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Sensitivity Analysis in Linear Fractional Programming with Optimality Condition

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Abstract In this paper, an overview of theoretical and methodological issues in simplex method-based sensitivity analysis is proposed. The paper focuses somewhat on developing shortcut methods to perform Linear Fractional Programming (LFP) sensitivity analysis manually and in particular changes in the parameter of the LFP model. Shortcut methods for conducting sensitivity analysis have been suggested. Simple examples are given to illustrate this proposed method.

Keywords: *LFP model, sensitivity analysis, simplex*

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1. Introduction

Many managerial decisions hinge on the issue of how to make the most of the company's resources of raw material, manpower, time, and facilities. LFP is a technique that aims at optimizing performance regarding combinations of resources. LFP can offer managers the capability of building scenarios through its extensive "what if" analysis and sensitivity analysis facilities. While most practical LFP problems would require a very long time to solve manually.

When we are dealing with sensitivity analysis, we are initially looking into changes might happen to the parameters of LFP model. These possible changes would imply to investigate the changes in Right-Hand-Side of the model constraints and the coefficient of the objective function [1,2]. Once the optimal solution to an LFP problem has been achieved using the Simplex algorithm, it may be desirable to study how current optimal solution stays optimal when one or more of the problem parameters may change. It is crucial to figuring out how sensitive the optimal solution is to some changes in the model parameters [1,2]. Sensitivity analysis, (post-optimality), therefore, looks at "what if" questions scenarios. What happens to the cash position, for example, if sales fall by 5%? What happens if primary supplier increases raw material prices by 12%?

When we deal with practical problems, sensitivity analysis is much more important than the result obtained from the optimal solution. Such an analysis transforms the LFP solution into a valuable tool to study the effect of changing conditions such as in management, business, and industry. When we include the organization's business plan with the sensitivity analysis report, it will show that we have thought about some of the potential risks - and that is halfway to avoiding them. Sensitivity analysis can help in making proper decisions. For example, if we may want to consider, the effect of increased labor force or decrease overhead charges, or reducing capacities, due to over-optimistic forecasts, what effect of these actions on counteracting competitors.

2. Literature Review

The literature on sensitivity analysis is enormous and diverse. In late 1980's and early 1990's several researchers and scientists were involved in the fields of operations research working on the L.P. sensitivity analysis topic. Some significant advances were produced in L.P. sensitivity analysis and related problems. The research in the field of sensitivity analysis was extensively carried out by many operational research specialists. It includes [5,9,13,16,17,18] worked on sensitivity analysis parameter but excluded the simultaneous changes in the LP parameter.

[3] studied the sensitivity analysis for the parameters of structured problems. Khan et al. (2011) studied the profit in products by using LP techniques and sensitivity analysis.

However, the existing literature concerning our research scope is limited. Most of the previous work on sensitivity analysis were focused on lengthy methodologies and procedures that consume a significant amount of time to arrive at a solution of sensitivity analysis. [21] introduced two kinds of sensitivity analysis. First one is defining the properties of sensitivity region while the second one is the positive sensitivity analysis.

More recent works on linear fractional programming theory and methods can be found in [1,2]. The suggested method in this paper depends mainly on the updated method in iterative manner then the optimality condition for a given basic feasible solution of (LFP) is defined.

3. Research Objectives

Calculations the ranges for optimality and feasibility in sensitivity analysis are covered in most operations research books and most quantitative textbooks. The methods used vary from one book to another although all of them achieve the same results. These methods may take a very long time and effort to solve the sensitivity issues manually. Many of these methods have the tendency to involve lengthy mathematical approaches that they require students to be well-equipped with advanced mathematical techniques such as matrices and vectors. Some other methods need a longer routine to get the results. It is much practical to use lighter and sounder methods to obtain the sensitivity analysis results with ease and with less time. Thus, the paper will explain how the shortcut methods can be used to derive the sensitivity analysis results.

4. Materials and Methods

This part is devoted to the study of the simplex method. This method is the main tool for solving linear programming problems. It consists of following a certain number of stages before obtaining the solution of a given problem. It is an iterative algebraic method which allows to find the exact solution of a linear programming problem in a finite number of steps.

4.1. Mathematical Formulation of LFP

A general problem of linear fractional programming can be formulated as follows: find the values of n variables x_j , $j = 1, 2, 3, \dots, n$ satisfying m inequalities or linear equations (constraints) of the form:

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_i : i = 1, 2, 3, \ldots, m$$

each constraint may have a different sign of inequality. In addition, the variables must be non-negative, that is to say $x_j \geq 0$, $j = 1, 2, 3, \dots, n$ (constraints of non-negativity) and must Maximize or Minimize a linear form (function objective) such as:

$$F(x) = \frac{\sum_{j=1}^{n} p_j x_j + p_0}{\sum_{j=1}^{n} q_j x_j + q_0}$$

where a_{ij} , b_i, p_j , q_j , p_0 and q_0 are known reals and

$$\sum_{j=1}^{n} q_j x_j + q_0 \neq 0.$$

The matrix form makes it possible to represent a problem of linear fractional programming in a more concise form.

Objective function to Maximize or Minimize

$$F(x) = \frac{P(x)}{Q(x)} = \frac{px + p_0}{qx + q_0}$$

Subject to $Ax \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b$ and $x \geq 0$

where we have the following matrices:

$$p = (p_1 p_2 \dots p_n),\ A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}},\ q = (q_1 q_2 \dots q_n),$$

$$x = (x_1 x_2 \dots x_n)^t,\ \text{and } b = (b_1 b_2 \dots b_m)^t.$$

4.2. Simplex Table and Iteration Procedure

When applying the simplex method by hand, it is best to work with a table that contains all the necessary data. Each iteration corresponds to a new table taking the following form:

4.2.1. Simplex Table: $\boldsymbol{T^{(s)}}$

Consider the linear fractional program (LFP):

$$F(x) = \frac{P(x)}{Q(x)} \to \max$$

subject to $Ax \leq b$ and $x \geq 0$.

We are going to transform the inequalities encountered into equality. This transformation is done simply by introducing non-negative variables (which verify the constraints of non-negativity) called slack variables. If the constraints are of the type: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i$, we introduce a slack variable $x_{n+i} \geq 0$ (slack variable for i-th contraint) and write the canonical form:

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + x_{n+i} = b_i.$$

So we have

$$F(x) = \frac{\sum_{j=1}^{n} p_j x_j + 0x_{n+1} + \ldots + 0x_{n+m} + p_0}{\sum_{j=1}^{n} q_j x_j + 0x_{n+1} + \ldots + 0x_{n+m} + q_0}$$

Subject to

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + x_{n+i} = b_i$$
$$\text{and } x_j \geq 0,\ j = 1, 2, 3, \ldots, n+m.$$

In the iteration (s) or in the s-th table called the simplex table $\boldsymbol{T^{(s)}}$.

In the simplex table $\boldsymbol{T^{(s)}}$ and for $1 \leq j \leq n+m$ we have: the basic variables column is $x_B^{(s)} = \{x_{J_1}, x_{J_2}, \dots, x_{J_m}\}$ and nonbasic variables is $x_N^{(s)} = \{x_d, x_d \notin x_B^{(s)}\}$, the matrix of the coefficients of basic variables in $P(x)$ is: $P_B^{(s)} = (p_{J_1}\, p_{J_2} \dots p_{J_m})$, the matrix of the coefficients of basic variables in $Q(x)$ is: $Q_B^{(s)} = (q_{J_1}\, q_{J_2} \dots q_{J_m})$, the solution matrix is $X_B^{(s)} = (x_{J_1} = b_1^{(s)}\ x_{J_2} = b_2^{(s)} \dots x_{J_m} = b_m^{(s)})^t$, the matrices of each column of the table are

$$A_j^{(s)} = \left(a_{1j}^{(s)} a_{2j}^{(s)} \dots a_{mj}^{(s)}\right)^t,\ B_B^{-1(s)} = \left(a_{ij}^{(s)}\right)_{\substack{1 \leq i \leq m \\ n+1 \leq j \leq n+m}}$$

$$\text{or } B_B^{-1(s)} = \left(A_{n+1}^{(s)} A_{n+2}^{(s)} \dots A_{n+m}^{(s)}\right)$$

and the opportunity and marginal costs of each activity $Z_j^{'(s)} = P_B^{(s)} A_j^{(s)}$, $Z_j^{"(s)} = Q_B^{(s)} A_j^{(s)}$, $\Delta_j^{'(s)} = Z_j^{'(s)} - p_j$ and $\Delta_j^{"(s)} = Z_j^{"(s)} - q_j$. The values of the functions F, P and Q are: $P(x) = P_B^{(s)} X_B^{(s)} + p_0$, $Q(x) = Q_B^{(s)} X_B^{(s)} + q_0$ and $F(x) = P(x)/Q(x)$. Moreover $A_j^{(s)} = B_B^{-1(s)} A_j$ and $X_B^{(s)} = B_B^{-1(s)} b$.

Simplex table $T^{(s)}$.

Basic variables $x_B^{(s)}$	Coefficients of basis in $P(x)$: $P_B^{(s)}$	Coefficients of basis in $Q(x)$: $Q_B^{(s)}$	p_1	p_2	. . .	p_{n+m}	Current values $X_B^{(s)}$
			q_1	q_2	. . .	q_{n+m}	
			$A_1^{(s)}$	$A_2^{(s)}$	. . .	$A_{n+m}^{(s)}$	
x_{J_1}	p_{J_1}	q_{J_1}	$a_{11}^{(s)}$	$a_{12}^{(s)}$	. . .	$a_{1(n+m)}^{(s)}$	$x_{J_1}=b_1^{(s)}$
x_{J_2}	p_{J_2}	q_{J_2}	$a_{21}^{(s)}$	$a_{22}^{(s)}$	. . .	$a_{2(n+m)}^{(s)}$	$x_{J_2}=b_2^{(s)}$
⋮	⋮	⋮	⋮	⋮	. . .	⋮	⋮
x_{J_m}	p_{J_m}	q_{J_m}	$a_{m1}^{(s)}$	$a_{m2}^{(s)}$	. . .	$a_{m(n+m)}^{(s)}$	$x_{J_m}=b_m^{(s)}$
$Z_j^{'(s)}=P_B^{(s)}A_j^{(s)}$			$Z_1^{'(s)}$	$Z_2^{'(s)}$	. . .	$Z_{n+m}^{'(s)}$	$P(x)$ $Q(x)$ $F(x)$
$Z_j^{''(s)}=Q_B^{(s)}A_j^{(s)}$			$Z_1^{''(s)}$	$Z_2^{''(s)}$	. . .	$Z_{n+m}^{'(s)}$	
$\Delta_j^{'(s)}=Z_j^{'(s)}-p_j$			$\Delta_1^{'(s)}$	$\Delta_2^{'(s)}$	. . .	$\Delta_{n+m}^{'(s)}$	
$\Delta_j^{''(s)}=Z_j^{''(s)}-q_j$			$\Delta_1^{''(s)}$	$\Delta_2^{''(s)}$	. . .	$\Delta_{n+m}^{''(s)}$	
$\Delta_j^{(s)}=\Delta_j^{'(s)}-F(x)\Delta_j^{''(s)}$			$\Delta_1^{(s)}$	$\Delta_2^{(s)}$	. . .	$\Delta_{n+m}^{(s)}$	
Basic variables $x_B^{(s)}$	Coefficients of basis in $P(x)$: $P_B^{(s)}$	Coefficients of basis in $Q(x)$: $Q_B^{(s)}$	p_1	p_2	. . .	p_{n+m}	Current values $X_B^{(s)}$
			q_1	q_2	. . .	q_{n+m}	
			$A_1^{(s)}$	$A_2^{(s)}$	. . .	$A_{n+m}^{(s)}$	
x_{J_1}	p_{J_1}	q_{J_1}	$a_{11}^{(s)}$	$a_{12}^{(s)}$	. . .	$a_{1(n+m)}^{(s)}$	$x_{J_1}=b_1^{(s)}$
x_{J_2}	p_{J_2}	q_{J_2}	$a_{21}^{(s)}$	$a_{22}^{(s)}$	. . .	$a_{2(n+m)}^{(s)}$	$x_{J_2}=b_2^{(s)}$
⋮	⋮	⋮	⋮	⋮	. . .	⋮	⋮
x_{J_m}	p_{J_m}	q_{J_m}	$a_{m1}^{(s)}$	$a_{m2}^{(s)}$	. . .	$a_{m(n+m)}^{(s)}$	$x_{J_m}=b_m^{(s)}$
$Z_j^{'(s)}=P_B^{(s)}A_j^{(s)}$			$Z_1^{'(s)}$	$Z_2^{'(s)}$	. . .	$Z_{n+m}^{'(s)}$	$P(x)$ $Q(x)$ $F(x)$
$Z_j^{''(s)}=Q_B^{(s)}A_j^{(s)}$			$Z_1^{''(s)}$	$Z_2^{''(s)}$	. . .	$Z_{n+m}^{'(s)}$	
$\Delta_j^{'(s)}=Z_j^{'(s)}-p_j$			$\Delta_1^{'(s)}$	$\Delta_2^{'(s)}$	. . .	$\Delta_{n+m}^{'(s)}$	
$\Delta_j^{''(s)}=Z_j^{''(s)}-q_j$			$\Delta_1^{''(s)}$	$\Delta_2^{''(s)}$	. . .	$\Delta_{n+m}^{''(s)}$	
$\Delta_j^{(s)}=\Delta_j^{'(s)}-F(x)\Delta_j^{''(s)}$			$\Delta_1^{(s)}$	$\Delta_2^{(s)}$	. . .	$\Delta_{n+m}^{(s)}$	

4.2.2. Iteration Procedure: Simplex Algorithm

Consider the following problem (LFP):

$$F(x)=\frac{P(x)}{Q(x)}\to \max$$

subjetc to $Ax\le b$ and $x\ge 0$ with $b>0$.

Simplex Algorithm (Maximization Form)

STEP (0) The problem is initially in canonical form and all $b_i\ge 0$ and construct the initial table of the simplex $\boldsymbol{T}^{(0)}$.

STEP (1) If $\Delta_j^{(s)}\ge 0$ for $j=1,2,\dots,n$ then *stop*; we are optimal $\boldsymbol{T}^{(s)}$. If we continue then there exists some $\Delta_j^{(s)}<0$.

STEP (2) Choose the column k to pivot in (i.e., the variable x_k to introduce into the basis) by

$$\Delta_k^{(s)}=\min_{x_d\in x_N^{(s)}}\left(\Delta_d^{(s)}\right).$$

If $a_{ik}^{(s)}\le 0$ for $i=1,2,\dots,m$ then *stop*; the primal problem is unbounded.

If we continue, then $a_{ik}^{(s)}>0$ for some $i=1,2,\dots,m$.

STEP (3) Choose row ℓ to pivot in (i.e., the variable x_ℓ to drop from the basis) by the ratio test:

$$\frac{b_\ell^{(s)}}{a_{\ell k}^{s}}=\min_{1\le i\le m}\left(\frac{b_i^{(s)}}{a(_{ik}^{s)}},\ a_{ik}^{(s)}>0\right).$$

STEP (4) Replace the basic variable in row ℓ with variable k and re-establish the canonical form (i.e., pivot on the coefficient $a_{\ell k}^{(s)}$).

STEP (5) do

$$\begin{cases} L_\ell^{(s+1)}=\dfrac{L_\ell^{(s)}}{a_{\ell k}^{(s)}} & \\ & \text{with } 1\le r\ne \ell\le m. \\ L_r^{(s+1)}=L_r^{(s)}-a_{rk}^{(s)}L_\ell^{(s+1)} & \end{cases}$$

STEP (6) Go to step (1).

These steps are the essential computations of the simplex method.

Optimal solution:

If $\boldsymbol{T}^{(s)}$ is optimal, then the current basis is $x_B^{(s)}=\{x_{J_1},x_{J_2},\dots,x_{J_m}\}$ and the corresponding solution is $x_B^{*}=\left\{x_{J_1}=b_1^{(s)},x_{J_2}=b_2^{(s)},\dots,x_{J_m}=b_m^{(s)}\right\}$. Moreover, the current nonbasic variables is $x_N^{(s)}=\left\{x_d,x_d\notin x_B^{(s)}\right\}$ and the corresponding solution is $x_N^{*}=\left\{x_d=0,x_d\in x_N^{(s)}\right\}$.

Hence the optimal solution to the problem can be written as $x^* = (x_1\ x_2\ \dots\ x_n\ \dots\ x_{n+m})^t$ with the associated value of the objective function $F\left(x^*\right) = \dfrac{P\left(x^*\right)}{Q\left(x^*\right)} = \dfrac{px^* + p_0}{qx^* + q_0}$.

Example: Simplex algorithm

Consider the following linear fractional program:

$$F(x) = \frac{8x_1 + 9x_2 + 4x_3 + 4}{2x_1 + 3x_2 + 2x_3 + 7} \to \max$$

Subject to
$$\begin{cases} x_1 + x_2 + 2x_3 \le 3 \\ 2x_1 + x_2 + 4x_3 \le 4 \\ 5x_1 + 3x_2 + x_3 \le 15 \\ x_j \ge 0,\ j = 1,2,3. \end{cases}.$$

To convert these inequality constraints to equalities, we add slack variables x_4, x_5 and x_6 to the left side of the inequality. The constraints becomes

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 + 0x_5 + 0x_6 = 3 \\ 2x_1 + x_2 + 4x_3 + 0x_4 + x_5 + 0x_6 = 4 \\ 5x_1 + 3x_2 + x_3 + 0x_4 + 0x_5 + x_6 = 15 \\ x_j \ge 0,\ j = 1,2,3,4,5,6. \end{cases}$$

Because slack variables represent unused resources (such as time on a machine or labor-hours available), they yield no profit, but we must add them to the objective function with zero profit coefficients.

Thus, the objective function becomes

$$F(x) = \frac{8x_1 + 9x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6 + 4}{2x_1 + 3x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 + 7}$$

After the addition of slack variables x_4, x_5 and x_6, the initial table can be written as: $\boldsymbol{T}^{(s=0)}$

Initial Simplex table $\boldsymbol{T}^{(0)}$

Basic variables $x_B^{(0)}$	Coefficients of basis in $P(x)$ $P_B^{(0)}$	Coefficients of basis in $Q(x)$ $Q_B^{(0)}$	8	9	4	0	0	0	Current values $X_B^{(0)}$
			2	3	2	0	0	0	
			$A_1^{(0)}$	$A_2^{(0)}$	$A_3^{(0)}$	$A_4^{(0)}$	$A_5^{(0)}$	$A_6^{(0)}$	
x_4	0	0	1	1	2	1	0	0	$x_4 = 3$
x_5	0	0	2	1	4	0	1	0	$x_5 = 4$
x_6	0	0	5	3	1	0	0	1	$x_6 = 15$
$Z_j^{'(0)} = P_B^{(0)} A_j^{(0)}$			0	0	0	0	0	0	$P(x) = 4$ $Q(x) = 7$ $F(x) = \frac{4}{7}$
$Z_j^{''(0)} = Q_B^{(0)} A_j^{(0)}$			0	0	0	0	0	0	
$\Delta_j^{'(0)} = Z_j^{'(0)} - p_j$			-8	-9	-4	0	0	0	
$\Delta_j^{''(0)} = Z_j^{''(0)} - q_j$			-2	-3	-2	0	0	0	
$\Delta_j^{(0)} = \Delta_j^{'(0)} - F(x)\Delta_j^{''(0)}$			$\frac{-48}{7}$	$\frac{-51}{7}$	$\frac{-20}{7}$	0	0	0	

$\boldsymbol{T}^{(0)}$ contains the problem formulation, which is in canonical form with x_4, x_5 and x_6 as basic variables and x_1, x_2 and x_3 as nonbasic variables at value zero. $\boldsymbol{T}^{(s=0)}$ is not optimal because $\Delta_j^{(0)} \le 0$. Hence now the table $\boldsymbol{T}^{(s=0)}$ is transformed and we obtain the table $\boldsymbol{T}^{(s=1)}$.

Simplex table $\boldsymbol{T}^{(1)}$:

Basic variables $x_B^{(1)}$	Coefficients of basis in $P(x)$ $P_B^{(1)}$	Coefficients of basis in $Q(x)$ $Q_B^{(1)}$	8	9	4	0	0	0	Current values $X_B^{(1)}$
			2	3	2	0	0	0	
			$A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$A_4^{(1)}$	$A_5^{(1)}$	$A_6^{(1)}$	
x_2	9	3	1	1	2	1	0	0	$x_2 = 3$
x_5	0	0	1	0	2	-1	1	0	$x_5 = 1$
x_6	0	0	2	0	-5	-3	0	1	$x_6 = 6$
$Z_j^{'(1)} = P_B^{(1)} A_j^{(1)}$			9	9	16	0	4	0	$P(x) = 31$ $Q(x) = 16$ $F(x) = \frac{31}{16}$
$Z_j^{''(1)} = Q_B^{(1)} A_j^{(1)}$			3	3	18	9	0	0	
$\Delta_j^{'(1)} = Z_j^{'(1)} - p_j$			1	0	14	9	0	0	
$\Delta_j^{''(1)} = Z_j^{''(1)} - q_j$			1	0	4	3	0	0	
$\Delta_j^{(1)} = \Delta_j^{'(1)} - F(x)\Delta_j^{''(1)}$			$\frac{-15}{16}$	0	$\frac{25}{4}$	$\frac{51}{16}$	0	0	

$\boldsymbol{T}^{(1)}$ contains the problem formulation, which is in canonical form with x_2, x_5 and x_6 as basic variables and x_1, x_3 and x_4 as nonbasic variables at value zero. $\boldsymbol{T}^{(1)}$ is not optimal because $\Delta_1^{(1)} < 0$. Hence now the table $\boldsymbol{T}^{(1)}$ is transformed and we obtain the table $\boldsymbol{T}^{(2)}$**.**

Simplex table $T^{(2)}$:

Basic variables $x_B^{(2)}$	Coefficients of basis in $P(x)$ $P_B^{(2)}$	Coefficients of basis in $Q(x)$ $Q_B^{(2)}$	8	9	4	0	0	0	Current values $X_B^{(2)}$
			2	3	2	0	0	0	
			$A_1^{(2)}$	$A_2^{(2)}$	$A_3^{(2)}$	$A_4^{(2)}$	$A_5^{(2)}$	$A_6^{(2)}$	
x_2	9	3	0	1	0	2	-1	0	$x_2 = 2$
x_1	8	2	1	0	2	-1	1	0	$x_1 = 1$
x_6	0	0	0	0	-9	-1	-2	1	$x_6 = 4$
$Z_j^{'(2)} = P_B^{(2)} A_j^{(2)}$			8	9	16	10	-1	0	$P(x) = 30$ $Q(x) = 15$ $F(x) = 2$
$Z_j^{''(2)} = Q_B^{(2)} A_j^{(2)}$			2	3	4	4	-1	0	
$\Delta_j^{'(2)} = Z_j^{'(2)} - p_j$			0	0	12	10	-1	0	
$\Delta_j^{''(2)} = Z_j^{''(2)} - q_j$			0	0	2	4	-1	0	
$\Delta_j^{(2)} = \Delta_j^{'(2)} - F(x)\Delta_j^{''(2)}$			0	0	8	2	1	0	

The table $\boldsymbol{T}^{(2)}$ is optimal because $\Delta_j^{(2)} \geq 0, j = 1,2,\ldots,6$ and $b_i^{(2)} \geq 0, i = 1,2,3$. Hence, thus the current basis is $x_B^{(2)} = \{x_2, x_1, x_6\}$ and the corresponding solution $x_B^* = \{x_2 = 2, x_1 = 1, x_6 = 4\}$. Moreover, the nonbasic variables $x_N^{(2)} = \{x_3, x_4, x_5\}$ and the corresponding solution $x_N^* = \{x_3 = 0, x_4 = 0, x_5 = 0\}$. Hence the optimal solution to the problem can be written as $x^* = (x_1 = 1\ x_2 = 2\ x_3 = 0\ x_4 = 0\ x_5 = 0\ x_6 = 4)^t$ with the associated value of the objective function $F\left(x^*\right) = \dfrac{P\left(x^*\right)}{Q\left(x^*\right)} = \dfrac{30}{15} = 2.$

$$B_B^{-1(2)} = \left(A_4^{(2)} A_5^{(2)} A_6^{(2)}\right)$$

$$or\ B_B^{-1(2)} = \left(a_{ij}^{(2)}\right)_{\substack{1\leq i\leq 3\\ 4\leq j\leq 6}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$$

5. Results and Discussion

To apply Sensitivity analysis of LFP problems, the optimal solution of Simplex method must be available. The essence of the sensitivity analysis is to examine how marginal changes in the parameter of the problem might affect the derived optimal solution. The most taught topics of sensitivity analysis at academic institutes comprise the following items:

1) Changes in the objective function Coefficients (p_j) and (q_j)

2) Changes in the Right-Hand-Side values of the constraints (b_i)

3) Changes in the Right-Hand-Side values of the constraints (a_{ij})

4) Adding a new decision variable (x_j)

As mentioned earlier, it is expected that the methods of performing sensitivity analysis taught in educational institutes should be easy to apply and short in procedures. In this paper, the author has developed and implemented simple methods for calculating sensitivity analysis that the author has used in teaching Operations Research courses for his long careers, in education. Demonstrations of these methods will be presented below.

5.1. Changes in the Right-Hand-Side Values of the Constraints b_i

If we replace $b = (b_1, \ldots, b_m)^t$ by $b' = (b_1, \ldots, b_i + \alpha_i, \ldots, b_m)^t$, The optimal solution will remain optimal if and only if

$$X_B^{(s)} + A_{n+i}^{(s)} \alpha_i \geq 0.$$

Example: Changes in the Right-Hand-Side values of the constraints b_2 into $b_2' = b_2 + \alpha_2$.

The optimal solution will remain optimal if and only if

$$X_B^{(2)} + \alpha_2 A_5^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow -1 \leq \alpha_2 \leq 2.$$

5.2. Changes in the Objective Function Coefficients p_j or q_j

Case 1: Changes in the objective function Coefficients p_j into $p_j' = p_j + \delta_j$ with $x_j \in x_B^{(s)}$.

The optimal solution will remain optimal if and only if

$$\left(a_{rd}^{(s)} Q(x^*) - x_j^* \Delta_d^{''(s)}\right)\delta_j - P(x^*)\Delta_d^{''(s)} + Q(x^*)\Delta_d^{'(s)} \geq 0$$

for all the $x_d \in x_N^{(s)}$ and r the number of the line $x_j \in x_B^{(s)}$ in Table $T^{(s)}$ optimal.

Example: Changes in the objective function Coefficients p_2 into $p_2' = p_2 + \delta_2$ with $x_2 \in x_B^{(s)}$.

In Table $T^{(2)}$ optimal: $x_B^{(2)} = \{x_2, x_1, x_6\}$, $x_N^{(2)} = \{x_3, x_4, x_5\}$, $d = 3,4,5$, $j = 2$, $r = 1$, $x_2^* = 2$, $Q(x^*) = 15$ and $P(x^*) = 30$.

The optimal solution will remain optimal if and only if

$$\begin{cases} \left(a_{13}^{(2)} Q(x^*) - x_2^* \Delta_3^{''(2)}\right)\delta_2 - P(x^*)\Delta_3^{''(2)} + Q(x^*)\Delta_3^{'(2)} \geq 0 \\ \left(a_{14}^{(2)} Q(x^*) - x_2^* \Delta_4^{''(2)}\right)\delta_2 - P(x^*)\Delta_4^{''(2)} + Q(x^*)\Delta_4^{'(2)} \geq 0 \\ \left(a_{15}^{(2)} Q(x^*) - x_2^* \Delta_5^{''(2)}\right)\delta_2 - P(x^*)\Delta_5^{''(2)} + Q(x^*)\Delta_5^{'(2)} \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} (0-4)\delta_2 - 60 + 180 \geq 0 \\ (30-8)\delta_2 - 120 + 150 \geq 0 \Rightarrow \\ (-15+2)\delta_2 + 30 - 15 \geq 0 \end{cases}$$

$$\begin{cases} (-4)\delta_2 + 120 \geq 0 \\ (22)\delta_2 + 30 \geq 0 \\ (-13)\delta_2 + 15 \geq 0 \end{cases} \Rightarrow \frac{-15}{11} \leq \delta_2 \leq \frac{15}{13}.$$

Case 2: Changes in the objective function Coefficients q_j into $q_j^{'} = q_j + \delta_j$ with $x_j \in x_B^{(s)}$.

The optimal solution will remain optimal if and only if

$$\left(x_j^* \Delta_d^{'(s)} - a_{rd}^{(s)} P(x^*)\right) \delta_j - P(x^*) \Delta_d^{''(s)} + Q(x^*) \Delta_d^{'(s)} \geq 0$$

for all the $x_d \in x_N^{(s)}$ and r the number of the line $x_j \in x_B^{(s)}$ in Table $T^{(s)}$ optimal.

Example: Changes in the objective function Coefficients q_2 into $q_2^{'} = q_2 + \delta_2$ with $x_2 \in x_B^{(s)}$.

In Table $T^{(2)}$ optimal: $x_B^{(2)} = \{x_2, x_1, x_6\}$, $x_N^{(2)} = \{x_3, x_4, x_5\}$, $d = 3,4,5$, $j = 2$, $r = 1$, $x_2^* = 2$, $Q(x^*) = 15$ and $P(x^*) = 30$.

The optimal solution will remain optimal if and only if

$$\begin{cases} (24-0)\delta_2 - 60 + 180 \geq 0 \\ (20-60)\delta_2 - 120 + 150 \geq 0 \\ (-2+30)\delta_2 + 30 - 15 \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} (24)\delta_2 + 120 \geq 0 \\ (-40)\delta_2 + 30 \geq 0 \\ (28)\delta_2 + 15 \geq 0 \end{cases} \Rightarrow \frac{-15}{28} \leq \delta_2 \leq \frac{3}{4}.$$

Case 3: Changes in the objective function Coefficients p_j into $p_j^{'} = p_j + \alpha_j$ with $x_j \in x_N^{(s)}$.

The optimal solution will remain optimal if and only if

$$\Delta_j^{(s)} - \alpha_j \geq 0 \text{ for } x_j \in x_N^{(s)}.$$

Example: Changes in the objective function Coefficients p_3 into $p_3^{'} = p_3 + \alpha_3$.

In Table $T^{(2)}$ optimal: $x_N^{(2)} = \{x_3, x_4, x_5\}$ and $j = 3$.

The optimal solution will remain optimal if and only if

$$\Delta_3^{(2)} - \alpha_3 = 8 - \alpha_3 \geq 0 \Rightarrow \alpha_3 \leq 8.$$

Case 4: Changes in the objective function Coefficients q_j into $q_j^{'} = q_j + \alpha_j$ with $x_j \in x_N^{(s)}$.

The optimal solution will remain optimal if and only if

$$\Delta_j^{(s)} + F(x^*)\alpha_j \geq 0 \text{ for } x_j \in x_N^{(s)}.$$

Example: Changes in the objective function Coefficients q_3 into $q_3^{'} = q_3 + \alpha_3$.

In Table $T^{(2)}$ optimal: $x_N^{(2)} = \{x_3, x_4, x_5\}$, $j = 3$ and $F(x^*) = 2$.

The optimal solution will remain optimal if and only if

$$\Delta_3^{(2)} + 2\alpha_3 = 8 + 2\alpha_3 \geq 0 \Rightarrow \alpha_3 \geq -4.$$

5.3. Changes in the Left-Hand-Side Values of the Constraints a_{ij} into $a_{ij}^{'} = a_{ij} + \alpha_i$

Let the new column matrix $A_j^{'} = \left(a_{1j}, \dots a_{ij} + \alpha_i, \dots, a_{mj}\right)^t$, The optimal solution will remain optimal if and only if

$$\Delta_j^{(s)} + \Delta_{n+i}^{(s)} \alpha_i \geq 0.$$

Example: Changes in the Left-Hand-Side values of the constraints a_{23} into $a_{23}^{'} = a_{23} + \alpha_2$

In Table $T^{(2)}$ optimal: $i = 2$, $j = 3$ $\Delta_3^{(2)} = 8$ and $\Delta_5^{(2)} = 1$.

The optimal solution will remain optimal if and only if

$$\Delta_3^{(2)} + \Delta_5^{(2)} \alpha_2 = 8 - \alpha_2 \geq 0 \Rightarrow \mu_2 \leq 8.$$

5.4. Adding a New Decision Variable x_{n+m+1}

Let the new matrix

$$A_{n+m+1} = \left(a_{1(n+m+1)}, \dots, a_{m(n+m+1)}\right)^t$$

coefficients of the variable x_{n+m+1}.

For $P(x)$, we have:

$$Z_{n+m+1}^{'(s)} = P_B^{(s)} \times A_{n+m+1}^{(s)} = P_B^{(s)} \times \left[B_B^{-1(s)} \times A_{n+m+1}\right]$$

and $\Delta_{n+m+1}^{'(s)} = Z_{n+m+1}^{'(s)} - p_{n+m+1}$ and for $Q(x)$, we have:

$$Z_{n+m+1}^{''(s)} = Q_B^{(s)} \times A_{n+m+1}^{(s)} = Q_B^{(s)} \times \left[B_B^{-1(s)} \times A_{n+m+1}\right]$$

and $\Delta_{n+m+1}^{''(s)} = Z_{n+m+1}^{''(s)} - p_{n+m+1}$. The optimal solution will remain optimal if and only if

$$\Delta_{n+m+1}^{(s)} \geq 0$$

else if

$$\Delta_{n+m+1}^{(s)} < 0,$$

the optimal solution will not remain optimal.

Example 1: The optimal solution will remain optimal if and only if $\Delta_{n+m+1}^{(s)} \geq 0$.

Adding a new decision variable x_7 with $\Delta_7^{(2)} \geq 0$.

By inserting the new decision variable x_7 the problem becomes:

$$F(x) = \frac{8x_1 + 9x_2 + 4x_3 + 5x_7 + 4}{2x_1 + 3x_2 + 2x_3 + x_7 + 7} \to \max$$

$$\text{Subject to} \begin{cases} x_1 + x_2 + 2x_3 + 2x_7 \leq 3 \\ 2x_1 + x_2 + 4x_3 + x_7 \leq 4 \\ 5x_1 + 3x_2 + x_3 + 2x_7 \leq 15 \\ x_j \geq 0, j = 1,2,3,7. \end{cases}$$

By inserting the following slack variables x_4, x_5 and x_6 in the constraints, we will have:

$$F(x) = \frac{8x_1 + 9x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6 + 5x_7 + 4}{2x_1 + 3x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 + x_7 + 7} \to \max$$

$$\text{Subject to} \begin{cases} x_1 + x_2 + 2x_3 + x_4 + 0x_5 + 0x_6 + 2x_7 = 3 \\ 2x_1 + x_2 + 4x_3 + 0x_4 + x_5 + 0x_6 + x_7 = 4 \\ 5x_1 + 3x_2 + x_3 + 0x_4 + 0x_5 + x_6 + 2x_7 = 15 \\ x_j \geq 0, j = 1,2,3,4,5,6,7. \end{cases}$$

The optimal solution will remain optimal if and only if $\Delta_7^{(s)} \geq 0$:

For $m = 3$, $n = 3$ and $F(x^*) = 2$, we have $\Delta_7^{(2)} \geq 0$.

For $\Delta_7^{'(2)} = Z_7^{'(2)} - p_7 = P_B^{(2)} \times \left[B_B^{-1(2)} \times A_7\right] - 5$ we have $\Delta_7^{'(2)} = (980)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 5 = 14$ and

For $\Delta_7^{''(2)} = Z_7^{''(2)} - q_7 = Q_B^{(2)} \times \left[B_B^{-1(2)} \times A_7 \right] - 1$ we have $\Delta_7^{''(2)} = (320)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 1 = 6.$ Therefore,

$$\Delta_7^{(2)} = \Delta_7^{'(2)} - F\left(x^*\right)\Delta_7^{''(2)} = 14 - 2(6) = 14 - 12 = 2 \geq 0.$$

So the activity (or the addition of a new decision variable x_7) is not profitable.

Example 2: The optimal solution will remain optimal if and only if $\Delta_{n+m+1}^{(s)} < 0.$

Adding a new decision variable x_7 with $\Delta_7^{(2)} < 0$:

By inserting the new decision variable x_7 the problem becomes:

$$F(x) = \frac{8x_1 + 9x_2 + 4x_3 + 10x_7 + 4}{2x_1 + 3x_2 + 2x_3 + x_7 + 7} \rightarrow \max$$

$$\text{Subject to } \begin{cases} x_1 + x_2 + 2x_3 + 2x_7 \leq 3 \\ 2x_1 + x_2 + 4x_3 + x_7 \leq 4 \\ 5x_1 + 3x_2 + x_3 + 2x_7 \leq 15 \\ x_j \geq 0,\ j = 1,2,3,7. \end{cases}$$

By inserting the following slack variables x_4, x_5 and x_6 in the constraints, we will have:

$$F(x) = \frac{8x_1 + 9x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6 + 10x_7 + 4}{2x_1 + 3x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 + x_7 + 7} \rightarrow \max$$

$$\text{Subject to } \begin{cases} x_1 + x_2 + 2x_3 + x_4 + 0x_5 + 0x_6 + 2x_7 = 3 \\ 2x_1 + x_2 + 4x_3 + 0x_4 + x_5 + 0x_6 + x_7 = 4 \\ 5x_1 + 3x_2 + x_3 + 0x_4 + 0x_5 + x_6 + 2x_7 = 15 \\ x_j \geq 0,\ j = 1,2,3,4,5,6,7. \end{cases}$$

$\Delta_7^{'(2)} = (980)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 10 = 9$ and

$\Delta_7^{''(2)} = (320)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 1 = 6.$ So

$$\Delta_7^{(2)} = \Delta_7^{'(2)} - F\left(x^*\right)\Delta_7^{''(2)} = -3 < 0.$$

In the table $T^{(2)}$, we will insert the new column $A_7^{(2)} = B_B^{-1(2)} A_7 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$ and we rewrite $T^{(2)}$ into $T^{new(2)}$

Simplex table $T^{new(2)}$:

Basic variables $x_B^{(2)}$	Coefficients of basis in $P(x)$ $P_B^{(2)}$	Coefficients of basis in $Q(x)$ $Q_B^{(2)}$	8	9	4	0	0	0	10	Current values $X_B^{(2)}$
			2	3	2	0	0	0	1	
			$A_1^{(2)}$	$A_2^{(2)}$	$A_3^{(2)}$	$A_4^{(2)}$	$A_5^{(2)}$	$A_6^{(2)}$	$A_7^{(2)}$	
x_2	9	3	0	1	0	2	-1	0	3	$x_2 = 2$
x_1	8	2	1	0	2	-1	1	0	-1	$x_1 = 1$
x_6	0	0	0	0	-9	-1	-2	1	-2	$x_6 = 4$
$Z_j^{'(2)} = P_B^{(2)} A_j^{(2)}$			8	9	16	10	-1	0	19	$P(x) = 30$ $Q(x) = 15$ $F(x) = 2$
$Z_j^{''(2)} = Q_B^{(2)} A_j^{(2)}$			2	3	4	4	-1	0	7	
$\Delta_j^{'(2)} = Z_j^{'(2)} - p_j$			0	0	12	10	-1	0	9	
$\Delta_j^{''(2)} = Z_j^{''(2)} - q_j$			0	0	2	4	-1	0	6	
$\Delta_j^{(2)} = \Delta_j^{'(2)} - F(x)\Delta_j^{''(2)}$			0	0	8	2	1	0	-3	

$\Delta_7^{(2)} = -3 < 0$, then the optimal $x_B^{(2)} = \{x_2, x_1, x_6\}$ is destabilized. Let's apply the simplex again to find the new optimal base or the new optimal solution. After two iterations, we see that the table $\boldsymbol{T^{new(s=4)}}$ is optimal. The new optimal solution is: $x_B^{(4)} = \{x_7, x_5, x_6\}$ and $x_B^* = \left\{x_7 = \frac{3}{2}, x_5 = \frac{5}{2}, x_6 = 12\right\}$. Moreover

$$x_N^{(4)} = \{x_1, x_2, x_3, x_4\}$$
$$and\ x_N^* = \{x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0\}$$

and

$$x^* = \begin{pmatrix} x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, \\ x_5 = \frac{5}{2}, x_6 = 12, x_7 = \frac{3}{2} \end{pmatrix}^t$$

with $P(x^*) = 19$, $Q(x^*) = \frac{17}{2}$ and

$$F(x^*) = \frac{P(x^*)}{Q(x^*)} = \frac{38}{17}.$$

6. Conclusions

Shortcut methods were presented in this paper to produce sensitivity analysis of linear fractional programming models. Four different topics on sensitivity were taken into account: changes in the model parameters, i.e., changes on objective function coefficients, Changes in the Right-Hand-Side values of the constraints, Changes in the Left-Hand-Side values of the constraints, and Adding a new decision variable.

The analysis has suggested a few shortcut methods to perform the sensitivity analysis that can be used in operations research and quantitative methods textbooks to be taught in educational institutes. It is very straightforward and less time is demanding to apply compared to current methods used by leading books around the world.

This research is a significant contribution in the sense that it will assist the management and business students at different universities in making correct decisions by using very short and easy-to-calculate methods concerning the sensitivity analysis of linear fractional programming problems.

The authors is planning to carry out research to analyze few well-known software in OR to support the investigation of such issues.

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