\frac{1}{\sqrt{3}}\sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \\ + \sum_{\gamma=0}^{N} C_\gamma e^{\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}h\gamma \end{bmatrix}$$

we get for $\beta \le 0$

$$v(h\beta) = -b_1 e^{-\frac{1}{2}h\beta}\sin\frac{\sqrt{3}}{2}h\beta - b_2 e^{-\frac{1}{2}h\beta}\cos\frac{\sqrt{3}}{2}h\beta$$

and for $\beta \ge N$

$$v(h\beta) = b_1 e^{-\frac{1}{2}h\beta}\sin\frac{\sqrt{3}}{2}h\beta + b_2 e^{-\frac{1}{2}h\beta}\cos\frac{\sqrt{3}}{2}h\beta.$$

Now, setting

$$d_1^- = d_1 - b_1, \quad d_2^- = d_2 - b_2,$$
$$d_1^+ = d_1 + b_1, \quad d_2^+ = d_2 + b_2$$

we have the following problem:

Problem 3. Find the solution of the equation

$$D(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1] \tag{2.11}$$

in the form:

$$u(h\beta) = \begin{cases} d_1^- e^{-\frac{1}{2}h\beta}\sin\frac{\sqrt{3}}{2}h\beta + d_2^- e^{-\frac{1}{2}h\beta}\cos\frac{\sqrt{3}}{2}h\beta, & \beta \le 0, \\ \varphi(h\beta), & 0 \le \beta \le N, \\ d_1^+ e^{-\frac{1}{2}h\beta}\sin\frac{\sqrt{3}}{2}h\beta + d_2^+ e^{-\frac{1}{2}h\beta}\cos\frac{\sqrt{3}}{2}h\beta, & \beta \ge N. \end{cases} \tag{2.12}$$

where d_1^-, d_2^-, d_1^+, d_2^+ are unknown coefficients.

It is clear that

$$d_1 = \frac{1}{2}\left(d_1^+ + d_1^-\right), \; d_2 = \frac{1}{2}\left(d_2^+ + d_2^-\right), \tag{2.13}$$

$$b_1 = \frac{1}{2}\left(d_1^+ - d_1^-\right), \; b_2 = \frac{1}{2}\left(d_2^+ - d_2^-\right).$$

These unknowns d_1^-, d_2^-, d_1^+, d_2^+ can be found from equation (2.11), using the function $D(h\beta)$. Then the explicit form of the function $u(h\beta)$ and coefficients C_β, d_1, d_2 can be found. Thus Problem 3 and respectively Problems 2 and 1 can be solved.

In the next section we realize this algorithm for computation of coefficients C_β, $\beta = 0,1,...,N$, d_1 and d_2 of the interpolation spline (1.6) for any $N = 1,2,...$.

3. Computation of Coefficients of Interpolation Spline (1.6)

In this section using the algorithm which is given in Section 2 we obtain explicit formulas for coefficients of interpolation spline (1.6) which is the solution of Problem 1.

It should be noted that the interpolation spline (1.6) which is the solution of Problem 1 is exact for the functions $e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$ and $e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$.

The following holds

Theorem 3.1. *Coefficients of interpolation spline (1.6) which minimizes the norm (1.4) with equal spaced nodes in the space $K_2(P_2)$ have the following form*

$$C_0 = KC\varphi(0) + K\begin{bmatrix}\varphi(h) - d_1^- e^{\frac{1}{2}h}\sin\frac{\sqrt{3}}{2}h \\ +d_2^- e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h\end{bmatrix} + \frac{KA_1}{\lambda_1}\left[\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma) + M_1 + \lambda_1^N N_1\right],$$

$$C_\beta = KC\varphi(h\beta) + K[\varphi(h(\beta-1)) + \varphi(h(\beta+1))] + \frac{KA_1}{\lambda_1}\left[\sum_{\gamma=0}^{N}\lambda_1^{|\beta-\gamma|}\varphi(h\gamma) + \lambda_1^{\beta}M_1 + \lambda_1^{N-\beta}N_1\right],$$

$$\beta = 1,2,...,N-1,$$

$$C_N = KC\varphi(1) + K\begin{bmatrix}\varphi(h(N-1)) + d_1^+ e^{-\frac{1}{2}(1+h)}\sin\frac{\sqrt{3}}{2}(1+h) \\ +d_2^+ e^{-\frac{1}{2}(1+h)}\cos\frac{\sqrt{3}}{2}(1+h)\end{bmatrix} + \frac{KA_1}{\lambda_1}\left[\sum_{\gamma=0}^{N}\lambda_1^{N-\gamma}\varphi(h\gamma) + \lambda_1^N M_1 + N_1\right],$$

$$d_1 = \frac{1}{2}(d_1^+ + d_1^-), \quad d_2 = \frac{1}{2}(d_2^+ + d_2^-),$$

where p, A_1, C and λ_1 are defined by (2.8), (2.9),

$$M_1 = \frac{\lambda_1 e^{\frac{1}{2}h}[d_2^-(\cos\frac{\sqrt{3}}{2}h - \lambda_1 e^{\frac{1}{2}h}) - d_1^-\sin\frac{\sqrt{3}}{2}h]}{\lambda_1^2 e^h - 2\lambda_1 e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h + 1}, \quad (3.1)$$

$$N_1 = \frac{\lambda_1\begin{bmatrix}d_2^+(e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}(h+1) - \lambda_1\cos\frac{\sqrt{3}}{2}) \\ +d_1^+({}_1e^{\frac{1}{2}h}\sin\frac{\sqrt{3}}{2}(h+1) - \lambda\sin\frac{\sqrt{3}}{2})\end{bmatrix}}{e^{\frac{h+1}{2}}(\lambda_1^2 e^{-\frac{h}{2}} - 2\lambda_1\cos\frac{\sqrt{3}}{2}h + e^{\frac{h}{2}})}, \quad (3.2)$$

d_1^+, d_1^-, d_2^+, d_2^- are defined by (3.3), (3.7).

Proof. First we find the expressions for d_2^- and d_2^+. From (2.12) when $\beta = 0$ and $\beta = N$ we get

$$d_2^- = \varphi(0), \quad d_2^+ = \frac{e^{\frac{1}{2}}}{\cos\frac{\sqrt{3}}{2}}\varphi(1) - d_1^+\tan\frac{\sqrt{3}}{2}. \quad (3.3)$$

Now we have 2 unknowns d_1^-, d_1^+. These unknowns we find from (2.11) when $\beta = -1$ and $\beta = N+1$.

Taking into account (2.12) and Definition 2.3 from (2.11) we have

$$\sum_{\gamma=-\infty}^{-1} D(h\beta - h\gamma)\begin{bmatrix}d_1^- e^{-\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \\ +d_2^- e^{-\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}(h\gamma)\end{bmatrix} + \sum_{\gamma=0}^{N} D(h\beta - h\gamma)\varphi(h\gamma) + \sum_{\gamma=N+1}^{\infty} D(h\beta - h\gamma)\begin{bmatrix}d_1^+ e^{-\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \\ +d_2^+ e^{-\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}h\gamma\end{bmatrix} = 0,$$

where $\beta < 0$ and $\beta > N$.

Hence for $\beta = -1$, $\beta = N+1$, taking into account (3.3) and (2.7), after some calculations we obtain

$$B_{11}d_1^- + B_{12}d_1^+ = T_1,$$
$$B_{21}d_1^- + B_{22}d_1^+ = T_2,$$

where

$$B_{11} = \lambda_1 e^{-\frac{1}{2}h}\sin\frac{\sqrt{3}}{2}h,$$

$$B_{12} = \frac{\lambda_1^{N+1}e^{-\frac{1}{2}(1+h)}\sin\frac{\sqrt{3}}{2}h}{\cos\frac{\sqrt{3}}{2}}, \quad (3.4)$$

$$B_{21} = \lambda_1^{N+1}e^{\frac{1}{2}h}\sin\frac{\sqrt{3}}{2}h,$$

$$B_{22} = \frac{\lambda_1 e^{\frac{1}{2}(h-1)}\sin\frac{\sqrt{3}}{2}h}{\cos\frac{\sqrt{3}}{2}}$$

$$T_1 = (\lambda_1^2 e^{-h} - 2\lambda_1 e^{-\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h + 1)\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma)$$

$$+(\lambda_1 e^{-\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h - 1)\varphi(0) \qquad (3.5)$$

$$+\frac{\lambda_1^{N+1}e^{-\frac{1}{2}h}(\cos\frac{\sqrt{3}}{2}(h+1) - \lambda_1 e^{-\frac{1}{2}h}\cos\frac{\sqrt{3}}{2})}{\cos\frac{\sqrt{3}}{2}}\varphi(1),$$

$$T_2 = \lambda_1^N(\lambda_1^2 e^{h} - 2\lambda_1 e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h + 1)\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}\varphi(h\gamma)$$

$$+\lambda_1^{N+1}e^{\frac{1}{2}h}(\cos\frac{\sqrt{3}}{2}h - \lambda_1 e^{\frac{1}{2}h})\varphi(0) \qquad (3.6)$$

$$+\frac{\lambda_1 e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}(h+1) - \cos\frac{\sqrt{3}}{2}}{\cos\frac{\sqrt{3}}{2}}\varphi(1).$$

Hence we get

$$d_1^- = \frac{T_1 B_{22} - T_2 B_{12}}{B_{11}B_{22} - B_{12}B_{21}}, \quad d_1^+ = \frac{T_2 B_{11} - T_1 B_{21}}{B_{11}B_{22} - B_{12}B_{21}}. \qquad (3.7)$$

Combaining (2.13), (3.3) and (3.7) we obtain d_1 and d_2 which are given in the statement of Theorem 3.1.

Now we calculate the coefficients C_β, $\beta = 0,1,...,N$. Taking into account (2.12) from (2.10) for C_β we have

$$C_\beta = D(h\beta)*u(h\beta) = \sum_{\gamma=-\infty}^{\infty} D(h\beta - h\gamma)u(h\gamma)$$

$$= \sum_{\gamma=1}^{\infty} D(h\beta + h\gamma)\left[\begin{array}{l} -d_1^- e^{\frac{1}{2}h\gamma}\sin\frac{\sqrt{3}}{2}h\gamma \\ +d_2^- e^{\frac{1}{2}h\gamma}\cos\frac{\sqrt{3}}{2}h\gamma \end{array}\right]$$

$$+\sum_{\gamma=0}^{N} D(h\beta - h\gamma)\varphi(h\gamma) + \sum_{\gamma=1}^{\infty} D(h(N+\gamma) - h\beta)$$

$$\times\left[\begin{array}{l} d_1^+ e^{-\frac{1}{2}(1+h\gamma)}\sin\frac{\sqrt{3}}{2}(1+h\gamma) \\ +d_2^+ e^{-\frac{1}{2}(1+h\gamma)}\cos\frac{\sqrt{3}}{2}(1+h\gamma) \end{array}\right].$$

From here using (2.7), taking into account notations (3.1), (3.2) when $\beta = 0,1,...,N$ for C_β we get expressions which are given in the statemant of Theorem 3.1.

Theorem 3.1 is proved.

4. Numerical Results

As numerical examples we consider the following functions

$$\varphi_1(x) = e^x, \ \varphi_2(x) = \tan(x), \varphi_3(x) = \frac{1}{1+x^2}.$$

Applying the interpolation spline (1.6) to the functions φ_1, φ_2, and φ_3, using Theorem 3.1 with $N = 5, 10$ we get corresponding interpolation splines denoted by $S(\varphi_1, x)$, $S(\varphi_2, x)$ and $S(\varphi_3, x)$. Graphs of absolute errors between functions and corresponding interpolation splines are displayed in the Figure 4.1 and Figure 4.2.

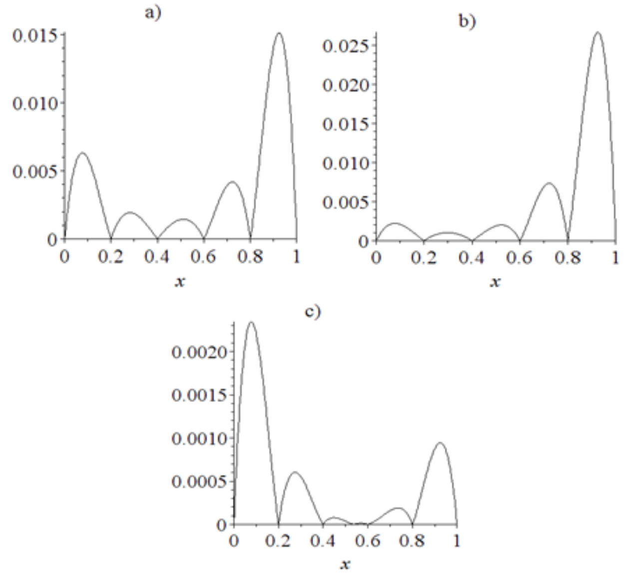

Figure 4.1. Graphs of the absolute errors for $N = 5$: a) $|\varphi_1 - S(\varphi_1, x)|$, b) $|\varphi_2 - S(\varphi_2, x)|$, c) $|\varphi_3 - S(\varphi_3, x)|$.

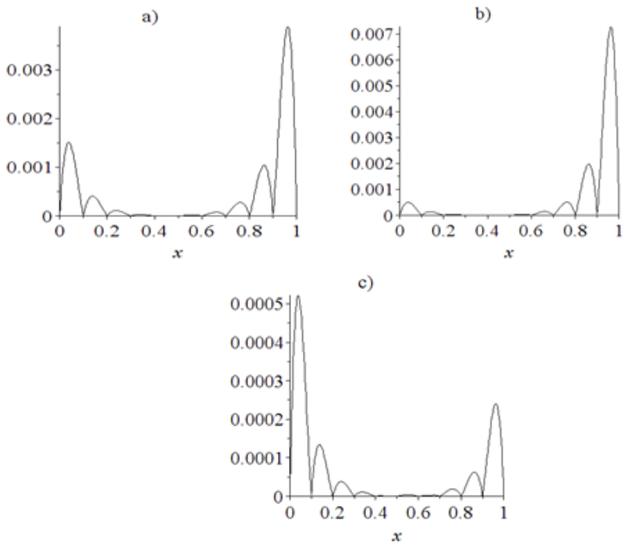

Figure 4.2. Graphs of the absolute errors for $N = 10$: a) $|\varphi_1 - S(\varphi_1, x)|$, b) $|\varphi_2 - S(\varphi_2, x)|$, c) $|\varphi_3 - S(\varphi_3, x)|$

In Figure 4.1, Figure 4.2 one can see that by increasing values of N the absolute errors between interpolation splines and given functions are decreasing.

It should be noted that in [15] the optimal quadrature formula of the following form

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta), \qquad (4.1)$$

was constructed in the space $K_2(P_2)$ and the following was proved

Theorem 4.1 *(Theorem 7 of [15]). The coefficients of the optimal quadrature formulas in the sense of Sard of the form (4.1) in the space* $K_2(P_2)$ *are*

$$C_0 = \frac{\sin\frac{\sqrt{3}}{2}h - \sqrt{3}\cos\frac{\sqrt{3}}{2}h}{2\sin\frac{\sqrt{3}}{2}h} + \frac{\sqrt{3}e^{-\frac{1}{2}h}\lambda_1(\lambda_1^{N-2}+1)}{2\sin\frac{\sqrt{3}}{2}h(\lambda_1^N+1)}$$

$$+\frac{e^{-\frac{1}{2}h}(\lambda_1-1)(\lambda_1^{N-1}-1)}{(\lambda_1^N+1)(\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}h+\sinh\frac{h}{2})},$$

$$C_\beta = \frac{2(\cosh\frac{h}{2}-\cos\frac{\sqrt{3}}{2}h)}{\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}h+\sinh\frac{h}{2}} + \frac{\sqrt{3}\sinh\frac{h}{2}-\sin\frac{\sqrt{3}}{2}h}{\sin\frac{\sqrt{3}}{2}h(\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}h+\sinh\frac{h}{2})}$$

$$\times\frac{\left[\begin{array}{l}\lambda_1^\beta(\lambda_1^2-2\lambda_1 e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h+e^h)\\ +\lambda_1^{N-\beta}(\lambda_1^2e^h-2\lambda_1e^{\frac{1}{2}h}\cos\frac{\sqrt{3}}{2}h+1)\end{array}\right]}{2\lambda_1(\lambda_1^N+1)e^{\frac{1}{2}h}},\quad \beta=\overline{1,N-1},$$

$$C_N = \frac{-\sin\frac{\sqrt{3}}{2}h - \sqrt{3}\cos\frac{\sqrt{3}}{2}h}{2\sin\frac{\sqrt{3}}{2}h} + \frac{\sqrt{3}e^{\frac{1}{2}h}\lambda_1(\lambda_1^{N-2}+1)}{2\sin\frac{\sqrt{3}}{2}h(\lambda_1^N+1)}$$

$$+\frac{e^{\frac{1}{2}h}(\lambda_1-1)(\lambda_1^{N-1}-1)}{(\lambda_1^N+1)(\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}h+\sinh\frac{h}{2})},$$

where λ_1 *is given in Theorem 2.1 and* $|\lambda_1|<1$.

In [15] in numerical results were considered the functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ and corresponding integrals

$$I=\int_0^1 e^x\,\mathrm{d}x = e-1 = 1.718281828459045\ldots,$$

$$J=\int_0^1 \tan x\,\mathrm{d}x = -\log(\cos 1)$$
$$=0.6156264703860142\ldots,$$

$$K=\int_0^1\frac{1}{1+x^2}\mathrm{d}x = 0.7853981633974483\ldots.$$

Applying the optimal quadrature formula (4.1), with $N=10,100,1000$, to the previous integrals were obtained their approximate values denoted by I_N, J_N, and K_N, respectively. The corresponding absolute errors are displayed in Table 4.1 (Table 4.1 of [15]). Numbers in parentheses indicate decimal exponents. Now applying the interpolation spline (1.6), with $N=10,100,1000$ to the functions $\varphi_1(x),\varphi_2(x),\varphi_3(x)$ using Theorem 3.1 we get corresponding interpolation splines $S(\varphi_1,x)$, $S(\varphi_2,x)$ and $S(\varphi_3,x)$. Further integrating of the differences

$$\varphi_1(x)-S(\varphi_1,x),\ \varphi_2(x)-S(\varphi_2,x),\ \varphi_3(x)-S(\varphi_3,x)$$

Table 4.1. Absolute errors of quadrature approximations I_N, J_N, K_N

N	$\lvert I_N-I\rvert$	$\lvert J_N-J\rvert$	$\lvert K_N-K\rvert$
10	2.642(-4)	3.767(-4)	1.356(-5)
100	2.679(-7)	3.987(-7)	1.214(-8)
1000	2.683(-10)	4.004(-10)	1.201(-11)

and taking their absolute values we get the results of the Table 4.1, i.e.

$$\left|\int_0^1(\varphi_1(x)-S(\varphi_1,x))\mathrm{d}x\right| = |I_N-I|,$$

$$\left|\int_0^1(\varphi_2(x)-S(\varphi_2,x))\mathrm{d}x\right| = |J_N-J|,$$

$$\left|\int_0^1(\varphi_1(x)-S(\varphi_1,x))\mathrm{d}x\right| = |K_N-K|.$$

Thus, we conclude that by integrating the interpolation spline of the form (1.6) which minimize the norm (1.4) in the space $K_2(P_2)$ we obtain optimal quadrature formula in the sense of Sard of the form (4.1) in the same space.

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