

Application of VIM Method for Nonlinear Porous Media Equations

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Abstract In this paper, we applied He's variational iteration method (VIM) to solve nonlinear porous media equations. The main advantage of this method is the flexibility to give approximate solutions to nonlinear problems without linearization or discretization. The results show that this method is simple and effective.

Keywords: nonlinear heat equation, degenerate parabolic differential equation, porous media equation, variational iteration method

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1. Introduction

The porous media equations arise in the nonlinear problems of heat and mass transfer, combination theory and in biological systems. The nonlinear heat equation called the porous media equation is given in [1] as;

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(u^m \frac{\partial u}{\partial x} \right), \quad (1)$$

where m is a rational number Eq. (1) is called a degenerate parabolic differential equation because the diffusion coefficient $D(u) = u^m$ does not satisfy the condition for classical diffusion equations, $D(u) > 0$.

A discussion of the formulation of these models is given in [1,2,3]. Equations of the form (1) admit traveling-wave solutions $u = u(Mx + Nt)$ where M and N are constants [3]. The variational iteration method is a new method for solving nonlinear problems and was introduced by a Chinese mathematician, He [4,5,6,7]. In [7] He, modified the general Lagrange multiplier method [8] and constructed an iterative sequence of functions which converges to the exact solution. The variational iteration method is used to address several problems, most of them well-known differential equations, which arise in several fields of physics and engineering. A criterion for convergence of the variational iteration method is introduced in [9] by means of the Banach fixed point theorem. Recently, many researchers have applied VIM method for different problems; see [10,11] and reference there in.

In this paper we use the variational iteration method (VIM) for solving porous media equations in order to find the approximate solution.

The article is organized as follows. In section 2, we describe the basic formulation of the variational iteration

method required for our subsequent development. Section 3 is devoted to the solution of Eq. (1) by using the variational iteration method. In section 4, we report our numerical findings and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Variational Iteration Method

The variational iteration method is used in [12] for finding the minimum of a functional over the specified domain. Using this technique the solution of the problem is provided in a closed form while the mesh point techniques provide the approximation at mesh points only. In [13] the variational iteration method is employed to solve time-dependent reaction – diffusion equation which has special importance in engineering and sciences and constitutes a good model for many systems in various fields. This method is employed in [14] to solve the Klein-Gordon equation which is the relativistic version of the Schrödinger equation. which is used to describe spinless particles. Application of He's variational iteration techniques to an inverse parabolic problem is described in [15]. The variational iteration method and Adomian decomposition method are used and compared in [16], for solving a biological population equation. The main advantage of the two methods over the mesh points methods [16] is the fact that they do not require discretization of the variables. Furthermore, the variational iteration method overcome the difficulty arising in calculating the Adomian polynomials which is an important advantage over the Adomian decomposition method [17,18].

The variational iteration method is applied in [19] to find solution of an inverse problem for the semi-linear parabolic partial differential equation. The modified variational iteration method is used in [20] to solve a

system of differential equations. The parabolic integro-differential equations arising in heat conduction in materials with memory is studied in [21] and the variational iteration method is used to find the solution of the problem the interested reader can see [22-29] for some other applications of the method.

Consider the following general nonlinear problem:

$$L(u(t)) + N(u(t)) = g(t), \tag{2}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytical function. The variational iteration method constructs an iterative sequence called the correction functional as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda (L(u_n(s)) + N(\tilde{u}_n(s)) - g(s)) ds \tag{3}$$

where λ is the general Lagrange multiplier [8] which can be identified optimally via the variational theory [8,22], $\tilde{u}_n(s)$ is considered as the restricted variation [7], i.e. $\delta \tilde{u}_n = 0$, and the index n denotes the n th iteration.

3. The Approximate Solutions of the Porous Media Equation

Consider the porous media Eq. (1). Applying the variational iteration method Eq. (3) to Eq. (1) we have the following iteration sequence:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \left[\frac{\partial u_n}{\partial s}(x, s) - \frac{\partial}{\partial x} (u_n^m(x, s) \frac{\partial u_n}{\partial x}(x, s)) \right] ds$$

According to the variational iteration method, we have correction functional:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \left[\frac{\partial u_n}{\partial s}(x, s) - \frac{\partial}{\partial x} (\tilde{u}_n^m(x, s) \frac{\partial \tilde{u}_n}{\partial x}(x, s)) \right] ds$$

Noticing that $\delta u_n(0) = 0$, we get :

$$\delta u_{n+1} = \delta u_n + \lambda \delta u_n \Big|_{s=t} - \int_0^t \lambda' \delta u_n ds = 0$$

Then, its stationary conditions can be obtained as:

$$1 + \lambda(s) \Big|_{s=t} = 0 \quad \lambda'(s) \Big|_{s=t} = 0$$

The general Lagrange multiplier can therefore be readily identified:

$$\lambda = -1,$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[\frac{\partial u_n}{\partial s}(x, s) - \frac{\partial}{\partial x} (u_n^m(x, s) \frac{\partial u_n}{\partial x}(x, s)) \right] ds$$

4. Illustrative Examples

In this section we applied the method presented in this paper to two examples to show the efficiency of the approach.

4.1. Example 1

Let us take $m = -1$ in Eq. (1). We obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(\frac{1}{u} \frac{\partial u}{\partial x} \right). \tag{4}$$

The exact solution is $u(x, t) = \frac{1}{x-t}$.

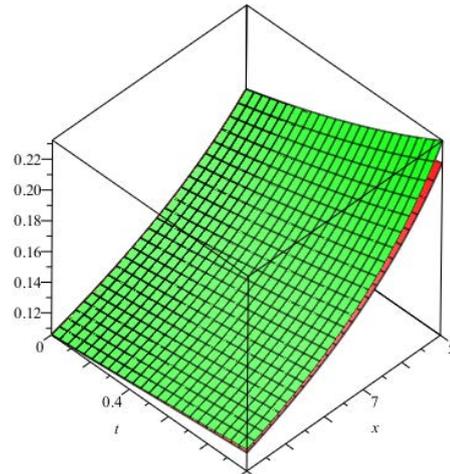


Figure 1. Exact and approximate solutions for Example2 and $u_1(x, t)$

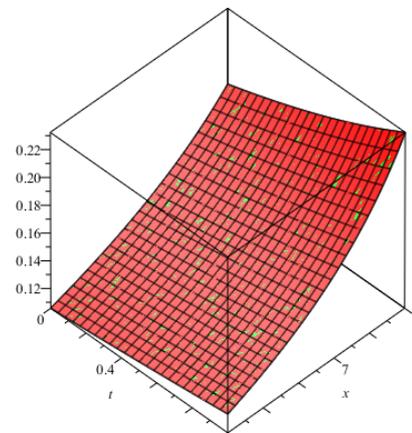


Figure 2. Exact and approximate solutions for Example 2 and $u_2(x, t)$

We applied the method presented in this paper and solved Eq. (4). The iteration formula for this example is

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[\frac{\partial u_n}{\partial s}(x, s) - \frac{\partial}{\partial x} \left(\frac{1}{u_n(x, s)} \frac{\partial u_n}{\partial x}(x, s) \right) \right] ds$$

Let $u_0(x, t) = \frac{1}{x}$. Applying the above iteration formula, we obtain:

$$u_1(x, t) = \frac{t+x}{x^2},$$

$$u_2(x, t) = \frac{1}{x-t}.$$

Therefore, we obtain the exact solution.

4.2. Example 2

Let us take $m = -4/3$, Eq. (1) we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(u^{-4/3} \frac{\partial u}{\partial x} \right) \quad (5)$$

The exact solution is $u(x, t) = (2x - 3t)^{-3/4}$. Then the iteration formula is

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[\frac{\partial u_n}{\partial s}(x, s) - \frac{\partial}{\partial x} \left(u_n^{-4/3}(x, s) \frac{\partial u_n}{\partial x}(x, s) \right) \right] ds \quad (6)$$

Let $u_0(x, t) = (2x)^{-3/4}$. Applying the iteration formula (6), we have;

$$u_1(x, t) = \frac{2^{1/4}(9t + 8x)}{16x^{7/4}},$$

And so on. The exact and approximate solutions for $u_1(x, t)$ and $u_2(x, t)$ are plotted in Figures. 1 and 2. In each figure "green color" stand for the exact solution and "blue color" stand for the approximate solutions.

5. Conclusion

In this paper the variational iteration method is used to solve the porous media equation. We described the method, used it on two test problems, and compared the results with their exact solutions in order to demonstrate the validity and applicability of the method. Moreover, only a small number of iterations are needed to obtain a satisfactory result the given numerical examples support this claim.

References

- [1] S. Pamuk, Solution of the porous media equation by Adomion's decomposition method, *phys.lett. A* 344 (2005) 184-188.
- [2] T. P. Witelski, *J. Math. Biol.* 35 (1997) 695.
- [3] A. D. Polyanin, V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall / CRC press, Boca Raton, 2004.
- [4] J. H. He, Variational iteration method for delay differential equations, *commun. Nonlinear Sci. Numer. Simul.* 2 (4) (1997) 235-236.
- [5] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Comput. Methods Appl. Mech. Engrg.* 167 (1998) 57-68.
- [6] J. H. He, Approximate solution of nonlinear differential equation with convolution product nonlinearities, *Comput. Methods Appl. Mech. Engrg.* 167 (1998) 69-73.
- [7] J. H. He, Variational iteration method-a kind of nonlinear analytical technique: some examples, *Int. J. Non-linear Mech.* 34 (1999) 699-708.
- [8] M. Inokuti, et al., General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Nasser (Ed.), *Variational Method in the Mechanics of solids*, pergamon Press, Oxford. 1978, pp. 156-162.
- [9] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, *J. Comput. Appl. Math.* 207 (2007) 121-128.
- [10] G.C. Wu, D. Baleanu, Variational iteration method for the Burgers' flow with fractional derivatives-New Lagrange multipliers, 37 (2013), 6183-6190.
- [11] A. Maida, J.P. Corriou, Open-loop optimal controller design using variational iteration method, *Applied Mathematics and Computation*, 219 (2013) 8632-8645.
- [12] M. Tatari, M. Dehghan, solution of problems in calculus of variations via He's variational iteration method, *phys. Lett. A* 362 (2007) 401-406.
- [13] M. Dehghan, F. Shakeri, Application of He's variational iteration method for solving the Cauchy reaction diffusion problem, *J. Comput. Appl. Math.* 214 (2008) 435-446.
- [14] F. Shakeri, M. Dehghan, Numerical solution of the Klien-Gordon equation via He's variational iteration Method, *Nonlinear Dynam.* 51 (2008) 89-97.
- [15] M. Dehghan, M. Tatari, Identifying an unknown function in a parabolic equation with over specified data via He's variational iteration method, *chaos solitons Fractals* 36 (2008) 157-166.
- [16] M. Dehghan, F. shakeri, Numerical solution of a biological Population model using He's variational iteration method, *comput. Math. Appl.* 54 (2007) 1197-1209.
- [17] M. Dehghan, Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices, *Math. Comput. Simulation* 71 (2006) 16-30.
- [18] M. shakourfar, M. Dehghan, on the numerical solution of nonlinear systems of volterra integro-differential equations with delay arguments, *computing* 82 (2008) 241-260.
- [19] M. Dehghan, M. Shakourifar, A. Hamidi, The solution of linear and nonlinear systems of Volterra functional equations using Adomian-Pade technique, *Chaos, solitons and fractals* 39 (2009) 2509-2521.
- [20] M. Tatari, M. Dehghan, He's variational iteration method for computing a control parameter in a semi-linear inverse parabolic equation, *chaos, solitons and Fractals* 33 (2007) 671-677.
- [21] M. Tatari, M. Dehghan, Improvement of He's variational iteration method for solving systems of differential equations, *computers and Mathematics with Applications* 58 (11-12) (2009) 2160-2166.
- [22] M. Dehghan, F. Shakeri, solution of parabolic integro-differential equations arising in heat conduction in materials with memory via He's variational iteration technique, *communication in Numerical Methods in Engineering* (2008) (inpress).
- [23] S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, *chaos solitons Fractals* 27 (5) (2006) 1119-1123.
- [24] H. Tari, D. D. Ganji, M. Rostamian, Approximate solutions of K (2, 2), kdv and modified kdv and modified kdv equations by variational iteration method, homotopy perturbation method and homotopy analysis method, *Int. J. Nonlinear Sci. Numer. Simul.* 8 (2) (2007) 203-210.
- [25] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, *chaos solution Fractals* 31 (2007) 1238-1255.
- [26] Z. M. Odibat, S. Momani, Application of variational iteration method to Nonlinear differential equations of fractional order, *Int. J. Nonlinear Sci. Numer. Simul.* 7 (2006) 27-34.
- [27] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, *Int. J. Nonlinear Sci. Numer. Simul.* 7 (2006) 65-70.
- [28] J. H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, *chaos solitons Fractals* 29 (2006) 108-113.
- [29] N. H. Sweilam, M. M. Khader, Variational iteration method for one dimensional nonlinear the thermoelasticity, *chaos solitons Fractals* 32 (2007) 145-149.
- [30] B. A. Finlayson, *The Method of Weighted Residuals and Variational principles*, Academic press, New York, 1972.
- [31] J. H. He, some asymptotic methods for strongly nonlinear equations, *Int. J. Mod phys. B* 20 (10) (2006) 1141-1199.