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Initial Value Technique for Singular Perturbation Boundary Value Problems

Hradyesh Kumar Mishra*

Department of Mathematics, Jaypee University of Engineering & Technology, Raghogarh, Guna(M.P) India
*Corresponding author: hk.mishra@jiet.ac.in, hkm1975@yahoo.co.in

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Abstract In this paper, an initial-value approach is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at left end point. The given singularly perturbed two-point boundary value problems replaced by first-order problem. This first-order problem is solved by classical Runge- Kutta method. Numerical examples have been solved to demonstrate the applicability of the method. Graphs are shown in support of results.

Keywords: *singular perturbation, ordinary differential equation, boundary layer, initial-value technique*

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1. Introduction

The numerical solution of singular perturbation problems is currently a field in which active research is going on. Singular perturbation problems are of common occurrence in fluid mechanics and other branches of applied mathematics. For detailed analytical discussion on singular perturbation problems, one can refer to Bender and Orsazag [1], Kevorkian and Cole [4], Nayfeh [12,13], O's Mally [7]. More recently, a non asymptotic method also called the boundary value technique has been introduced by Robert [15] to solve certain classes of singular perturbation problems. He also discussed the analytical and approximate solutions of several numerical examples. The concept of replacing singularly perturbed two-point boundary value problem by initial-value technique is presented by many authors; see for examples (cf. [2,3,5,6,8,9,10,11,16,17]).

In this paper, we present an approximate method for the numerical solution of a class of singularly perturbed two-point boundary value problems with the boundary layer at left end point. The original problem is replaced by an asymptotically equivalent first-order problem which is solved by using boundary condition. Numerical examples have been solved to demonstrate the applicability of the method.

2. Description of the Method

We consider a class of singular perturbation problems of the form

$$\varepsilon\, y''(x) + [a(x)y(x)]' = k(x); \quad 0 \le x \le 1 \tag{1}$$

with

$$y(0) = \alpha \quad and \quad y(1) = \beta \tag{2}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$); α, β are given constants; a(x) and k(x) are assumed to be sufficiently continuously differentiable function in [0, 1]. Furthermore, we assume that $a(x) \ge M > 0$ through out the interval [0, 1], where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of x=0. Now, we replace the original second-order problem by an asymptotically equivalent first-order problem as follows: By integrating Eq. (1), we obtain

$$\varepsilon\, y'(x) + a(x)y(x) = h(x) + C \tag{3}$$

where $h(x) = \int k(x)dx$

and C is a constant to be determined. Setting $\varepsilon = 0$ in (3) and using boundary condition, $y(1) = \beta$; i.e.

$$y(1) = \frac{1}{a(1)}[\,h(1) + C\,] = \beta$$

Or

$$C = a(1)\,\beta - h(1). \tag{4}$$

This value of C ensures that the solution of the reduced problem of Eq. (1) satisfies the reduced equation of the Eq. (3).

Now, we adjoin the condition $y(0) = \alpha$ to Eq. (3) to obtain a first-order problem

$$\varepsilon\, y'(x) + a(x)y(x) = h(x) + C \tag{5}$$

$$y(0) = \alpha \tag{6}$$

where the constant C is given by Eq. (4). Thus, we have replaced the original second-order problem (1)-(2) with

the asymptotically equivalent first-order problem (5)-(6). Now we solve Eq. (5) using Runge Kutta method. In fact, any standard analytically or numerical method can be used.

3. Numerical Examples

To demonstrate the applicability of the method, we will discuss two examples; a homogeneous singular perturbation problem (SPP), a non-homogeneous SPP. These examples have been chosen because either analytical or approximate solutions are available for comparison.

Example 3.1. Consider the following homogeneous SPP from Kevorkian and Cole [4], p.33, Eqs. (2.3.26) and (2.3.27) with $\alpha = 0$],

$$\varepsilon y''(x) + y'(x) = 0 \tag{7}$$

$$\text{for } 0 \le x \le 1 \text{ with } y(0) = 0 \text{ and } y(1) = 1. \tag{8}$$

The exact solution is given by

$$y(x) = \frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}. \tag{9}$$

By integrating Eq. (7), we get

$$\varepsilon y'(x) + y(x) = C. \tag{10}$$

The constant C is determined by using Eq. (4), as

$$C = y(1) = 1. \tag{11}$$

Therefore,

$$\varepsilon y'(x) + y(x) = 1 \ , y(0) = 0 \tag{12}$$

On solving Eq. (12), we get approximate value of y(x).

The computational results are presented in Table 1 and Table 2 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 1. Numerical results of Example 3.1 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	y(x)	Exact solution
0.000	0.0000000	0.0000000
0.001	0.6250000	0.63212056
0.010	0.9999450	0.99995460
0.020	1.0000000	1.0000000
0.030	1.0000000	1.0000000
0.040	1.0000000	1.0000000
0.050	1.0000000	1.0000000
0.100	1.0000000	1.0000000
0.200	1.0000000	1.0000000
0.300	1.0000000	1.0000000
0.400	1.0000000	1.0000000
0.500	1.0000000	1.0000000
0.600	1.0000000	1.0000000
0.700	1.0000000	1.0000000
0.800	1.0000000	1.0000000
0.900	1.0000000	1.0000000
1.000	1.0000000	1.0000000

Table 2. Numerical results of Example 3.1 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	y(x)	Exact solution
0.0000	0.0000000	0.0000000
0.0001	0.6250000	0.6321205
0.0010	0.9999450	0.9999546
0.0020	1.0000000	1.0000000
0.0030	1.0000000	1.0000000
0.0040	1.0000000	1.0000000
0.0050	1.0000000	1.0000000
0.1000	1.0000000	1.0000000
0.2000	1.0000000	1.0000000
0.3000	1.0000000	1.0000000
0.4000	1.0000000	1.0000000
0.5000	1.0000000	1.0000000
0.6000	1.0000000	1.0000000
0.7000	1.0000000	1.0000000
0.8000	1.0000000	1.0000000
0.9000	1.0000000	1.0000000
1.0000	1.0000000	1.0000000

Example 3.2. Consider the following non-homogeneous SPP from fluid dynamics for fluid of small viscosity, Reinhardt [[14], Example2],

$$\varepsilon y''(x) + y'(x) = 1 + 2x \tag{13}$$

$$\text{for } 0 \le x \le 1 \text{ with } y(0) = 0 \textit{ and } y(1) = 1 \tag{14}$$

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + (2\varepsilon)\frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}. \tag{15}$$

By integrating Eq. (13), we get

$$\varepsilon y'(x) + y(x) = x + x^2 + C. \tag{16}$$

The constant C is determined by using Eq. (4), as

$$C = y(1) - 1 - 1^2 = -1. \tag{17}$$

Therefore,

$$\begin{aligned} &\varepsilon y'(x) + y(x) = x + x^2 - 1 \\ &y(0) = 0 \end{aligned} \tag{18}$$

On solving Eq. (18), we get approximate value of y(x).

The computational results are presented in Table 3 and Table 4 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

4. General Problems

Here, we consider a more general class of problems of the form

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y(x) = k(x) \tag{19}$$

$$\text{for } 0 \le x \le 1 \text{ with } y(0) = \alpha \text{ and } y(1) = \beta, \tag{20}$$

where ε is a small positive parameter; α, β are given constants: $a(x), b(x)$, $k(x)$ are assumed to be sufficiently continuously differentiable functions in [0, 1]; and

$a(x) \geq M > 0$ throughout the interval [0, 1], where M is some positive constant.

Table 3. Numerical results of Example 3.2 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	y(x)	Exact solution
0.000	0.0000000	0.0000000
0.001	-0.6246247	-0.6298573
0.010	-0.9908629	-0.9878747
0.020	-0.9806380	-0.9776400
0.030	-0.9701580	-0.9671600
0.040	-0.9583989	-0.9564800
0.050	-0.9474990	-0.9456000
0.100	-0.8911980	-0.8882000
0.200	-0.7613977	-0.7584000
0.300	-0.6099995	-0.6086000
0.400	-0.4400019	-0.4388000
0.500	-0.2500047	-0.2490000
0.600	-0.0400080	-0.0391999
0.700	0.1899880	0.1906001
0.800	0.4399836	0.4404000
0.900	0.7099786	0.7102001
1.000	1.0000000	1.0000000

Table 4. Numerical results of Example 3.2 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	y(x)	Exact solution
0.0000	0.0000000	0.0000000
0.0001	-0.6249625	-0.6318942
0.0010	-0.9989784	-0.9987538
0.0020	-0.9979960	-0.9977964
0.0030	-0.9970916	-0.9967916
0.0040	-0.9960847	-0.9957848
0.0050	-0.9949750	-0.9947760
0.1000	-0.8901186	-0.8898200
0.2000	-0.7601361	-0.7598400
0.3000	-0.6100047	-0.6098600
0.4000	-0.4400291	-0.4398800
0.5000	-0.2500587	-0.2499000
0.6000	-0.0400281	-0.0399199
0.7000	0.1897691	0.1900600
0.8000	0.4397930	0.4400400
0.9000	0.7098235	0.7100199
1.0000	1.0000000	1.0000000

It is clear that there will be difficulty in applying the integration process due to the presence of the term $b(x)y(x)$. To overcome this difficulty, we first modify Eq. (19) and then apply our method.

Let y_0 be the solution of the reduced problem of (19)-(20); that is,

$$[a(x)y_0(x)]' + b(x)y_0(x) = k(x) \tag{21}$$

with

$$y_0(1) = \beta. \tag{22}$$

Now set up the approximate equation to the given equation (19) as

$$\varepsilon\, y''(x) + [a(x)y(x)]' + b(x)y_0(x) = k(x), \tag{23}$$

where we have simply replaced the $y(x)$ -term by $y_0(x)$, the solution of the reduced problem (21)-(22). Then, we rewrite Eq. (23) in the form of Eq. (1) as

$$\varepsilon\, y''(x) + [a(x)y(x)]' = K(x), \tag{24}$$

where $K(x) = k(x) - b(x)y_0(x)$.

Finally, we apply our method, from Eq. (1) to Eq. (6), to the modified problem (24).

Example 4.1. Consider the following singular perturbation problem from Bender and Orszag [[1], p. 480, problem 9.17 with $\alpha = 0$]:

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \tag{25}$$

$$\text{for } 0 \leq x \leq 1 \text{ with } y(0) = 1 \text{ and } y(1) = 1. \tag{26}$$

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}, \tag{27}$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon)$.

From the previous step, we get an approximate equation to (25) as

$$\varepsilon y''(x) + y'(x) - y_0(x) = 0, \tag{28}$$

Where $y_0(x) = e^{x-1}$ is the solution of the reduced problem of (25)-(26);
i.e.

$$y_0'(x) - y_0(x) = 0 \tag{29}$$

with

$$y_0(1) = 1. \tag{30}$$

Then rewrite the Eq. (28) in the form of (1) as

$$\varepsilon y''(x) + y'(x) = e^{x-1}. \tag{31}$$

By integrating Eq. (31), we get

$$\varepsilon y''(x) + y'(x) = e^{x-1} + C. \tag{32}$$

The constant C is determined by using Eq. (4), as

$$C = y(1) - e^{1-1} = 0. \tag{33}$$

Therefore,

$$\begin{aligned} &\varepsilon\, y'(x) + y(x) = e^{x-1} \\ &y(0) = 1. \end{aligned} \tag{34}$$

On solving Eq. (34), we get approximate value of y(x).

The computational results are presented in Tables 5 and 6 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 5. Numerical results of Example 4.1 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	y(x)	Exact solution
0.000	1.0000000	1.0000000
0.001	0.6050626	0.6007918
0.010	0.3712403	0.3719724
0.020	0.3749362	0.3756784
0.030	0.3787043	0.3794502
0.040	0.3825104	0.3832599
0.050	0.3863547	0.3871079
0.100	0.4061635	0.4069350
0.200	0.4488802	0.4496879
0.300	0.4960891	0.4969324
0.400	0.5488111	0.5491404
0.500	0.6065292	0.6068334
0.600	0.6703176	0.6705877
0.700	0.7408146	0.7410401
0.800	0.8187257	0.8188942
0.900	0.9048307	0.9049277
1.000	1.0000000	1.0000000

Table 6. Numerical results of Example 4.1 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	y(x)	Exact solution
0.0000	1.0000000	1.0000000
0.0001	0.6049384	0.6004604
0.0010	0.3682455	0.3683130
0.0020	0.3685791	0.3686527
0.0030	0.3689478	0.3690215
0.0040	0.3693170	0.3693907
0.0050	0.3696865	0.3697602
0.1000	0.4065295	0.4066062
0.2000	0.4492853	0.4493649
0.3000	0.4965838	0.4966200
0.4000	0.5488028	0.5488446
0.5000	0.6065128	0.6065609
0.6000	0.6703115	0.6703469
0.7000	0.7407470	0.7408404
0.8000	0.8186656	0.8187471
0.9000	0.9047804	0.9048464
1.0000	1.0000000	1.0000000

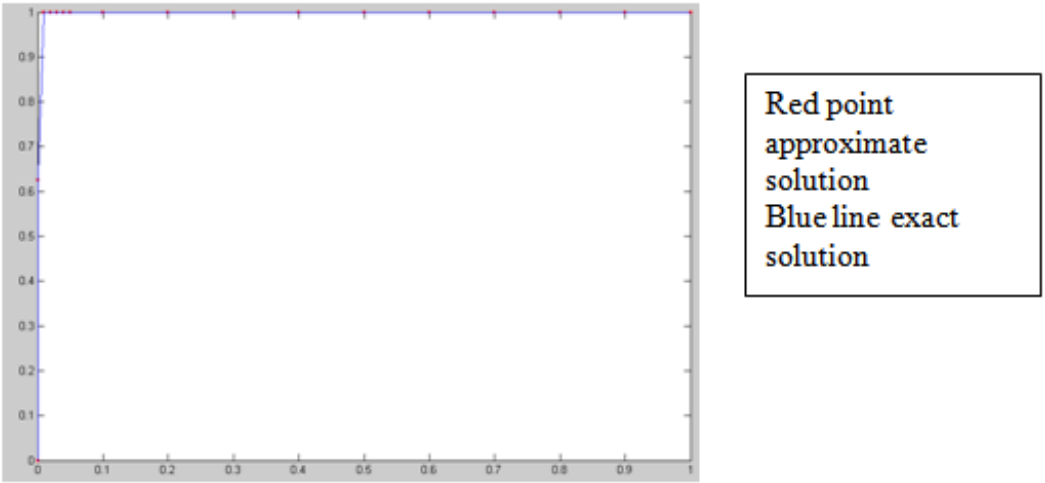


Figure 1. for Table 1

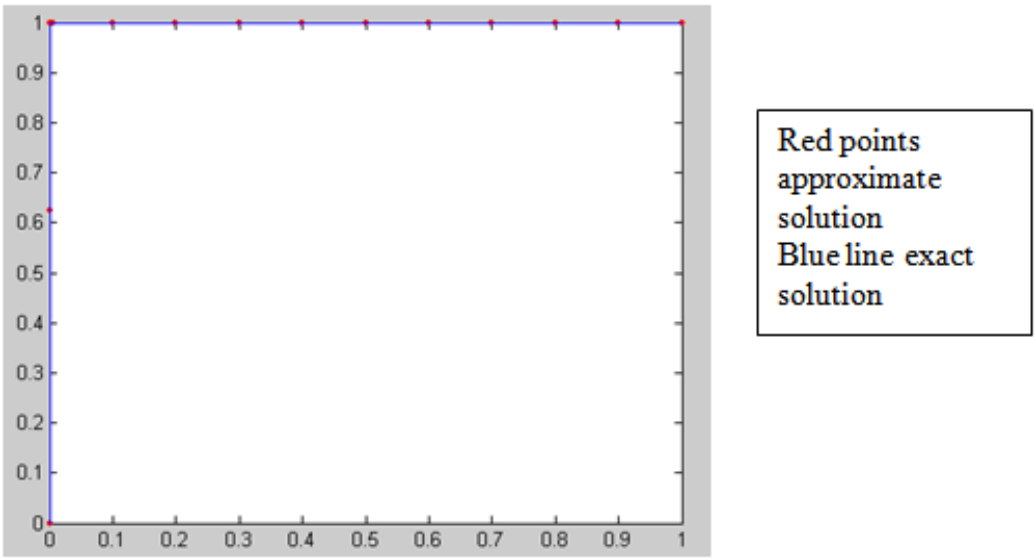


Figure 2. for Table 2

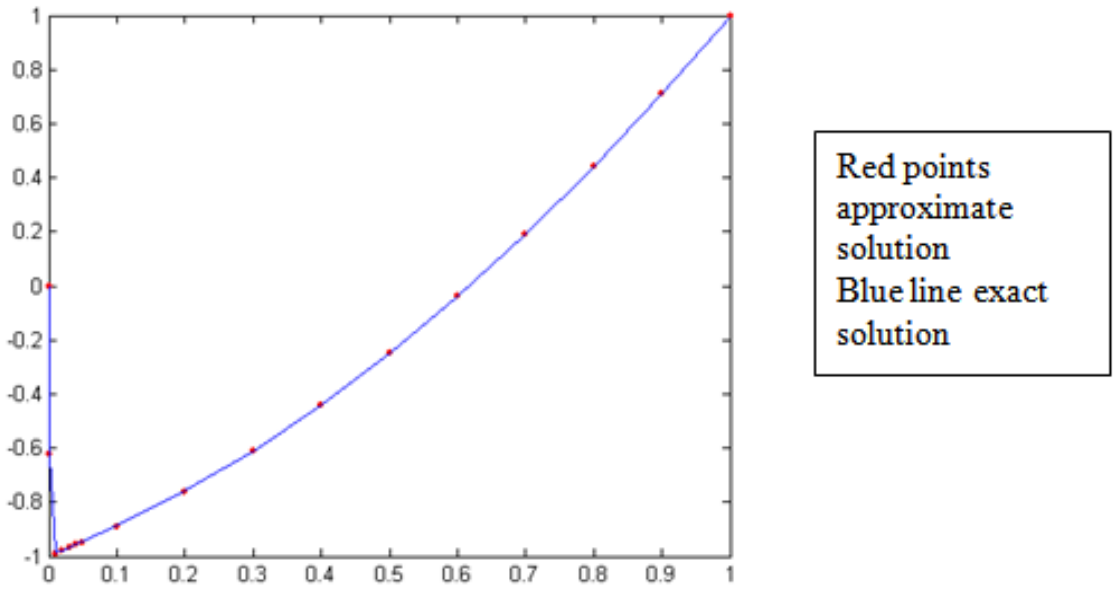


Figure 3. for Table 3

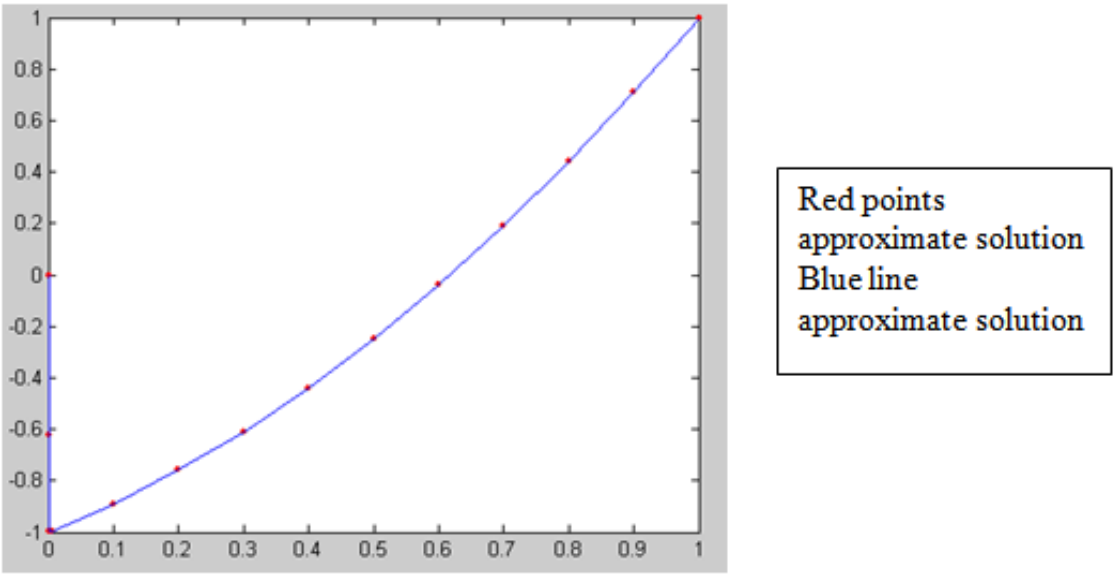
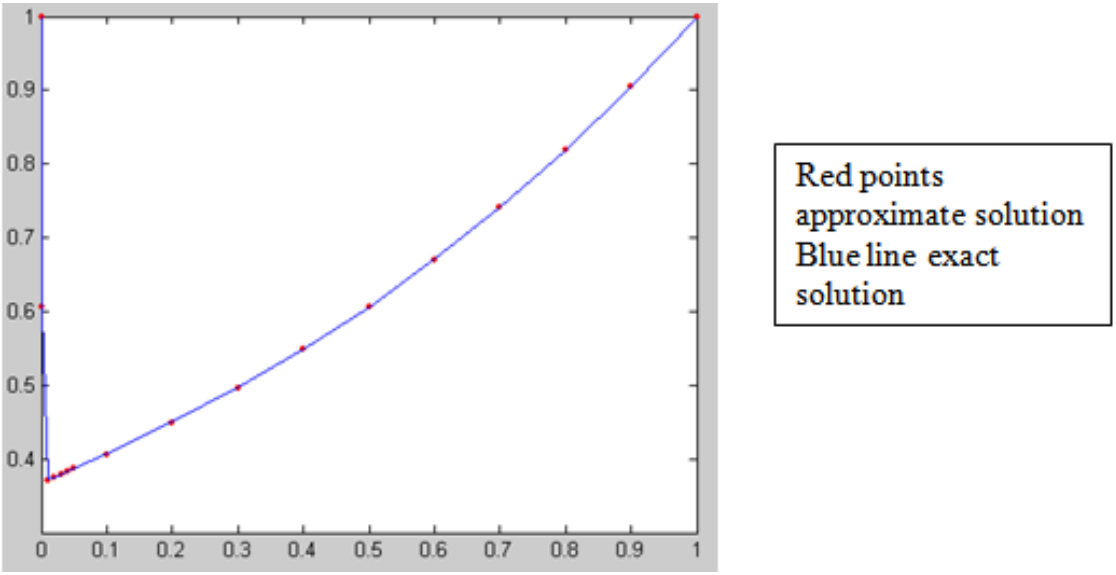


Figure 4. for Table 4



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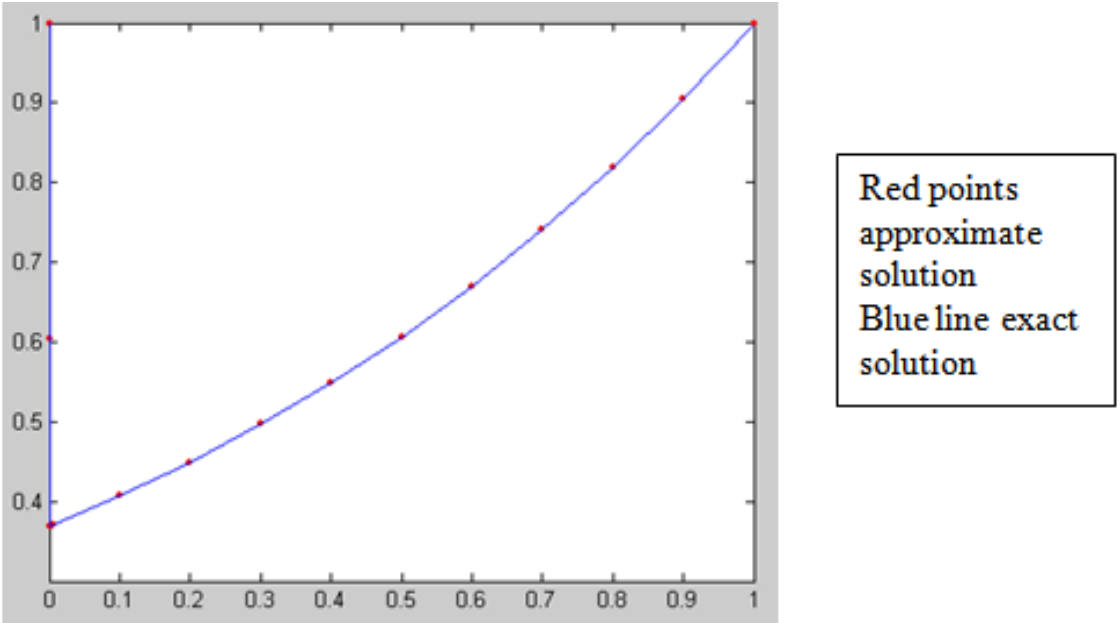

Figure 6. for Table 6

5. Discussion and Conclusions

We have presented an initial-value approach for solving singularly perturbed two-point boundary value problems. In general, the numerical solution of a boundary value problem will be more difficult matter than the numerical solution of the corresponding initial-value problems. Hence, we always prefer to convert the second-order problem into first-order problems. Here, the given singularly perturbed two-point boundary value problem is replaced by first-order problem. This first-order problem is solved by classical Runge-Kutta method. We have tabulated the numerical results obtained by present method as well as the exact solution. Graphs are plotted for the tabulated results.

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