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A Modification of Newton Method with Third-Order Convergence

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Abstract In this paper, we present a new modification of Newton method for solving non-linear equations. Analysis of convergence shows that the new method is cubically convergent. Per iteration the new method requires two evaluations of the function and one evaluation of its first derivative. Thus, the new method is preferable if the computational costs of the first derivative are equal or more than those of the function itself. Finally, we give some numerical examples to demonstrate our method is more efficient than other classical iterative methods.

Keywords: *Newton method, third-order convergence, non-linear equations, iterative method, quadrature formulas*

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1. Introduction

Iterative methods for finding the roots of nonlinear equations $f(x)=0$, are common yet important problem in science and engineering. Analytical methods for solving such equations are difficult or almost non-existent. Solving non linear equations is an important issue in pure and applied mathematics. Researchers have developed various effective methods to find a single root x^* of the non-linear equation $f(x)=0$, where $f: D \subset R \to R$ for an open interval D is a scalar function. We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed-point iteration method.

In the fixed-point iteration method for solving the nonlinear equation $f(x)=0$ the equation is usually rewritten as

$$x = g(x) \tag{1}$$

where

(1) there exist $[a,b]$ such that $g(x) \in [a,b]$, for all $x \in [a,b]$,

(2) there exist $[a,b]$ such that $|g'(x)| \le L < 1$, for all $x \in [a,b]$,

Considering the following iteration scheme:

$$x_{n+1} = g(x_n) \tag{2}$$

and starting with a suitable initial approximation x_k, we build up a sequence of approximations, say $\{x_n\}$, for the solution of the nonlinear equation, say χ .The scheme will converge to the root χ , provide that the initial approximation x_k is chosen in the interval $[a,b]$, g has a continuous derivative on (a,b), $|g'(x)| < 1$, for all $x \in [a,b]$, $a \le g(x) \le b$ for all $x \in [a,b]$.

The order of convergence for the sequence of approximations derived from an iteration method is defined in the literature, as follows.

Definition 1. Let $\{x_n\}$ converge to χ . If there exist an integer constant p and real positive constant $\mathbb{C}$ such that

$$\lim_{n\to\infty}\left|\frac{x_{n+1}-\chi}{(x_n-\chi)^p}\right| = \mathbb{C} \tag{3}$$

then p is called the order and $\mathbb{C}$ the constant of convergence.

To determine the order of convergence of the sequence $\{x_n\}$, let us consider the Taylor expansion of $g(x_n)$

$$\begin{aligned} g(x_n) &= g(x) + \frac{g'(x)}{1!}(x_n-x) + \frac{g''(x)}{2!}(x_n-x)^2 \\ &+ \frac{g'''(x)}{3!}(x_n-x)^3 + \ldots + \frac{g^l(x)}{l!}(x_n-x)^l + \ldots \end{aligned} \tag{4}$$

Using (1) and (2) in (4) we have

$$\begin{aligned} x_{n+1} - x &= g'(x)(x_n-x) + \frac{g''(x)}{2!}(x_n-x)^2 \\ &+ \frac{g'''(x)}{3!}(x_n-x)^3 + \ldots + \frac{g^l(x)}{l!}(x_n-x)^l + \ldots \end{aligned} \tag{5}$$

and we can state the following result [1]

2. Preliminaries

We firstly introduce a lemma before proposing our Newton-type iterative method for solving non-linear equations.

Lemma 1. Assume that $f \in C^p(D)$ and there is a simple root χ of the nonlinear equation $f(x)=0$, where $f: D \subset R \to R$ is a scalar function on an open interval D. If there is an iterative method $\Phi(x)$ with the order of convergence p (p is a integer) which produces the sequence $\{x_n\}$, then we have

$$x_{n+1} - \chi = \Lambda(x_n - \chi)^p + O((x_n - \chi)^{p+1}) \quad (6)$$
$$x_n \subset \Pi(\chi)$$

where constant $\Lambda \neq 0$ and $\Pi(\chi)$ is a neighborhood of χ.

Proof. Using the Taylor expansion and the definition of the convergence order of the iterative method, we know

$$x_{n+1} = \Phi(x_n) = \Phi(\chi) + \frac{\Phi'(\chi)}{1!}(x_n - \chi)$$
$$+ \frac{\Phi''(\chi)}{2!}(x_n - \chi)^2 + \frac{\Phi'''(\chi)}{3!}(x_n - \chi)^3$$
$$+ \ldots + \frac{\Phi^p(\chi)}{p!}(x_n - \chi)^l + O((x_n - \chi)^{p+1})$$
$$= \chi + \frac{\Phi^p(\chi)}{p!}(x_n - \chi)^l + O((x_n - \chi)^{p+1})$$
$$= \chi + \Lambda(x_n - \chi)^l + O((x_n - \chi)^{p+1})$$

Then, we get (6).This ends the proof.

3. Iterative Methods

Let $f: D \subset R \to R$, be $r-$ times Fréchet differentiable function on an open interval $D \subset R$ and χ be a real zero of the non-linear equation $f(x)=0$. For any $x, x_k \in D \subset R$ we may write the Taylor's expansion for f as follows:

$$f(x) = f(x_n) + \frac{f'(x_n)}{1!}(x - x_n) + \frac{f''(x_n)}{2!}(x - x_n)^2$$
$$+ \frac{f'''(x_n)}{3!}(x - x_n)^3 + \ldots + \frac{f^{l-1}(x_n)}{l-1!}(x - x_n)^{l-1}$$
$$+ \int_0^1 \frac{(1-t)^{l-1}}{(l-1)!} f^r(x_n + t*(x - x_n))(x - x_n)^l dt$$

For $l = 1$, we have

$$f(x) = f(x_n) + \int_0^1 f'(x_n + t*(x - x_n))(x - x_n)dt \quad (7)$$

Approximating the integral in (7), we have

$$\int_0^1 f'(x_n + t*(x - x_n))(x - x_n)dt = f'(x_n)(x - x_n) \quad (8)$$

By using (7) and (8) and $f(x)=0$, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This allows us to suggest the following one-step iterative method for solving the non-linear equations.

Algorithm 1. For a given x_0, compute the approximate solution x_{n+1}, by the iterative scheme where $f'(x_n)$ is the derivate at point x_n. **Algorithm 1.** is known as Newton's method for the non-linear equations $f(x)=0$ and has quadratic convergence.

If we approximate the integral in (7) by using the Closed-Open quadrature formula [3], then

$$\int_0^1 f'(x_n + t*(x_n - x))(x_n - x)dt$$
$$\cong \frac{1}{4}\left[f'(x_n) + 3f'\left(\frac{x_n + 2x}{3}\right)\right](x_n - x) \quad (9)$$

By using (7) and (9) and $f(x)=0$, we have

$$x_{n+1} = x_n - \frac{4f(x_n)}{\left[f'(x_n) + 3f'\left(\frac{x_n + 2x}{3}\right)\right]}$$
$$x_{n+1} = x_n - 4\left[f'(x_n) + 3f'\left(\frac{x_n + 2x}{3}\right)\right]^{-1} f(x_n)$$

Using this relation, we can suggest the following two-step iterative method for solving the nonlinear system of equation (1) as:

Algorithm 2. For a given x_0, compute the approximate solution x_{n+1},

Predictor step:

$$\rho_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (10)$$

Corrector step:

$$x_{n+1} = x_n - 4\left[f'(x_n) + 3f'\left(\frac{x_n + 2x}{3}\right)\right]^{-1} f(x_n) \quad (11)$$

Algorithm 2 is another iterative method for solving the nonlinear equation (1).

These modifications of Newton method are very important and interesting because per iteration they require one evaluation of the function f and two of the first derivative f', not requiring the second derivative f'', but they can converge cubically. Thus, the research of the third-order methods with free second derivatives becomes very active now. The efficiency of the new method is demonstrated by numerical examples.

4. Convergence Analysis

In this section, we consider the convergence criteria of the **Algorithm 2** using the Taylor series technique as (4)

Theorem 1. Let $f: D \subset R \to R$, be $r-$ times Fréchet differentiable function on an open interval $D \subset R$ containing the root χ of $f(x)=0$. The iterative method defined by **Algorithm 2** has cubic convergence and satisfies the error equation

$$\left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right]e_{n+1} = [f''(x_n)(f'(x_n))^{-1} f''(x_n)]e^3_n + O(\|e^4_n\|) \qquad (12)$$

Proof. The technique is given by

$$\rho_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - 4\left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right]^{-1} f(x_n)$$

Defining $e_n = x_n - \chi$ and from equation (11), we have

$$e_{n+1} - e_n = -4\left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right]^{-1} f(x_n) \qquad (13)$$

from which we have

$$e_{n+1}\left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right] = \left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right]e_n - 4f(x_n) \qquad (14)$$

from equation (7') with $x = \chi$,we have

$$f(\chi) = f(x_n) + \frac{f'(x_n)}{1!}(\chi - x_n) + \frac{f''(x_n)}{2!}(\chi - x_n)^2 + \frac{f'''(x_n)}{3!}(\chi - x_n)^3 + O(\|\chi - x_n\|^4) \qquad (15)$$

If χ be the zero of the non-linear equation $f(x) = 0$

$$f(\chi) = f(x_n) + \frac{f'(x_n)}{1!}(e_n) + \frac{f''(x_n)}{2!}(e_n)^2 + \frac{f'''(x_n)}{3!}(e_n)^3 + O(\|e^4_n\|) \qquad (16)$$

Pre-multiplying equation (16) by $\frac{1}{f'(x_n)} = [f'(x_n)]^{-1}$, we have

$$f(x_n)[f'(x_n)]^{-1} = (e_n) + \frac{f''(x_n)}{2!}[f'(x_n)]^{-1}(e_n)^2 + \frac{f'''(x_n)}{3!}[f'(x_n)]^{-1}(e_n)^3 + O(\|e^4_n\|) \qquad (17)$$

Now, applying Taylor's expansion for $f'\left(\frac{x_n+2\rho_n}{3}\right)$ at point x_n , we have

$$f'\left(\frac{x_n+2\rho_n}{3}\right) = f'(x_n) - \frac{2}{3}f''(x_n)(f(x_n)[f'(x_n)]^{-1}) + \frac{2}{9}f'''(x_n)(f(x_n)[f'(x_n)]^{-1})^2 + \ldots \qquad (18)$$

From equation (17) and (18), we obtain

$$f'\left(\frac{x_n+2\rho_n}{3}\right) = f'(x_n) - \frac{2}{3}f''(x_n)\left((e_n) + \frac{f''(x_n)}{2!}[f'(x_n)]^{-1}(e_n)^2 + \frac{f'''(x_n)}{3!}[f'(x_n)]^{-1}(e_n)^3 + O(\|e^4_n\|)\right) + \frac{2}{9}f'''(x_n)\left((e_n) + \frac{f''(x_n)}{2!}[f'(x_n)]^{-1}(e_n)^2 + \frac{f'''(x_n)}{3!}[f'(x_n)]^{-1}(e_n)^3 + O(\|e^4_n\|)\right)^2 + \ldots$$

Thus

$$f'\left(\frac{x_n+2\rho_n}{3}\right) = f'(x_n) - \frac{2}{3}f''(x_n)(e_n) + \frac{1}{3}f''(x_n)[f'(x_n)]^{-1}f''(x_n)(e_n)^2 + \frac{2}{9}f'''(x_n)(e_n)^3 + O(\|e^3_n\|) \qquad (19)$$

Now, from equations (14), (16) and (19)

$$e_{n+1}\left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right] = \left[f'(x_n) + 3\left(f'(x_n) - \frac{2}{3}f''(x_n)(e_n) + \frac{1}{3}f''(x_n)[f'(x_n)]^{-1}f''(x_n)(e_n)^2 + \frac{2}{9}f'''(x_n)(e_n)^3 + O(\|e^4_n\|)\right)\right]e_n - 4\left(f(x_n) + \frac{f'(x_n)}{1!}(e_n) + \frac{f''(x_n)}{2!}(e_n)^2 + \frac{f'''(x_n)}{3!}(e_n)^3 + O(\|e^4_n\|)\right)$$

Thus, from above equation, we have

$$\left[f'(x_n)+3f'\left(\frac{x_n+2x}{3}\right)\right]e_{n+1} = [f''(x_n)(f'(x_n))^{-1} f''(x_n)]e^3_n + O(\|e^4_n\|) \qquad (20)$$

Equation (20) shows that the **Algorithm 2** has cubic convergence.

It is easy to know that per iteration the number of function evaluation of iterative method defined by (11) is three. We consider the definition of efficiency index [11] as $p^{\frac{1}{w}}$ where p is the order of the method and w the number of function evaluations per iteration required by the method. We have that similar to the results obtained in [4] the method defined by (11) has the efficiency index equal to $3^{\frac{1}{3}} = 1.442$ which is better than the one of Newton method $2^{\frac{1}{2}} = 1.414$ .Thus, the method defined by (11) is preferable if the computational costs of the first derivative are equal or more than those of the function itself.

5. Numerical Results

Now we employ the new method (10-11) obtained in this paper to solve some non-linear equations Also compared are the Newton method (NM), the method of Soheili (SM), and the method, introduced in this present paper (**Algorithm 2**). Table 1 presents iteration number comparison of **Algorithm 2** with (NM) and (SM), in given precision. The computational results show that the cubically convergent methods, especially the new method (10-11), can compete with Newton method. Also the new method (10-11) seems to be superior in some cases where Newton method fails in convergence, as f_5: We used $\varepsilon = 10^{-14}$ The following stopping criteria is used for computer programs

$$\|x_{n+1} - x_n\|_\infty < \varepsilon, \| f(x_n)\|_\infty < \varepsilon,$$

$$f_1(x) = e^{x^2+7x-29} - 1,\ f_2(x) = \sin^2(x) - x^2 + 1,$$

$$f_3(x) = 11x^{11} - 1,\ f_4(x) = \sin\left(\frac{1}{x}\right) - x,\ f_5(x) = \tan^{-1}(x).$$

The numerical computations listed in Table 1 are performed using Matlab© programs.

Table 1. Examples and comparison between other methods

	IT	NFE	x_n
f_1 x_0=4			
NM	20	53	3.000000000000000
SM	13	84	3.000000000000000
Algorithm 2	6	18	3.000000000000000
f_2 x_0=-1			
NM	7	12	-1.40449164821534
SM	5	21	-1.40449164821534
Algorithm 2	3	15	-1.40449164821534
F_3 x_0=0.7			
NM	8	12	0.804133097503664
SM	6	18	0.804133097503664
Algorithm 2	4	12	0.804133097503664
F_4 x_0=4			
NM	6	12	0.257530285439861
SM	4	18	0.257530285439861
Algorithm 2	3	12	0.257530285439861
F_5 x_0=3			
NM			Divergence
SM			Divergence
Algorithm 2	4	12	1.5598640272247e-14

6. Conclusions

We have obtained a new modification of Newton method for solving non-linear equations. From Theorem 1, we prove that the order of convergence of the new method is three. The numerical results in the Table 1 show that the new method is very effective and provide highly accurate results in a less number of iterations as compared with Newton method (NM) and the method of Soheili (SM) (Soheili et al., 2008). Analysis of efficiency shows that the new method is preferable to the well-known Newton method, especially in the case where the computational costs of the first derivative are equal or more than those of the function itself.

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