American Journal of Modeling and Optimization, 2014, Vol. 2, No. 1, 1-7
Available online at http://pubs.sciepub.com/ajmo/2/1/1

DOI:10.12691/ajmo-2-1-1

Performance Analysis of M^X / (G_1, G_2) / 1 Retrial Queueing Model with Second Phase Optional Service and Bernoulli Vacation Schedule Using PGF Approach[22]

Vishwa Nath Maurya*

Department of Mathematics, School of Science & Technology, University of Fiji, Saweni / Suva, Fiji Islands
*Corresponding author: prof.drvnmaurya@gmail.com

Received October 19, 2013; Revised January 09, 2014; Accepted January 20, 2014

Abstract Present paper describes with the bulk arrival retrial queueing M^X / G_1, G_2, / 1 model with two phase service and Bernoulli vacation schedule wherein first phase service is essential and the next second phase service is optional. If the second phase service is not demanded by arriving customer then the single server takes a vacation period according to Bernoulli vacation schedule in order to utilize it to complete some supplementary work and such type of vacation is assumed working vacation. In the queueing model taken into present consideration, the concepts of Bernoulli vacation schedule and next optional service have been incorporated along with realistic provision that the server has an option to avail a vacation with probability p (q) or may continue to serve the next customer, if any with complementary probability $\overline{p}(\overline{q})$ just after the completion of first phase essential service i.e. before the commencement of second phase optional service. In the present paper, our central goal is to investigate the steady state behavior of the bulk arrival retrial queueing M^X / G_1, G_2, / 1 model with two phase service and Bernoulli vacation schedule. By introducing supplementary variables, Chapman Kolmogorov equations are established and then the probability generating functions (PGFs) for first phase essential service (FPES), next phase optional service (NPOS) and working vacation and for the number of the customers in the orbit at an arbitrary epoch are investigated successfully. Besides investigating PGF for different states of the queueing system taken into consideration, performance measures such as the long run probabilities for which the server is in idle state, FPES state, NPOS state and in working vacation state are also explored in order to focus the application aspect of the investigated PGFs. By the end of the present paper, some numerical illustrations of the investigated results for M^X / E_k / 1 model as special case of M^X / G_1, G_2, / 1 have been presented in Table 1 and Table 2 for varying parameters.

Keywords: *bulk arrival retrial queue, state dependent rate, supplementary variables, chapman-kolmogorov equations, probability generating function approach, Bernoulli vacation schedule, principle of maximum entropy, performance measures*

Cite This Article: Vishwa Nath Maurya, "Performance Analysis of M^X / (G_1, G_2) / 1 Retrial Queueing Model with Second Phase Optional Service and Bernoulli Vacation Schedule Using PGF Approach." *American Journal of Modeling and Optimization*, vol. 2, no. 1 (2014): 1-7. doi: 10.12691/ajmo-2-1-1.

1. Introduction

Recent research studies in queueing literature reveal that performance analysis of bulk arrival retrial queue with server vacation or server breakdown and repair model has become an integral measure of queueing problems in order to enable significant inferences which are utmost useful for system designers and decision makers. Policy of server vacation schedule in bulk arrival retrial queueing model is reasonably considered to complete some additional work so that the server cost can be ultimately reduced. In other words, provision of server vacation is an alternate policy to reduce the idle period of the server and to utilize the server at the maximum extent if vacation period of server is used to do some additional work. These are the basic motives why the servers are allowed to take vacations for fixed or variable periods. In order to make more realistic and versatile for analyzing the real-world congestion problems, we consider here a bulk arrival retrial queueing M^X / G_1, G_2, / 1 model with two phase service and Bernoulli vacation schedule wherein first phase service is essential and the next second phase service is optional. Applications of retrial queueing model with server vacation can be found in many congestion situations such as in production systems, manufacturing systems, data communication networks, call centers, distribution and service sectors etc. Thus retrial phenomenon in queueing systems is common in our day-to-day life. Any arriving batch enters a virtual pool of blocked customers called 'orbit' when the server is busy or in working vacation; otherwise one customer from the arriving batch gets the service immediately while the rest customers join the retrial group (orbit). A wide range of works done on retrial queue with server vacation model can be found in queueing literature. Here it is worth mentioning to reveal the earlier work done by several noteworthy researchers [1,2-17,19,20-27] who have studied rigorously the retrial queueing models in diverse frameworks. For evidence in this connection, we refer the noble book by Tian and Zhang [24]. The comprehensive survey on this topic has been established in Doshi [10,11]. Chakravarthy and Dudin [4] studied a retrial queueing model with two types of customers wherein arrival pattern follows Markovian process. Subsequently, Choudhury and Madan [7] considered a bulk arrival queueing system wherein the server delivers two phases of heterogeneous service and succeeded to investigate the queue size distribution at random epochs of the system states along with various vital performance measures. Atencia and Moreno [2] studied an $M / G / 1$ retrial queue with general retrial times and Bernoulli schedule and they derived the generating function of the system size distribution and explored also the stochastic decomposition law. Wu *et al.* [27] examined the retrial queues with general service times and non exponential retrial time distribution. Sherman and Kharoufeh [23] analyzed an unreliable $M / M / 1$ retrial queue with infinite-capacity orbit and succeeded to investigate the stability conditions as well as several stochastic decomposability results. Moreover, Wang *et al.* [26] examined an $M / G / 1$ retrial queueing system with disasters and unreliable server and they established the Laplace transforms both of the transient solutions and steady-state solutions for queueing and reliability measures of interest. Choudhury and Deka [5] considered $M^X / G / 1$ queueing model with two phases of heterogeneous service under Bernoulli vacation schedule and classical retrial policy. By making use of the embedded Markov chain technique, Choudhury and Deka [5] determined the steady state distribution of the server state and the number of the customers in the retrial group. Furthermore, Amandor and Artalejo [1] have focused their attention to study on $M / G / 1$ retrial queue to determine the distribution of the successful and blocked events made by the primary customers and the retrial customers. However, by the same time Boualem *et al.* [3] considered $M / G / 1$ retrial queue with server vacations and they explored several stochastic comparison properties for the stationary queue length distribution. Recently, Choudhury and Deka [6] analyzed rigorously the steady state behavior of $M^X / G / 1$ unreliable retrial queue with Bernoulli admission mechanism.

It has been keenly observed that in many realistic queueing situations, usually jobs demand the first phase "essential" service, whereas only some of them demand the next phase "optional" service. Wang [25] analyzed $M / G / 1$ queue with second phase optional service and unreliable server and they achieved to establish both the transient and steady-state solutions by using a supplementary variable technique. Later, a single server queue with two phases of heterogeneous service and linear retrial policy under Bernoulli vacation schedule was analyzed by Madan and Choudhury [17]. Moreover, Choudhury and Paul [8] considered to examine a queueing model wherein the server provides two phases of heterogeneous service to each customer in succession with Bernoulli vacation schedule under different vacation policies. Furthermore, Choudhury *et al.* [9] analyzed the steady state behavior of a bulk arrival queue and Bernoulli schedule vacation under multiple vacation policy and they obtained successfully the queue size distribution of idle period process. Of late, the steady state behavior of a $M / G / 1$ retrial queue with an additional second phase of optional service was also examined by Choudhury and Deka [5]. Besides these significant research works, some other noteworthy researchers have paid their keen interest to explore a variety of retrial queue with different versions. Among them, Ke and Chang [14] investigated a bulk arrival retrial queue with general retrial times where the server offers two phases of heterogeneous service to all the customers under Bernoulli vacation schedules. Ke and Lin [15] examined the $M^X / G / 1$ queueing system with server vacations and they investigated a comparative analysis between the approximate results with established exact results for vacation time, service time and repair time distributions by using the principle of maximum entropy.

In the present paper, our keen interest is to deal with state dependent $M^X / G_1, G_2, / 1$ retrial queueing system under Bernoulli vacation schedule with first phase essential service and second phase optional service wherein our central aim is to investigate probability generating functions for first phase essential service (FPES), next phase optional service (NPOS) and working vacation and for the number of the customers in the orbit at an arbitrary epoch along with some significant performance measures. It is remarkable here that our present investigation is motivated by the work of Kumar and Arumuganathan [16], Atencia and Moreno [2] and Maurya [18] wherein following three additional features are embraced:

(i) State dependent arrival rates.
(ii) working vacation and.
(iii) second phase optional service.

2. Description and Assumptions of the $M^X / (G_1, G_2) / 1$ Model

In the present paper, we envisage a single server retrial queueing system with first phase essential and second phase optional service. As it is obvious here that the server offers customers to provide his services in two phases, where service of the first phase is essential, however, service of second phase is optional. We remark here that all the arriving customers have to get the essential service whereas next phase optional service (NPOS) is provided only to those customers who demand for the same. As soon as the FPES (NPOS) of the customers is completed, the server may go for vacation with probability $p(q)$ or may continue to serve the next customer, if any with probability $\overline{p}(\overline{q})$. During his vacation period, the server may do some additional work with a different service rate and such type of vacation of server is assumed as working vacation. After the completion of FPES if the customer demands for the NPOS, then server may provide the NPOS with probability σ or becomes idle with probability $\overline{\sigma}$. We assume that the customers arrive in batches with a fixed batch size according to Poisson process with batch size distribution Cj and service times of FPES, NPOS and working vacation are distributed according to general service time distribution with mean service times $\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\mu_3}$ respectively. In the retrial group, the time between the two successive attempts of each customer is considered to be exponentially distributed with rate *v*. For the sake of presentation and mathematical formulation of the model, Let us consider a set of following assumptions:

X; the random variable denoting the batch size with batch size distribution as defined by

$$c_j = \Pr[c = k], k = 1, 2, \ldots\ldots, d$$

and the generating function for the batch size distribution is given by

$$C(z) = \sum_{k=1}^{\infty} z^k c_k$$

possessing its mean and variance respectively $C'(1) = E[X]$ and $C''(1) = E[X^2]$

N(t); the number of customers present in the system at time *t*.

A(t); the random variable denoting the server's state at time *t*; where *A(t)* is defined as following for different states:

$$A(t) = \begin{cases} 0, & \text{if the server is in idle state} \\ 1, & \text{if the server is in FPES state} \\ 2, & \text{if the server is in NPOS state} \\ 3, & \text{if the server is on working vacation state} \end{cases}$$

Moreover, λ_i; the state dependent arrival rates of the customers are given as follows:

$$\lambda_i = \begin{cases} \lambda_0, & \text{if the server is in idle state} \\ \lambda_1, & \text{if the server is in FPES state} \\ \lambda_2, & \text{if the server is in NPOS state} \\ \lambda_3, & \text{if the server is on working vacation state} \end{cases}$$

In addition to these, we use following notations for cumulative distribution function (CDF), probability distribution function (PDF), Laplace-Stieltijes transformation (LST) and the remaining service time (RST) or remaining working vacation time (RVT), respectively of FPES, NPOS and working vacation.

State	CDF	PDF	LST	RST/RVT
FPES	$S_1(x)$	$s_1(x)$	$\tilde{S}_1(\theta)$	$S_1^0(x)$
NPOS	$S_2(x)$	$s_2(x)$	$\tilde{S}_2(\theta)$	$S_2^0(x)$
Working Vacation	$S_3(x)$	$s_3(x)$	$\tilde{S}_3(\theta)$	$S_3^0(x)$

The steady state probabilities to construct the governing equations are defined as following:

$$P_{0,n}(t)dt = \Pr\{N(t) = n, A(t) = 0\}, \quad n \geq 0$$

$$P_{i,n}(x,t)dt = \Pr\{N(t) = n, A(t) = i, x \leq S_i^0(t) \leq x + dx\}, \quad n \geq 0,$$

$i = 1, 2, 3.$

The r[th] moment of FPES, NPOS and working vacation states are denoted by $E[S_1^r]$, $E[S_2^r]$, and $E[S_3^r]$, where $r \geq 1$. Thus, we have following expressions to obtain $E[S_i^r]$; $i = 1, 2, 3$.

$$E[S_1^r] = (-1)^r S_1^{'(r)}(0);$$

$$E[S_2^r] = (-1)^r S_2^{'(r)}(0);$$

$$E[S_3^r] = (-1)^r S_3^{'(r)}(0);$$

$$r \geq 1$$

The Laplace transforms of probabilities $P_{1,n}(x)$, $P_{2,n}(x)$, and $P_{3,n}(x)$ are denoted by $\tilde{P}_{1,n}(\theta)$, $\tilde{P}_{2,n}(\theta)$, and $\tilde{P}_{3,n}(\theta)$ respectively, so that $\tilde{P}_{i,n}(\theta)$; $i = 1, 2, 3$ can be expressed as follows

$$\tilde{P}_{1,n}(\theta) = \int_0^{\infty} e^{-x\theta} P_{1,n}(x)dx, \quad \tilde{P}_{2,n}(\theta) = \int_0^{\infty} e^{-x\theta} P_{2,n}(x)dx,$$

and $\tilde{P}_{3,n}(\theta) = \int_0^{\infty} e^{-x\theta} P_{3,n}(x)dx$.

3. Chapman Kolmogorov Equations Governing the States of the Model

Using the supplementary variable technique, we construct the Chapman Kolmogorov equations as follows:

$$(\lambda_0 + nv)P_{0,n} = P_{3,n}(0) + \bar{q}P_{2,n}(0) + (p\sigma + \bar{p}\bar{\sigma})P_{1,n}(0) \quad (3.1)$$

$$-\frac{d}{dx}P_{1,n}(x) = -\lambda_1 P_{1,n}(x) + \lambda_0 \sum_{k=1}^{n+1} c_k P_{0,n-k+1} S_1(x) + (n+1)vP_{0,n+1}S_1(x) + \lambda_1 \sum_{k=1}^{n} c_k P_{1,n-k}(x) \quad (3.2)$$

$$-\frac{d}{dx}P_{2,n}(x) = -\lambda_2 P_{2,n}(x) + \lambda_2 \sum_{k=1}^{n} c_k P_{2,n-k}(x) + \bar{p}\sigma P_{1,n}(0) S_2(x) \quad (3.3)$$

$$-\frac{d}{dx}P_{3,n}(x) = -\lambda_3 P_{3,n}(x) + \lambda_3 \sum_{k=1}^{n} c_k P_{3,n-k}(x) + qP_{2,n}(0)S_3(x) + p\bar{\sigma}P_{1,n}(0)S_3(x) \quad (3.4)$$

Define the probability generating functions (PGF)

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{0,n} \quad (3.5)$$

$$\tilde{P}_i(z,\theta) = \sum_{n=0}^{\infty} z^n \tilde{P}_{i,n}(\theta), \quad i = 1, 2, 3 \quad (3.6)$$

$$P_i(z,0) = \sum_{n=0}^{\infty} z^n P_{i,n}(0), \quad i = 1, 2, 3 \quad (3.7)$$

$$\tilde{P}_i(z,0) = \sum_{n=0}^{\infty} z^n \tilde{P}_{i,n}(0), \quad i = 1, 2, 3 \quad (3.8)$$

4. Determination of Probability Generating Functions

In this section, our central attention is to find out the partial probability generating functions for different cases of FPES, NPOS and working vacation. In order to serve our present goal, we state following theorems 4.1-4.3:

Theorem 4.1: The partial probability generating functions for the state of FPES is given by

$$\tilde{P}_1(z,0)=\frac{\left(\tilde{S}_1(\lambda_1-\lambda_1C(z)-1)\right)\left(\lambda_0P_0(z)\frac{C(z)}{z}+vP_0^{'}(z)\right)}{(\lambda_1-\lambda_1C(z))} \quad (4.1)$$

Theorem 4.2: The partial probability generating functions for the state of NPOS is given by

$$\tilde{P}_2(z,0)=\frac{\overline{p}\sigma P_1(z,0)\left(\tilde{S}_2(\lambda_2-\lambda_2C(z)-1\right)}{(\lambda_2-\lambda_2C(z))} \quad (4.2)$$

Theorem 4.3: The partial probability generating functions for the state of working vacation is given by

$$\tilde{P}_3(z,0)=\frac{\left(\tilde{S}_3(\lambda_3-\lambda_3C(z))-1\right)\left(q\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))+\overline{\sigma}\right)P_1(z,0)}{(\lambda_3-\lambda_3C(z))} \quad (4.3)$$

Proof:

Taking the LST of equations (3.2)-(3.4), we have

$$\theta\tilde{P}_{1,n}(\theta)-P_{1,n}(0)=\lambda_1\tilde{P}_{1,n}(\theta)-\lambda_0\sum_{k=1}^{n+1}c_kP_{0,n-k+1}\tilde{S}_1(\theta)-(n+1)vP_{0,n+1}\tilde{S}_1(\theta)-\lambda_1\sum_{k=1}^{n}c_k\tilde{P}_{1,n-k}(\theta) \quad (4.4)$$

$$\theta\tilde{P}_{2,n}(\theta)-P_{2,n}(0)=\lambda_2\tilde{P}_{2,n}(\theta)-\lambda_2\sum_{k=1}^{n}c_k\tilde{P}_{2,n-k}(\theta)-\overline{p}\sigma P_{1,n}(0)\tilde{S}_2(\theta) \quad (4.5)$$

$$\theta\tilde{P}_{3,n}(\theta)-P_{3,n}(0)=\lambda_3\tilde{P}_{3,n}(\theta)-\lambda_3\sum_{k=1}^{n}c_k\tilde{P}_{3,n-k}(\theta)-qP_{2,n}(0)\tilde{S}_3(x)-p\overline{\sigma}P_{1,n}(0)\tilde{S}_3(\theta) \quad (4.6)$$

We can easily obtain following equations on multiplying equations (3.1) and (4.4)-(4.6) by appropriate powers of *z* and then summing over *n*:

$$\lambda_0P_0(z)+vzP_0^{'}(z)=P_3(z,0)+\overline{q}P_2(z,0)+(p\sigma+\overline{p}\overline{\sigma})P_1(z,0) \quad (4.7)$$

$$(\theta-\lambda_1+\lambda_1C(z))\tilde{P}_1(z,\theta)=P_1(z,0)-\frac{\lambda_0C(z)}{z}P_0(z)\tilde{S}_1(\theta)-vP_0^{'}(z)\tilde{S}_1(\theta) \quad (4.8)$$

$$(\theta-\lambda_2+\lambda_2C(z))\tilde{P}_2(z,\theta)=P_2(z,0)-\overline{p}\sigma P_1(z,0)\tilde{S}_2(\theta) \quad (4.9)$$

$$(\theta-\lambda_3+\lambda_3C(z))\tilde{P}_3(z,\theta)=P_3(z,0)-qP_2(z,0)\tilde{S}_3(\theta)-p\overline{\sigma}P_1(z,0)\tilde{S}_3(\theta) \quad (4.10)$$

Substituting $\theta=\lambda_1-\lambda_1C(z),\lambda_2-\lambda_2C(z),\lambda_3-\lambda_3C(z)$ in equations (4.8)-(4.10), respectively, we have

$$P_1(z,0)=\frac{\lambda_0C(z)}{z}P_0(z)\tilde{S}_1(\lambda_1-\lambda_1C(z))+vP_0^{'}(z)\tilde{S}_1(\lambda_1-\lambda_1C(z)) \quad (4.11)$$

$$P_2(z,0)=\overline{p}\sigma P_1(z,0)\tilde{S}_2(\lambda_2-\lambda_2C(z)) \quad (4.12)$$

$$P_3(z,0)=qP_2(z,0)\tilde{S}_3(\lambda_3-\lambda_3C(z))+p\overline{\sigma}P_1(z,0)\tilde{S}_3(\lambda_3-\lambda_3C(z)) \quad (4.13)$$

Substituting the values of $P_1(z,0)$, $P_2(z,0)$, and $P_3(z,0)$ into equation (4.7), we get

$$\lambda_0P_0(z)+vzP_0^{'}(z)=\left[\begin{array}{l}\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(z))+\overline{q})\\+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(z))+(p\sigma+\overline{p}\overline{\sigma})\end{array}\right]\times\left[\frac{\lambda_0C(z)}{z}P_0(z)+vP_0^{'}(z)\right]\tilde{S}_1(\lambda_1-\lambda_1C(z)) \quad (4.14)$$

Equation (4.14) yields,

$$P_0^{'}(z)=\frac{\lambda_0}{v}\left(\frac{\begin{array}{l}1-[\tilde{S}_1(\lambda_1-\lambda_1C(z))(\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(z))+\overline{q})+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))\\+(p\sigma+\overline{p}\overline{\sigma}))]\frac{C(z)}{z}\end{array}}{\begin{array}{l}[\tilde{S}_1(\lambda_1-\lambda_1C(z))(\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(z))\\+\overline{q})+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(z))\\+(p\sigma+\overline{p}\overline{\sigma}))]-z\end{array}}\right)P_0(z) \quad (4.15)$$

which is a linear differential equation. On integrating (4.15), we obtain following solution of equation (4.15) in equation (4.16)

$$P_0(z)=P_0(1)\times\exp\left[-\frac{\lambda_0}{v}\int_z^1\left(\frac{\begin{array}{l}1-[\tilde{S}_1(\lambda_1-\lambda_1C(u))\\(\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(u))(q\tilde{S}_3(\lambda_3-\lambda_3C(u))+\overline{q})\\+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))+(p\sigma+\overline{p}\overline{\sigma}))]\frac{C(u)}{u}\end{array}}{\begin{array}{l}[\tilde{S}_1(\lambda_1-\lambda_1C(u))(\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(u))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(u))+\overline{q})+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))\\+(p\sigma+\overline{p}\overline{\sigma}))]-u\end{array}}\right)du\right] \quad (4.16)$$

Using result from equation (4.12) into equation (4.9), we have

$$(\theta-\lambda_1+\lambda_1C(z))\tilde{P}_1(z,\theta)=\left(\tilde{S}_1(\lambda_1-\lambda_1C(z))-\tilde{S}_1(\theta)\right)\left[\begin{array}{l}\frac{\lambda_0C(z)}{z}P_0(z)\\+vP_0^{'}(z)\end{array}\right] \quad (4.17)$$

Proceeding in similar way, from equations (4.13)-(4.14) and equations (4.10)-(4.11) respectively, we get

$$(\theta-\lambda_2+\lambda_2C(z))\tilde{P}_2(z,\theta)=\overline{p}\sigma P_1(z,0)\left(\tilde{S}_2(\lambda_2-\lambda_2C(z))-\tilde{S}_2(\theta)\right) \quad (4.18)$$

$$(\theta-\lambda_3+\lambda_3C(z))\tilde{P}_3(z,\theta)=\left(qP_2(z,0)+p\overline{\sigma}P_1(z,0)\right)\left(\tilde{S}_3(\lambda_3-\lambda_3C(z))-\tilde{S}_3(\theta)\right) \quad (4.19)$$

Using the partial generating functions $\tilde{P}_i(z,0)=\sum_{n=0}^{\infty}z^n\tilde{P}_{i,n}(0), i=1,2,3$ and equations (4.17)-(4.19), we finally obtain the theorems 4.1-4.3.

Theorem 4.4: The probability generating function of the number of the customers in the orbit is given by

$$P(z)=\frac{P_0(z)\begin{bmatrix}\psi-\lambda_0(\lambda_2\lambda_3(\tilde{S}_1(\lambda_1-\lambda_1C(z)-1)\\+\lambda_1\lambda_3\overline{p}\sigma\tilde{S}_1(\lambda_1-\lambda_1C(z)(\tilde{S}_2(\lambda_2-\lambda_2C(z)-1)\\+\lambda_1\lambda_2\tilde{S}_1(\lambda_1-\lambda_1C(z)(\tilde{S}_3(\lambda_3-\lambda_3C(z)-1)\cdot\\(\overline{p}\sigma q\tilde{S}_2(\lambda_2-\lambda_2C(z))+\overline{\sigma})\end{bmatrix}}{\psi}$$

where $P_0(z)$, $P_0(1)$, ψ and ρ are as follows

$$P_0(z)=P_0(1)\times\exp\left[-\frac{\lambda_0}{v}\int_z^1\left(\frac{\begin{matrix}1-[\tilde{S}_1(\lambda_1-\lambda_1C(u))\\(\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(u))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(u))+\overline{q})\\+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))\\+(p\sigma+\overline{p}\overline{\sigma}))]\frac{C(u)}{u}\end{matrix}}{\begin{matrix}[\tilde{S}_1(\lambda_1-\lambda_1C(u))(\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(u))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(u))+\overline{q})\\+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))\\+(p\sigma+\overline{p}\overline{\sigma}))]-u\end{matrix}}\right)du\right]$$

$$P_0(1)=\frac{1-\rho}{\left[1-\rho+\lambda_0E[X][E[S_1]+\overline{p}\sigma E[S_2]+(\overline{p}\sigma q+\overline{\sigma})E[S_3]]\right]}$$

$$\psi=\lambda_1\lambda_2\lambda_3\left[\tilde{S}_1(\lambda_1-\lambda_1C(z))\begin{pmatrix}\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))\\(q\tilde{S}_3(\lambda_3-\lambda_3C(z))+\overline{q})\\+p\overline{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(z))\\+(p\sigma+\overline{p}\overline{\sigma}))\end{pmatrix}\right]$$

$$\rho=\overline{p}\sigma E[X]\left[\lambda_1E[S_1]+\lambda_2E[S_2]+q\lambda_3E[S_3]\right]+p\overline{\sigma}E[X]\left[\lambda_1E[S_1]+\lambda_3E[S_3]\right]+(p\sigma+\overline{p}\overline{\sigma})\lambda_1E[S_1]E[X]$$

Proof of Theorem 4.4:

The probability generating function of the number of the customers in the orbit at an arbitrary epoch can be expressed as

$$P(z)=P_0(z)+\tilde{P}_1(z,0)+\tilde{P}_2(z,0)+\tilde{P}_3(z,0) \quad (4.20)$$

Substituting the values of $\tilde{P}_1(z,0)$, $\tilde{P}_2(z,0)$ and $\tilde{P}_3(z,0)$ from theorems 4.1-4.3 in equation (4.20) we readily get

$$P(z)=P_0(z)+\frac{\left(\tilde{S}_1(\lambda_1-\lambda_1C(z)-1\right)\left(\lambda_0P_0(z)\frac{C(z)}{z}+vP_0'(z)\right)}{(\lambda_1-\lambda_1C(z))}+\frac{\overline{p}\sigma P_1(z,0)\left(\tilde{S}_2(\lambda_2-\lambda_2C(z)-1\right)}{(\lambda_2-\lambda_2C(z))}+\frac{\left(\tilde{S}_3(\lambda_3-\lambda_3C(z)-1\right)\left(q\overline{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))+\overline{\sigma}\right)P_1(z,0)}{(\lambda_3-\lambda_3C(z))} \quad (4.21)$$

Now one can obtain value of $P_0(1)$ by the normalizing condition given by

$$P(1)=P_0(1)+\tilde{P}_1(1,0)+\tilde{P}_2(1,0)+\tilde{P}_3(1,0) \quad (4.22)$$

Using some algebraic manipulations and the normalizing condition, it is fairly easy to establish the theorem-4.4.

5. Performance Evaluation of the M^X / (G_1, G_2) / 1 Model

In this section, we derive the expressions for some performance measures to envisage the behavior of the system taken into present consideration. To serve our present purpose, some significant performance measures are evaluated and are stated in the following theorems 5.1-5.4.

Theorem 5.1: The long run probability of the server in idle state is denoted by *P* (*I*) and given as follows

$$P(I)=P_0(1)$$

Theorem 5.2: The long run probability of the server in FPES state is denoted by *P* (*E*) and given as follows

$$P(E)=\tilde{P}_1(1,0)=\frac{\lambda_0E[X]E[S_1]}{1-\rho}P_0(1)$$

Theorem 5.3: The long run probability for which the server is in NPOS state is denoted by *P* (*S*) and can be expressed as follows

$$P(S)=\tilde{P}_2(1,0)=\frac{\lambda_0\overline{p}\sigma E[X]E[S_2]}{1-\rho}P_0(1)$$

Theorem 5.4: The long run probability for which the server is in working vacation state is denoted by *P* (*V*) and can be expressed as follows

$$P(V)=\tilde{P}_3(1,0)=\frac{\lambda_0(\overline{p}\sigma q+\overline{\sigma})E[X]E[S_3]}{1-\rho}P_0(1)$$

Proof:

In order to prove theorems 5.1-5.4 or alternatively to obtain the values of *P* (*I*), *P* (*E*), *P* (*S*) and *P* (*V*), we apply L-hospital rule once in equations (4.22) and (4.1) to (4.3) respectively in limiting case $z\to1$ and then it is fairly easy to establish the theorems 5.1 to 5.4.

6. Numerical Illustration for Special Case of M^X / (G_1, G_2) / 1 Model

In this section, our keen interest is to find long run probabilities of the server in different states when λ and μ are varying for some special cases of the M^X / (G_1, G_2) / 1 model. Here, we are presenting long run probabilities of the server in different states when λ and μ are varying for M^X / E_k / 1 model only as a numerical illustration in following Table 1and Table 2.

Table 1. Long run probabilities of the server in different states when λ varies for M^X / E_k / 1 model

Case I: (p, q, σ) = (.6, .5, .6)					Case II: (p, q, σ) = (.7, .6, .5)			
λ	P(I)	P(E)	P(S)	P(V)	P(I)	P(E)	P(S)	P(V)
0.500	0.654	0.140	0.084	0.121	0.670	0.139	0.052	0.116
0.600	0.590	0.166	0.099	0.143	0.609	0.165	0.062	0.137
0.700	0.528	0.191	0.114	0.165	0.550	0.190	0.071	0.158
0.800	0.467	0.215	0.129	0.187	0.492	0.215	0.080	0.179
0.900	0.408	0.239	0.143	0.207	0.436	0.238	0.089	0.199
1.00	0.350	0.263	0.157	0.228	0.382	0.262	0.098	0.218

Table 2. Long run probabilities of the server in different states when μ varies for M^X / E_k / 1 model

Case I: (p, q, σ) = (.6, .5, .6)					Case II: (p, q, σ) = (.7, .6, .5)			
μ	P(I)	P(E)	P(S)	P(V)	P(I)	P(E)	P(S)	P(V)
2.00	0.044	0.543	0.130	0.282	0.061	0.539	0.080	0.269
3.00	0.318	0.387	0.092	0.201	0.329	0.385	0.057	0.192
4.00	0.470	0.300	0.072	0.156	0.478	0.299	0.044	0.149
5.00	0.566	0.246	0.059	0.127	0.573	0.245	0.036	0.122
6.00	0.633	0.208	0.049	0.108	0.638	0.207	0.031	0.103
7.00	0.682	0.180	0.043	0.093	0.686	0.179	0.026	0.089

7. Discussions and Conclusions

In the present paper, probability generating functions in idle state, FPES state, NPOS state and in working vacation state of the server for the bulk arrival retrial queueing M^X / (G_1, G_2) / 1 model with two phase service and Bernoulli vacation schedule are explored in theorems 4.1-4.4. In addition to this, some significant performance measures of the model have also been presented successfully in theorems 5.1-5.4 by way of using probability generating function approach. Moreover, to emphasize the application aspect, numerical illustrations have also been provided for long run probabilities of the server in different states when λ and μ are varying for a special case in M^X / E_k / 1 model. Furthermore, with passing remarks in significant theorems established in earlier sections, some valuable conclusions based on observations and theorems are drawn as following;

- The partial probability generating functions for different states of FPES, NPOS and working vacation have been successfully established. Likewise, the probability generating function of the number of the customers in the orbit is also explored.
- The long run probabilities of the server in different states of idle state, FPES, NPOS and working vacation have been proposed.
- As a special case of the M^X / (G_1, $G2$) / 1 model, long run probabilities of the server in different states for M^X / E_k / 1 model， when λ and μ varies, have been computed in Table 1 and Table 2 respectively for a numerical illustration.
- With varying parameters λ and μ, sensitivity analysis can be done for the long run probabilities of the server in different states of idle state, FPES, NPOS and working vacation from Table 1 and Table2.
- It is highly expected that significant results established in this paper are utmost useful for system designers, researchers and decision makers in many real life congestion situations including those in computer and communication networks, production and manufacturing systems, service and distribution systems. In this context, it is worth mentioning that Maurya [19,20] has recently used the investigated results of our present paper to explore various measures pertaining to system length, orbit length, waiting time etc. along with numerical illustrations for the bulk arrival retrial queueing M^X / (G_1, G_2) / 1 model with two phase service and Bernoulli vacation schedule. Finally, it is remarkable here that the research of the present investigation can be further extended by incorporating the concept of server breakdown or multi-optional services.

References

[1] Amandor, J. and Artalejo, J.R., “The M/G/1 retrial queue: New descriptors of the customer’s behavior,” Computers and Applied Mathematics, Vol. 223, No. 1, 2009, pp. 15-26.

[2] Atencia and Moreno, P., “A single server retrial queue with general retrial times and Bernoulli schedule,” Applied Mathematics and Computation, Vol. 190, No. 2, 2005, pp. 1612-1626.

[3] Boualem, M. Djellab, N. and Aissani, D., “Stochastic inequalities for M/G/1 retrial queues with vacations and constant retrial policy,” Mathematical and Computer Modelling, Vol. 50, No. 1-2, 2009, pp. 207-212.

[4] Chakravarthy, S.R. and Dudin, A., “Analysis of retrial queueing model with MAP arrivals and two type of customers,” Mathematical and Computer Modelling, Vol. 37, No. 3-4, 2003, pp. 343-363.

[5] Choudhury, G. and Deka, K., “An M/G/1 retrial queueing system with two phases of service subject to the server breakdown and repair,” Performance Evaluation, Vol. 65, No. 10, 2008, pp. 714-724.

[6] Choudhury, G. and Deka, K., “An MX/G/1 unreliable retrial queue with two phase of service and Bernoulli admission mechanism,” Applied Mathematical Modelling, Vol. 215, No. 3, 2009, pp. 936-949.

[7] Choudhury, G. and Madan, K.C., “A two phase batch arrival queueing system with vacation time under Bernoulli schedule,” Applied Mathematics and Computation, Vol. 149, No. 2, 2004, pp. 337-349.

[8] Choudhury, G. and Paul, M., “A two phase queueing system with Bernoulli vacation schedule under multiple vacation policy,” Statistical Methods, Vol. 3, 2006, pp. 174-185.

[9] Choudhury, G., Tadj., L. and Paul., M., "Steady state analysis of an $M^X/G/1$ queue with two phase service and Bernoulli vacation schedule under multiple vacation policy," Applied Mathematical Modelling, Vol. 31, No. 6, 2007, pp. 1079-1091.

[10] Doshi, B.T., "Queueing systems with vacations: A survey," Queueing Systems, Vol. 1, 1986, pp. 29-66.

[11] Doshi, B.T., "Single server with vacations, in: H. Takagi (Ed.), Stochastic Analysis of Computer and Communication System," North Holland, Amsterdam, 1990, pp. 217-265.

[12] Jain, M. and Dhakad, M.R., "Queueing distribution MX/G/1 model using principle of maximum entropy," Journal of Decision and Mathematical Sciences, Vol. 8, No. 1-3, 2003, pp. 37-44.

[13] Jain, M., Sharma, G.C. and Sharma, R., "Unreliable server M/G/1 queue with multi-optional services and multi-optional vacations," International Journal of Mathematical Operational Research, Vol. 5, No. 2, 2013 pp. 145-169.

[14] Ke, J.C. and Chang, F.M., "$M^X/(G_1,G_2)/1$ retrial queue under Bernoulli vacation schedules with general repeated attempts and starting failures," Applied Mathematical Modelling, Vol. 33, No. 7, 2009, pp. 3186-3196.

[15] Ke, J.C. and Lin, C.H., "Maximum entropy approach for batch arrival queue with an unreliable server and delaying vacations," Computers and Applied Mathematics, Vol. 183, No. 2, 2006, pp. 1328-1340.

[16] Kumar, M.S. and Arumuganathan, R., "On the single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service," Quality Technology and Quantitative Management, Vol. 5, No. 2, 2008, pp. 145-160.

[17] Madan, K.C. and Choudhury, G., "A single server queue with two phase of heterogeneous service under Bernoulli schedule and a generalized vacation time," International Journal of Information and Management Sciences, Vol. 16, No. 2, 2005, pp. 1-16.

[18] Maurya V.N., "Investigation of probability generating function in an interdependent M/M/1: (∞; GD) queueing model with controllable arrival rates using Rouche's theorem", Open Journal of Optimization, Scientific Research Publishing, Irvine, California, USA, Vol.1, Issue 2, 2012, pp. 34-38.

[19] Maurya V.N., "Sensitivity analysis on significant performance measures of bulk arrival retrial queueing $M^X/(G_1,G_2)/1$ model with second phase optional service and Bernoulli vacation schedule", International Open Journal of Operations Research, Academic and Scientific Publishing, New York, USA, Vol. 1, No. 1, 2013, pp. 1-15.

[20] Maurya V.N., "Maximum entropy analysis of $M^X/(G_1,G_2)/1$ retrial queueing model with second phase optional service and Bernoulli vacation schedule", American Journal of Operational Research, Scientific and Academic Publishing, Rosemead, California, USA, Vol. 3, No. 1, 2013, pp. 1-12.

[21] Maurya V.N., Arora D.K. and Maurya A.K., "Elements of Advanced Probability Theory and Statistical Techniques", Scholar's Press Publishing Co., Saarbrucken, Germany, 2013.

[22] Maurya V.N., "Performance analysis of $M^X/(G_1,G_2)/1$ retrial queueing model with second phase optional service and Bernoulli vacation schedule using PGF approach", International Conference on Statistical Sciences (ICSS), Dubai, UAE; World Academy of Science, Engineering & Technology (WASET), Italy, 2014 (Communicated).

[23] Sherman, N.P. and Kharoufeh, J.P., "An M/M/1 retrial queue with unreliable server," Operational Research Letters, Vol. 34, No. 6, 2006, pp. 697-705.

[24] Tian, N. and Zhang, Z.G., "Vacations queueing models-Theory and Applications," Springer-Verlag, New York, 2006.

[25] Wang, J., "An M/G/1 queue with second optional service and server breakdowns," Computers and Mathematical Application, Vol. 47, No. 10-11, 2004, pp. 1713-1723.

[26] Wang, J., Liu, B. and Li, J., "Transient analysis of an *M*/*G*/1 retrial queue subject to disasters and server failures," Operational Research, Vol. 189, No. 3, 2008, pp. 1118-1132.

[27] Wu, X., Brill, P., Hlynka, M. and Wang, J., "An M/G/1 retrial queue with balking and retrials during service," International Journal of Operations Research, Vol. 1, 2005, pp. 30-51.