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Partial Differential Problems of Four Types of Two-Variables Functions

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Abstract This article takes the mathematical software Maple as the auxiliary tool to study the partial differential problems of four types of two-variables functions. We can obtain the infinite series forms of any order partial derivatives of these two-variables functions by using differentiation term by term theorem. In addition, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying the answers by using Maple.

Keywords: *partial derivatives, infinite series forms, differentiation term by term theorem, Maple*

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1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics curricula, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. For example, Laplace equation, wave equation, as well as other important physical equations are involved the partial derivatives. On the other hand, evaluating the *n*-th order partial derivative value of a multivariable function at some point, in general, needs to go through two procedures: firstly determining the *n*-th order partial derivative of this function, and then taking the point into the *n*-th order partial derivative. These two procedures will make us face with increasingly complex calculations when calculating higher order partial derivative values (i.e. *n* is large), and hence to obtain the answers by manual calculations is not easy. In this article, we study the partial differential problem of the following four types of two-variables functions

$$f(x,y)=\frac{\cos(ax+b)\cosh(cy+d)}{\cos 2(ax+b)+\cosh 2(cy+d)} \tag{1}$$

$$g(x,y)=\frac{\sin(ax+b)\sinh(cy+d)}{\cos 2(ax+b)+\cosh 2(cy+d)} \tag{2}$$

$$p(x,y)=\frac{\sin(ax+b)\cosh(cy+d)}{\cosh 2(cy+d)-\cos 2(ax+b)} \tag{3}$$

$$q(x,y)=\frac{\cos(ax+b)\sinh(cy+d)}{\cosh 2(cy+d)-\cos 2(ax+b)} \tag{4}$$

where a, b, c, d are real numbers. We can obtain the infinite series forms of any order partial derivatives of these four types of two-variables functions using differentiation term by term theorem; these are the major results of this study (i.e., Theorems [1, 2, 3, 4]), and hence greatly reduce the difficulty of calculating their higher order partial derivative values. As for the study of related partial differential problems can refer to [1-13]. On the other hand, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce some notations and formulas used in this article.

2.1. Notations

2.1.1. Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, a, b are real numbers. We denote a the real part of z by Re(z), and b the imaginary part of z by Im(*z*).

2.1.2. Suppose *m*, *n* are non-negative integers. For the two-variables function *f* (*x*, *y*), its *n*-times partial derivative with respect to *x*, and *m*-times partial derivative with respect to *y*, forms a *m* + *n*-th order partial derivative, and denoted by $\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x, y)$.

2.2. Formulas

2.2.1. Euler's Formula

$e^{ix} = \cos x + i \sin x$, where *x* is any real number.

2.2.2. Complex Euler's Formula

$e^{iz} = \cos z + i \sin z$, where *z* is any complex number.

2.2.3. ([14])

$\sin(\alpha + i\beta) = \sin\alpha \cosh\beta + i\cos\alpha \sinh\beta$, where α, β are real numbers.

2.2.4. ([14])

$\cos(\alpha + i\beta) = \cos\alpha \cosh\beta - i\sin\alpha \sinh\beta$, where α, β are real numbers.

2.2.5. Geometric Series

$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, where *z* is a complex number, $|z| < 1$.

Next, we introduce an important theorem used in this paper.

2.3. Differentiation Term by Term Theorem ([15])

For all non-negative integers *k*, if the functions $g_k : (a,b) \to R$ satisfy the following three conditions: (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (*a*, *b*), (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (*a*, *b*). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (*a*, *b*). Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

Before deriving the first and second major results in this study, we need two lemmas.

2.4. Lemma 1

Suppose the domain of the complex secant function $\sec z$ is $\left\{ z \in C \middle| z \neq \frac{(2m-1)\pi}{2}, m \in Z \right\}$. If Re(*iz*) > 0, then

$$\sec z = 2 \cdot \sum_{k=0}^{\infty} (-1)^k e^{-i(2k+1)z} \tag{5}$$

Proof $\sec z$

$$= \frac{1}{\cos z}$$

$$= \frac{1}{\frac{1}{2}\left(e^{iz} + e^{-iz}\right)}$$

(By complex Euler's formula)

$$= 2 \cdot \frac{e^{-iz}}{1 + e^{-i2z}}$$

$$= 2 \cdot \sum_{k=0}^{\infty} (-1)^k e^{-i(2k+1)z}$$

(Because $\operatorname{Re}(iz) > 0$, we can use geometric series) □

2.5. Lemma 2

Suppose *a*, *b*, *c*, *d*, *x*, *y*, are real numbers and $cy + d \neq 0$, then

$$\sec[(ax+b) + i(cy+d)]$$
$$= \frac{2\cos(ax+b)\cosh(cy+d) + i2\sin(ax+b)\sinh(cy+d)}{\cos 2(ax+b) + \cosh 2(cy+d)} \tag{6}$$

Proof $\sec[(ax+b) + i(cy+d)]$

$$= \frac{1}{\cos[(ax+b) + i(cy+d)]}$$

$$= \frac{1}{\cos(ax+b)\cosh(cy+d) - i\sin(ax+b)\sinh(cy+d)}$$

(By Formula 2.2.4)

$$= \frac{\cos(ax+b)\cosh(cy+d) + i\sin(ax+b)\sinh(cy+d)}{\cos^2(ax+b)\cosh^2(cy+d) + \sin^2(ax+b)\sinh^2(cy+d)}$$

$$= \frac{2\cos(ax+b)\cosh(cy+d) + i2\sin(ax+b)\sinh(cy+d)}{\cos 2(ax+b) + \cosh 2(cy+d)}$$

□

The following is the first major result in this paper, we obtain the infinite series forms of any order partial derivatives of the two-variables function (1).

2.6. Theorem 1

Suppose *a*, *b*, *c*, *d* are real numbers, $c \neq 0$ and *m*, *n* are non-negative integers. If the domain of the two-variables functions

$$f(x, y) = \frac{\cos(ax+b)\cosh(cy+d)}{\cos 2(ax+b) + \cosh 2(cy+d)}$$

is $\left\{(x,y)\in R^2 \,\middle|\, cy+d\neq 0\right\}$.

Case (1) If $cy+d<0$, then the $m+n$-th order partial derivative of $f(x, y)$,

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y) = a^n \cdot c^m \cdot \sum_{k=0}^{\infty}(-1)^k (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right] \quad (7)$$

Case (2) If $cy+d>0$, then

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y) = a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty}(-1)^k (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right] \quad (8)$$

Proof Case (1) If $cy+d<0$, because

$$f(x,y) = \frac{\cos(ax+b)\cosh(cy+d\ )}{\cos 2(ax+b)+\cosh 2(cy+d\ )} = \frac{1}{2}\operatorname{Re}\left\{\sec\left[(ax+b)+i(cy+d\ )\right]\right\}$$

(By Lemma 2)

$$=\operatorname{Re}\left\{\sum_{k=0}^{\infty}(-1)^k e^{-i(2k+1)[(ax+b)+i(cy+d)]}\right\}$$

(By Lemma 1)

$$=\sum_{k=0}^{\infty}(-1)^k e^{(2k+1)(cy+d)}\cos[(2k+1)(ax+b)] \quad (9)$$

(By Euler's formula)

On both sides of (9), by differentiation term by term theorem, differentiating m-times with respect to x, and n-times with respect to y, we obtain

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y) = a^n \cdot c^m \cdot \sum_{k=0}^{\infty}(-1)^k (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right]$$

Case (2) If $cy+d>0$, then $-cy-d<0$. Because

$$f(x,y)=\frac{\cos(ax+b)\cosh(-cy-d)}{\cos 2(ax+b)+\cosh 2(-cy-d)} \quad (10)$$

By (7), we obtain

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y) = a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty}(-1)^k (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right]$$

□

Next, we determine the infinite series forms of any order partial derivatives of the two-variables function (2).

2.7. Theorem 2

If the assumptions are the same as Theorem 1, and the domain of the two-variables function

$$g(x,y)=\frac{\sin(ax+b)\sinh(cy+d)}{\cos 2(ax+b)+\cosh 2(cy+d)}$$

is $\left\{(x,y)\in R^2 \,\middle|\, cy+d\neq 0\right\}$.

Case (1) If $cy+d<0$, then the $m+n$-th order partial derivative of $g(x, y)$,

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y) = -a^n \cdot c^m \cdot \sum_{k=0}^{\infty}(-1)^k (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \sin\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right] \quad (11)$$

Case (2) If $cy+d>0$, then

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y) = a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty}(-1)^k (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \sin\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right] \quad (12)$$

Proof Case (1) If $cy+d<0$, because

$$g(x,y) = \frac{\sin(ax+b)\sinh(cy+d)}{\cos 2(ax+b)+\cosh 2(cy+d)} = \frac{1}{2}\operatorname{Im}\left\{\sec[(ax+b)+i(cy+d)]\right\} \quad (13)$$

$$=\operatorname{Im}\left\{\sum_{k=0}^{\infty}(-1)^k e^{-i(2k+1)[(ax+b)+i(cy+d)]}\right\} = -\sum_{k=0}^{\infty}(-1)^k e^{(2k+1)(cy+d)}\sin[(2k+1)(ax+b)]$$

By differentiation term by term theorem, differentiating n-times with respect to x, and m-times with respect to y. On both sides of (13), we have

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y)$$
$$= -a^n \cdot c^m \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^{m+n} e^{(2k+1)(cy+d)}$$
$$\times \sin\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right]$$

Case (2) If $cy+d>0$, then $-cy-d<0$. Because

$$g(x,y)$$
$$= \frac{\sin(ax+b)\sinh(cy+d)}{\cos 2(ax+b)+\cosh 2(cy+d)}$$
$$= -\frac{\sin(ax+b)\sinh(-cy-d)}{\cos 2(ax+b)+\cosh 2(-cy-d)} \quad (14)$$
$$= \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)(cy+d)} \sin[(2k+1)(ax+b)]$$

Therefore,

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y)$$
$$= a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^{m+n} e^{-(2k+1)(cy+d)}$$
$$\times \sin\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right]$$

□

Before deriving the third and fourth major results in this article, we also need two lemmas.

2.8. Lemma 3

Assume the domain of the complex cosecant function $\sec z$ is $\{z \in C \,|\, z \neq m\pi, m \in Z\}$. If $\mathrm{Re}(iz)>0$, then

$$\csc z = 2i \cdot \sum_{k=0}^{\infty} e^{-i(2k+1)z} \quad (15)$$

Proof $\csc z$

$$= \frac{1}{\sin z}$$
$$= \frac{1}{\frac{1}{2_i}(e^{iz}-e^{-iz})}$$

(By complex Euler's formula)

$$= 2i \cdot \frac{e^{-iz}}{1-e^{-i2z}}$$
$$= 2i \cdot \sum_{k=0}^{\infty} e^{-i(2k+1)z}$$

□

2.9. Lemma 4

Suppose *a, b, c, d, x, y*, are real numbers and $cy+d \neq 0$, then

$$\csc[(ax+b)+i(cy+d)]$$
$$= \frac{2\sin(ax+b)\cosh(cy+d) - i2\cos(ax+b)\sinh(cy+d)}{\cosh 2(cy+d) - \cos 2(ax+b)} \quad (16)$$

Proof $\csc\left[(ax+b)+i(cy+d)\right]$

$$= \frac{1}{\sin\left[(ax+b)+i(cy+d)\right]}$$
$$= \frac{1}{\sin(ax+b)\cosh(cy+d)+i\cos(ax+b)\sinh(cy+d)}$$

(By Formula 2.2.3)

$$= \frac{\sin(ax+b)\cosh(cy+d)-i\cos(ax+b)\sinh(cy+d)}{\sin^2(ax+b)\cosh^2(cy+d)-\cos^2(ax+b)\sinh^2(cy+d)}$$

$$= \frac{2\sin(ax+b)\cosh(cy+d)-i2\cos(ax+b)\sinh(cy+d)}{\cosh 2(cy+d)-\cos 2(ax+b)}$$

□

In the following, we determine the infinite series forms of any order partial derivatives of the two-variables function (3).

2.10. Theorem 3

If the assumptions are the same as Theorem 1, and the domain of the two-variables function

$$p(x,y) = \frac{\sin(ax+b)\cosh(cy+d)}{\cosh 2(cy+d)-\cos 2(ax+b)}$$

is $\left\{(x,y) \in R^2 \,\middle|\, cy+d \neq 0\right\}$.

Case (1) If $cy+d<0$, then the $m+n$-th order partial derivative of $p\,(x,y)$,

$$\frac{\partial^{m+n} p}{\partial y^m \partial x^n}(x,y)$$
$$= a^n \cdot c^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \quad (17)$$
$$\sin\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right]$$

Case (2) If $cy+d>0$, then

$$\frac{\partial^{m+n} p}{\partial y^m \partial x^n}(x,y)$$
$$= a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \quad (18)$$
$$\sin\left[(2k+1)(ax+b)+\frac{n\pi}{2}\right]$$

Proof Case (1) If $cy+d<0$, because

$$p(x,y)$$
$$= \frac{\sin(ax+b)\cosh(cy+d)}{\cosh 2(cy+d)-\cos 2(ax+b)}$$
$$= \frac{1}{2}\mathrm{Re}\left\{\csc\left[(ax+b)+i(cy+d)\right]\right\}$$

(By Lemma 4)

$$= \operatorname{Re}\left\{ i \sum_{k=0}^{\infty} e^{-i(2k+1)[(ax+b)+i(cy+d)]} \right\}$$

(By Lemma 3)

$$= \sum_{k=0}^{\infty} e^{(2k+1)(cy+d)} \sin[(2k+1)(ax+b)] \tag{19}$$

By differentiation term by term theorem, we obtain

$$\frac{\partial^{m+n} p}{\partial y^m \partial x^n}(x,y) = a^n \cdot c^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \sin\left[(2k+1)(ax+b) + \frac{n\pi}{2}\right]$$

Case (2) If $cy+d>0$, then $-cy-d<0$. Because

$$p(x,y) = \frac{\sin(ax+b)\cosh(-cy-d)}{\cosh 2(-cy-d) - \cos 2(ax+b)} \tag{20}$$

By (17), we have

$$\frac{\partial^{m+n} p}{\partial y^m \partial x^n}(x,y) = a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \sin\left[(2k+1)(ax+b) + \frac{n\pi}{2}\right]$$

□

Finally, we find the infinite series forms of any order partial derivatives of the two-variables function (4).

2.11. Theorem 4

If the assumptions are the same as Theorem 1, and the domain of the two-variables function

$$q(x,y) = \frac{\cos(ax+b)\sinh(cy+d)}{\cosh 2(cy+d) - \cos 2(ax+b)}$$

is $\left\{(x,y) \in R^2 \,|\, cy+d \neq 0\right\}$.

Case (1) If $cy+d<0$, then the $m+n$-th order partial derivative of $q(x,y)$,

$$\frac{\partial^{m+n} q}{\partial y^m \partial x^n}(x,y) = -a^n \cdot c^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b) + \frac{n\pi}{2}\right] \tag{21}$$

Case (2) If $cy+d>0$, then

$$\frac{\partial^{m+n} q}{\partial y^m \partial x^n}(x,y) = a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b) + \frac{n\pi}{2}\right] \tag{22}$$

Proof Case (1) If $cy+d<0$, because

$$\begin{aligned} q(x,y) &= \frac{\cos(ax+b)\sinh(cy+d)}{\cosh 2(cy+d) - \cos 2(ax+b)} \\ &= -\frac{1}{2}\operatorname{Im}\left\{\csc\left[(ax+b)+i(cy+d)\right]\right\} \\ &= -\operatorname{Im}\left\{ i \sum_{k=0}^{\infty} e^{-i(2k+1)[(ax+b)+i(cy+d)]} \right\} \\ &= -\sum_{k=0}^{\infty} e^{(2k+1)(cy+d)} \cos[(2k+1)(ax+b)] \end{aligned} \tag{23}$$

Also, by differentiation term by term theorem, we obtain

$$\frac{\partial^{m+n} q}{\partial y^m \partial x^n}(x,y) = -a^n \cdot c^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b) + \frac{n\pi}{2}\right]$$

Case (2) If $cy+d>0$, then $-cy-d<0$. Because

$$q(x,y) = -\frac{\cos(ax+b)\sinh(-cy-d)}{\cosh 2(-cy-d) - \cos 2(ax+b)} \tag{24}$$

Using (21), we obtain

$$\frac{\partial^{m+n} q}{\partial y^m \partial x^n}(x,y) = a^n \cdot (-c)^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{-(2k+1)(cy+d)} \times \cos\left[(2k+1)(ax+b) + \frac{n\pi}{2}\right]$$

□

3. Examples

In the following, for the partial differential problem of the four types of two-variables functions in this study, we provide four examples and use Theorems 1-4 to determine the infinite series forms of any order partial derivatives of these functions. In addition, we evaluate some higher order partial derivative values of these functions and employ Maple to calculate the approximations of these higher order partial derivative values and their solutions for verifying our answers.

3.1. Example 1

Suppose the domain of the two-variables function

$$f(x,y)=\frac{\cos\left(2x+\pi/6\right)\cosh\left(3y+4\right)}{\cos(4x+\pi/3)+\cosh(6y+8)} \tag{25}$$

is $\left\{(x,y)\in R^2\left|3y+4\neq 0\right.\right\}$.

If $3y+4<0$, i.e., $y<-\frac{4}{3}$. By (7), we obtain the $m+n$-th order partial derivative of $f(x,y)$,

$$\begin{aligned}&\frac{\partial^{m+n}f}{\partial y^m\partial x^n}(x,y)\\&=2^n\cdot 3^m\cdot\sum_{k=0}^{\infty}(-1)^k(2k+1)^{m+n}e^{(2k+1)(3y+4)}\times\\&\quad\cos\left[(2k+1)\left(2x+\frac{\pi}{6}\right)+\frac{n\pi}{2}\right]\end{aligned} \tag{26}$$

Hence, we can determine the 14-th order partial derivative value of $f(x,y)$ at $\left(\frac{\pi}{3},-2\right)$,

$$\begin{aligned}&\frac{\partial^{14}f}{\partial y^8\partial x^6}\left(\frac{\pi}{3},-2\right)\\&=-2^6 3^8\cdot\sum_{k=0}^{\infty}(-1)^k(2k+1)^{14}e^{-4k-2}\cos\frac{(2k-5)\pi}{6}\end{aligned} \tag{27}$$

Next, we use Maple to verify the correctness of (27).

```
>f:=(x,y)->cos(2*x+Pi/6)*cosh(3*y+4)/(cos(4*x+Pi/3)
+cosh(6*y+8));
>evalf(D[1$6,2$8](f)(Pi/3,-2),22);
```

$$7.30152250285790997\cdot 10^{10}$$

```
>evalf(-2^6*3^8*sum((-1)^k*(2*k+1)^14*exp(-4*k-
2)*cos((2*k-5)*Pi/6),k=0..infinity),22);
```

$$7.30152250285790999\cdot 10^{10}$$

3.2. Example 2

Assume the domain of the two-variables function

$$g(x,y)=\frac{\sin\left(4x-\pi/3\right)\sinh\left(5y-2\right)}{\cos(8x-2\pi/3)+\cosh(10y-4)} \tag{28}$$

is $\left\{(x,y)\in R^2\left|5y-2\neq 0\right.\right\}$.

If $5y-2>0$, i.e. $y>2/5$. By (12), we have

$$\begin{aligned}&\frac{\partial^{m+n}g}{\partial y^m\partial x^n}(x,y)\\&=4^n\cdot(-5)^m\cdot\sum_{k=0}^{\infty}(-1)^k(2k+1)^{m+n}e^{-(2k+1)(5y-2)}\times\\&\quad\sin\left[(2k+1)\left(4x-\frac{\pi}{3}\right)+\frac{n\pi}{2}\right]\end{aligned} \tag{29}$$

Thus,

$$\begin{aligned}&\frac{\partial^{10}g}{\partial y^7\partial x^3}\left(\frac{\pi}{2},1\right)\\&=-4^3\cdot(-5)^7\cdot\sum_{k=0}^{\infty}(-1)^k(2k+1)^{10}e^{-6k-3}\sin\frac{(4k+5)\pi}{6}\end{aligned} \tag{30}$$

```
>g:=(x,y)->sin(4*x-Pi/3)*sinh(5*y-2)/(cos(8*x-2*Pi/3)
+cosh(10*y-4));
>evalf(D[1$3,2$7](g)(Pi/2,1),22);
```

$$4.34603766442614248\cdot 10^{7}$$

```
>evalf(-4^3*(-5)^7*sum((-1)^k*(2*k+1)^10*exp(-6*k-
3)*sin((4*k+5)*Pi/6),k=0..infinity),22);
```

$$4.34603766442614249\cdot 10^{7}$$

3.3. Example 3

Suppose the domain of the two-variables function

$$p(x,y)=\frac{\sin\left(2x+3\pi/4\right)\cosh\left(4y+3\right)}{\cosh(8y+6)-\cos(4x+3\pi/2)} \tag{31}$$

is $\left\{(x,y)\in R^2\left|3y+4\neq 0\right.\right\}$.

If $3y+4<0$, i.e., $y<-4/3$. By (17), we have

$$\begin{aligned}&\frac{\partial^{m+n}p}{\partial y^m\partial x^n}(x,y)\\&=2^n\cdot 4^m\cdot\sum_{k=0}^{\infty}(2k+1)^{m+n}e^{(2k+1)(4y+3)}\times\\&\quad\sin\left[(2k+1)\left(2x-\frac{3\pi}{4}\right)+\frac{n\pi}{2}\right]\end{aligned} \tag{32}$$

Hence,

$$\begin{aligned}&\frac{\partial^{11}p}{\partial y^3\partial x^8}\left(\frac{\pi}{2},-3\right)\\&=-2^8\cdot 4^3\cdot\sum_{k=0}^{\infty}(2k+1)^{11}e^{-18k-9}\sin\frac{(2k+1)\pi}{4}\end{aligned} \tag{33}$$

```
>p:=(x,y)->sin(2*x+3*Pi/4)*cosh(4*y+3)/(cosh(8*y+6)-
cos(4*x+3*Pi/2));
>evalf(D[1$8,2$3](p)(Pi/2,-3),18);
```

$$-1.43358921266771880$$

```
>evalf(-2^8*4^3*sum((2*k+1)^11*exp(-18*k-9)*sin((
2*k+1)*Pi/4),k=0..infinity),18);
```

$$-1.43358921266771889$$

3.4. Example 4

Let the domain of the two-variables function

$$q(x,y)=\frac{\cos\left(5x-\pi/6\right)\sinh\left(6y+1\right)}{\cosh(12y+2)-\cos(10x-\pi/3)} \tag{34}$$

be $\left\{(x,y)\in R^2\left|6y+1\neq 0\right.\right\}$.

If $6y+1>0$, i.e., $y>-1/6$. Using (22), we obtain

$$\frac{\partial^{m+n} q}{\partial y^m \partial x^n}(x, y) = 5^n \cdot (-6)^m \cdot \sum_{k=0}^{\infty} (2k+1)^{m+n} e^{-(2k+1)(6y+1)} \times \cos\left[(2k+1)\left(5x - \frac{\pi}{6}\right) + \frac{n\pi}{2}\right] \quad (35)$$

Thus,

$$\frac{\partial^{16} q}{\partial y^{11} \partial x^5}(\pi, 1) = -5^5 \cdot (-6)^{11} \cdot \sum_{k=0}^{\infty} (2k+1)^{16} e^{-14k-7} \sin\frac{(10k+5)\pi}{6} \quad (36)$$

```
>q:=(x,y)->cos(5*x-Pi/6)*sinh(6*y+1)/(cosh(12*y+2)-cos(10*x-Pi/3));
>evalf(D[1$5,2$11](q)(Pi,1),28);
```

$$3.757724112380406779 \cdot 10^{10}$$

```
>evalf(-5^5*(-6)^11*sum((2*k+1)^16*exp(-14*k-7)*sin((10*k+5)*Pi/6),k=0..infinity),28);
```

$$3.757724112380406798 \cdot 10^{10}$$

4. Conclusion

In this paper, we provide a new technique to determine any order partial derivatives of four types of two-variables functions. We hope this technique can be applied to solve another partial differential problems. On the other hand, the differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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