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Variational Iteration Method for a Singular Perturbation Boundary Value Problems

Hradyesh Kumar Mishra*, Sonali saini*

Department of Mathematics, Jaypee University of Engineering & Technology, Guna (M.P), India
*Corresponding author: hk.mishra@juet.ac.in, sonali.saini1386@gmail.com

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Abstract In this paper, the author used He's variational iteration method for solving singularly perturbed two-point boundary value problems. Few examples are solved to demonstrate the applicability of the method. It is observed that a good choice of the freely selected initial approximation in the variational iteration method leads to closed form solutions by using only one or two iterations. It is also observed that the variational iteration method can be easily applied to the initial and boundary value problems. Graphs are also plotted for the numerical examples.

Keywords: *Singular perturbation, two-point, boundary value problems; Variational Iteration method*

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1. Introduction

We consider the following class of singularly perturbed two-point boundary value problems (BVPs)

$$\varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = g(x),\ x \in [0,1] \qquad (1)$$

Boundary conditions are

$$y(0) = \alpha,\ y(1) = \beta,\ \ \alpha, \beta \in \Re, \qquad (2)$$

Where ε is a small positive parameter ($0 < \varepsilon << 1$), α, β are given constants. and $a(x), b(x)$ *and* $g(x)$ are assumed to be sufficiently continuously differentiable function in [0,1] and $b(x) \ge 0,\ a(x) \ge M > 0$ on [0,1] where M is some positive constant. Under these assumptions (1)-(2) has a unique solution. This type of boundary value problems are generally occurs in optimal control theory, reaction-diffusion process, fluid mechanics and quantum mechanics. There are so many technique for solving singular perturbation problems [5,22,24,26].

The variational iteration method (VIM) was proposed by Ji-Huan He [7-14]. The idea of VIM is to construct a correction functional using generalized Lagrange multipliers. The multipliers can be defined using the calculus of variations. The multipliers in the correction functional should be chosen such that the corrected solution is superior to the initial approximation (trial function) and is the best for the given type of trial function. The trial function can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions. The method has been found to give rapidly convergent successive approximations for the solution if such a solution exists. The VIM is used by many researchers to study different type of equation for finding both analytical and approximate solution. VIM method is used for solving autonomous ordinary differential system in He, Fisher's equation, Klein-Gorden equation, Voletra and Fredholm Equation and many linear and non-linear equations. Some author used the variational iteration method in many other problems, for example Dehghan and Shakeri [19] applied VIM to solve the Cauchy reaction-diffusion problem. Jinbo and Jiang [17] used VIM to solve an inverse parabolic equation. H.K.Mishra [20] studied a comparative study of variational iteration method and He Laplace method. Ramos [15] applied VIM to solve nonlinear partial differential equations. Das [25] used VIM to obtain the solution of a fractional-diffusion equation. Assas [16] applied VIM to solve coupled-KdV equations. Mustafa etc [23] used VIM to solve space and time-fractional Burgers equations with initial conditions. Javidi and Jalilian [18] applied VIM to calculate the wave solution of the Boussinesq equation. Wazwaz [1] applied VIM to some linear and nonlinear systems of PDEs. Batiha et al. [2] used VIM to solve systems of PDEs. Salkuyeh [4] showed convergence of VIM for linear systems of ODEs with constant coefficients.

In this paper, the goal of our study is to introduce He's Variational Itreation Method to solve a singular singularly perturbed two-point boundary value problems (BVPs). In the section 2, analysis of variation iteration method is given. Section 3 of the paper is based on some numerical examples for showing the utility of the presented method. In the last section 4, we will discuss our results with conclusion.

2. Analysis of Variational Iteration Method

By He's method we introduce the following correction functional corresponding to equation (1)

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t) \begin{Bmatrix} \varepsilon y''(t) + a(t)y'(t) \\ -b(t)y(t) - g(t) \end{Bmatrix} dt \quad (3)$$

Where λ is a Lagranges multiplier that can be identify by VIM method. After using the restricted variations $\delta y_n(x) = 0, \quad \delta y_n'(x) = 0$, we obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(x,t) \begin{Bmatrix} \varepsilon y''(t) + a(t)y'(t) \\ -b(t)y(t) - g(t) \end{Bmatrix} dt$$

$$\delta y_{n+1}(x) = \delta y_n(x) + \varepsilon \int_0^x \lambda(x,t) \frac{d^2}{dt^2} \delta y_n(t) dt + \int_0^x \lambda(x,t) \frac{d}{dt} a(t) \delta y_n(t) dt - \int_0^x \lambda(x,t) \delta b(t) y_n(t) dt \quad (4)$$

Integrating (4) by parts, we get

$$\delta y_{n+1}(x) = \left(1 - \varepsilon \frac{\partial \lambda(x,t)}{\partial t} + a(t)\lambda(x,t)\right) \delta y_n(t) \Big]_{t=x} + \varepsilon \lambda(x,t) \frac{d}{dt} \delta y_n(t) \Big]_{t=x} + \int_0^x \left(\varepsilon \frac{\partial^2 \lambda(x,t)}{\partial t2} - a(t) \frac{\partial \lambda(x,t)}{\partial t} - b(t)(t)\lambda(x,t) \right) \delta y_n(t) dt \quad (5)$$

After solving the equation with imposing the above restricted variation terms to (5) we obtain the following Euler- Langrange equation

$$\varepsilon \frac{\partial^2 \lambda(x,t)}{\partial t2} - a(t) \frac{\partial \lambda(x,t)}{\partial t} - b(t)\lambda(x,t) = 0,$$

$$\left(1 - \varepsilon \frac{\partial \lambda(x,t)}{\partial t} + a(t)\lambda(x,t)\right)_{t=x} = 0,$$

$$\lambda(x,t)_{t=x} \quad (6)$$

3. Numerical Results

To incorporate our discussion above two special cases of the singularly perturbed two-point boundary value problem (1)-(2) will be studied. Here we made comparison between one iteration Variational iteration method and the exact solution for each problem. All the results are calculated by using Mathematica software.

Example 3.1. Consider the following singular perturbation problem from Bender and Orszag [3, p. 480, problem 9.17 with $\alpha = 0$]:

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad (7)$$

$$\text{for } 0 \le x \le 1 \text{ with } y(0) = 1 \text{ and } y(1) = 1. \quad (8)$$

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]} \quad (9)$$

where $m_1 = (-1 + \sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1+4\varepsilon})/(2\varepsilon)$.

Solving (6) using the coefficients $a(x) = 1, \quad b(x) = -1,$ then λ can easily identified as

$$\lambda(x,t) = \frac{e^{m_1(t-x)} - e^{m_2(t-x)}}{\varepsilon(m_1 - m_2)}$$

Here

$$m_1 = \frac{1 + \sqrt{1+4\varepsilon}}{2\varepsilon}, \quad m_2 = \frac{1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$$

Therefore, we have the following iteration formula,

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t) \begin{Bmatrix} \varepsilon y_n''(t) + a(t)y_n'(t) \\ -b(t)y_n(t) - g(t) \end{Bmatrix} dt$$

Now, we begin with the following initial approximation,

$$y_0(x) = d_1 + d_2 e^{-x/\varepsilon}$$

where d_1 and d_2 are constants to be determined.

By the above iteration formula, we have

$$y_1(x) = d_1 + d_2 e^{-x/\varepsilon} - \frac{1}{\varepsilon(m_1 - m_2)} \begin{bmatrix} \frac{d_1}{m_1} - \frac{d_1}{m_2} + \frac{d_2 e^{-x/\varepsilon}}{\left(m_1 - \frac{1}{\varepsilon}\right)} - \frac{d_2 e^{-x/\varepsilon}}{\left(m_2 - \frac{1}{\varepsilon}\right)} \\ -\frac{d_1 e^{-m_1 x}}{m_1} + \frac{d_1 e^{-m_2 x}}{m_2} \\ -\frac{d_2 e^{-m_1 x}}{\left(m_1 - \frac{1}{\varepsilon}\right)} - \frac{d_2 e^{-m_2 x}}{\left(m_2 - \frac{1}{\varepsilon}\right)} \end{bmatrix}$$

Applying the boundary conditions,

$$y(0) = 1, \quad y(1) = 1,$$

we have at $y(0) = 1$,

$$\therefore\ d_1 + d_2 = 1 \Rightarrow d_1 = 1 - d_2$$

at $y(1) = 1$,

$$1 = 1 - d_2 + d_2 e^{-1/\varepsilon} - \frac{1}{\varepsilon(m_1 - m_2)} \begin{bmatrix} (1 - d_2) \begin{Bmatrix} \frac{1}{m_1} - \frac{e^{-m_1}}{m_1} + \frac{e^{-m_2}}{m_2} \\ -\frac{1}{m_2} \end{Bmatrix} + \\ d_2 \begin{Bmatrix} \frac{e^{-1/\varepsilon}}{(m_1 - 1/\varepsilon)} - \frac{e^{-1/\varepsilon}}{(m_2 - 1/\varepsilon)} \\ -\frac{e^{-m_1}}{(m_1 - 1/\varepsilon)} + \frac{e^{-m_2}}{(m_2 - 1/\varepsilon)} \end{Bmatrix} \end{bmatrix}$$

$$d_2 = d_2 e^{-1/\varepsilon} - \frac{1}{\varepsilon(m_1 - m_2)} \left[\begin{array}{l} \left\{ \begin{array}{l} \frac{1}{m_1} - \frac{e^{-m_1}}{m_1} \\ + \frac{e^{-m_2}}{m_2} - \frac{1}{m_2} \end{array} \right\} \\ -d_2 \left\{ \frac{1}{m_1} - \frac{e^{-m_1}}{m_1} + \frac{e^{-m_2}}{m_2} - \frac{1}{m_2} \right\} \\ d_2 \left\{ \begin{array}{l} \frac{e^{-1/\varepsilon}}{(m_1 - 1/\varepsilon)} - \frac{e^{-1/\varepsilon}}{(m_2 - 1/\varepsilon)} \\ - \frac{e^{-m_1}}{(m_1 - 1/\varepsilon)} + \frac{e^{-m_2}}{(m_2 - 1/\varepsilon)} \end{array} \right\} \end{array} \right]$$

$$\therefore d_2 = \frac{\frac{1}{\varepsilon(m_1 - m_2)} \left\{ \frac{1}{m_1} - \frac{e^{-m_1}}{m_1} + \frac{e^{-m_2}}{m_2} - \frac{1}{m_2} \right\}}{\left[\begin{array}{l} e^{-1/\varepsilon} - 1 + \left\{ \frac{1}{\varepsilon(m_1 - m_2)} \right\} \left\{ \begin{array}{l} \frac{1}{m_1} - \frac{e^{-m_1}}{m_1} \\ + \frac{e^{-m_2}}{m_2} - \frac{1}{m_2} \end{array} \right\} \\ - \left\{ \frac{1}{\varepsilon(m_1 - m_2)} \right\} \left\{ \begin{array}{l} \frac{e^{-1/\varepsilon}}{(m_1 - 1/\varepsilon)} - \frac{e^{-1/\varepsilon}}{(m_2 - 1/\varepsilon)} \\ - \frac{e^{-m_1}}{(m_1 - 1/\varepsilon)} + \frac{e^{-m_2}}{(m_2 - 1/\varepsilon)} \end{array} \right\} \end{array} \right]}$$

Hence

$$d_1 = 1 - d_2$$

Table 1.

X	Method by [21]			Present Method		
	Y(x) Approx	Exact solution	Error	Y(x)	Exact solution	Error
0.000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000	0.0000000
0.001	0.6050626	0.6007918	0.0042708	0.59953028	0.6007918	0.0012615
0.010	0.3712403	0.3719724	0.0007321	0.37194399	0.3719724	0.00002841
0.020	0.3749362	0.3756784	0.0007422	0.37567835	0.3756784	0.00000005
0.030	0.3787043	0.3794502	0.0007459	0.37945019	0.3794502	0.00000001
0.040	0.3825104	0.3832599	0.0007495	0.38325991	0.3832599	0.00000001
0.050	0.3863547	0.3871079	0.0007532	0.38710787	0.3871079	0.00000003
0.100	0.4061635	0.4069350	0.0007715	0.40693501	0.4069350	0.00000001
0.200	0.4488802	0.4496879	0.0008077	0.44968785	0.4496879	0.00000005
0.300	0.4960891	0.4969324	0.0008433	0.49693234	0.4969324	0.00000006
0.400	0.5488111	0.5491404	0.0003293	0.54914036	0.5491404	0.00000004
0.500	0.6065292	0.6068334	0.0003042	0.60683340	0.6068334	0.00000000
0.600	0.6703176	0.6705877	0.0002701	0.67058770	0.6705877	0.00000000
0.700	0.7408146	0.7410401	0.0002255	0.74104006	0.7410401	0.00000004
0.800	0.8187257	0.8188942	0.0001685	0.81889419	0.8188942	0.00000001
0.900	0.9048307	0.9049277	0.0000970	0.90492772	0.9049277	0.00000002
1.000	1.0000000	1.0000000	0.0000000	1.00000000	1.0000000	0.00000000

The numerical results of examples 3.1 are given in the Table 1 for $\varepsilon = 10^{-3}$.

The numerical results of examples 3.1 are given in the Table 2 for $\varepsilon = 2^{-4}, 2^{-6}, 2^{-8}, 2^{-10}, 2^{-12}$.

Table 2.

x	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-12}$
0.000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.001	0.9241463	0.9421089	0.8525521	0.5940484	0.3785345
0.010	0.8419333	0.6852981	0.4163505	0.3707146	0.3713549
0.020	0.7646326	0.5325240	0.3754578	0.3744141	0.3750862
0.030	0.6998921	0.4544860	0.3757677	0.3781733	0.3788549
0.040	0.6457588	0.4154867	0.3792655	0.3819703	0.3826616
0.050	0.6005844	0.3968878	0.3830421	0.3858054	0.3865065
0.100	0.4673137	0.3923263	0.4026013	0.4055663	0.4063181
0.200	0.4108459	0.4318915	0.4447709	0.4481764	0.4490400
0.300	0.4326568	0.4765895	0.4913573	0.4952633	0.4962538
0.400	0.4720351	0.5259152	0.5428234	0.5472972	0.5484319
0.500	0.5181434	0.5803460	0.5996802	0.6047980	0.6060962
0.600	0.5693376	0.6404103	0.6624922	0.6683401	0.6698236
0.700	0.6256972	0.7066910	0.7318834	0.7385580	0.7402515
0.800	0.6876555	0.7798316	0.8085428	0.8161533	0.8180845
0.900	0.7557527	0.8605421	0.8932317	0.9019010	0.9041011
1.000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

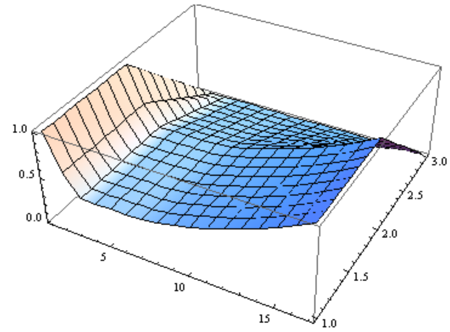


Figure 1. Error analysis for Table 1 at $\varepsilon = 10^{-3}$

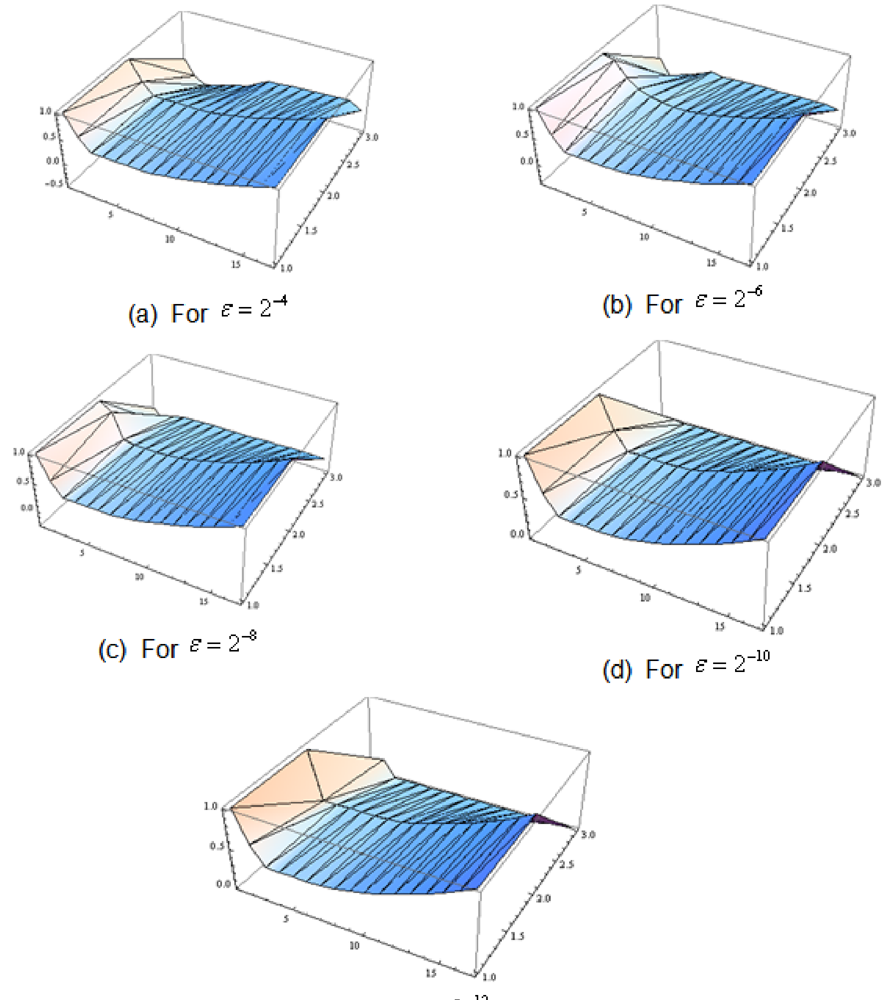
Example 3.2. Consider the following homogeneous SPP from Kevorkian and Cole [6, p.33, Eqs. (2.3.26) and (2.3.27) with $\alpha = 0$],

$$\varepsilon\, y''(x) + y'(x) = 0 \tag{10}$$

$$\text{for } \ 0 \le x \le 1 \ \text{ with } y(0) = 0 \ \text{ and } y(1) = 1 \tag{11}$$

The exact solution is given by

$$y(x) = \frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}. \tag{12}$$



(a) For $\varepsilon = 2^{-4}$ (b) For $\varepsilon = 2^{-6}$ (c) For $\varepsilon = 2^{-8}$ (d) For $\varepsilon = 2^{-10}$ (e) For $\varepsilon = 2^{-12}$

Figure 2. Error Analysis for Table 2 (a),(b),(c),(d),(e)

Solving (6) using the coefficients

$$a(x) = 1, \quad b(x) = g(x) = 0,$$

then λ can easily identified as

$$\lambda(x,t) = -1 + e^{(t-x)/\varepsilon}$$

$$y_0(x) = d_1 e^{-x/\varepsilon} + d_2$$

where d_1 and d_2 are constants to be determined.

By the above iteration formula, we have

$$y_1(x) = d_1 e^{-x/\varepsilon} + d_2$$

Applying the boundary conditions yield,

$$d_1 = \frac{1}{\left(e^{-1/\varepsilon} - 1\right)}, \quad d_2 = \frac{1}{\left(e^{-1/\varepsilon} - 1\right)},$$

Thus,

$$y_1(x) = \frac{e^{-x/\varepsilon}}{\left(e^{-1/\varepsilon} - 1\right)} - \frac{1}{\left(e^{-1/\varepsilon} - 1\right)} = \frac{\left(-1 + e^{-x/\varepsilon}\right)}{\left(e^{-1/\varepsilon} - 1\right)}$$

$$y_1(x) = \frac{\left(1 - e^{-x/\varepsilon}\right)}{\left(1 - e^{-1/\varepsilon}\right)}$$

which is the exact solution.

4. Conclusion

We have described a variational iteration method for solving singularly perturbed boundary value problem. In Table 1 and Table 2 we have shown the maximum error for the problems using variational iteration method. As we decrease the value of ε, to study the behavior of the solution at the boundary layer we need a large number of mesh points in that region. Therefore the uniform mesh is not a good technique for such problems because the number of mesh points in the boundary layer region should be much higher than that from the outer region. Figure 1-Figure 2 has been plotted for values of x belong to $[0,1]$ versus the computed solution obtained at the different value of x for different value of ε.

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