American Journal of Modeling and Optimization, 2013, Vol. 1, No. 1, 1-5
Available online at http://pubs.sciepub.com/ajmo/1/1/1
© Science and Education Publishing
DOI:10.12691/ajmo-1-1-1

General Characterization of a Linear Complementarity Problem

Youssef Elfoutayeni[1,2,*], Mohamed Khaladi[1,2]

[1]Analysis, Modeling and Simulation Laboratory, University Hassan II Casablanca, Morocco
[2]MPD Laboratory, UCAM, Marrakech, Morocco and UMI UMMISCO, IRD - UPMC, Paris, France
*Corresponding author: youssef_foutayeni@yahoo.fr

Received December 13, 2012; Revised May 04, 2013; Accepted May 05, 2013

Abstract For a given $n \times n$ matrix A and a vector b of IR^n, the linear complementarity problem *LCP(A,b)* is to find a vector x in IR^n satisfying $x \geq 0$, $Ax + b \geq 0$ and $x^T(Ax+b) = 0$ or showing that such a vector x does not exist. Under various hypotheses on the matrix A, *LCP (A,b)* was studied by many authors in the last decade. In previous papers we have developed algorithms for solving some classes of *LCP (A,b)*. In this work, we give a general characterization of the solutions of *LCP (A,b)*, we show under what conditions the problem has a solution or not and how to calculate the solution when they exist. We then apply this characterization to some examples and find the solutions or show that the problem *LCP (A,b)* has no solution.

Keywords: *linear complementarity problem, system of linear equations, P-matrix, principal sub-matrix, general characterization*

1. Introduction

For a given $n \times n$ matrix A and a given vector $b \in \mathrm{IR}^n$, the linear complementarity problem *LCP(A,b)* is to find a vector x in IR^n satisfying

$$\begin{cases} x \geq 0 \\ y = Ax + b \geq 0 \\ x^T y = 0 \end{cases} \qquad (1)$$

or to show that such a vector x does not exist.

The constraint $x^T y = \sum_{i=1}^{n} x_i y_i = 0$ is known as the complementarity constraint. Since the variables are restricted to be nonnegative, this implies that $x_i y_i = 0$ for each $i = 1$ to n and hence, for any solution to (1) at least one of the variables in the pair x_i, y_i should be equal to zero for $i = 1$ to n.

The problem (1) can be equivalently stated as

$$\begin{aligned} &x_i \geq 0,\ y = Ax + b,\ y_i \geq 0 \text{ and} \\ &x_i y_i = 0 \text{ for all } i = 1, 2, \ldots, n \end{aligned} \qquad (2)$$

The **l**inear **C**omplementarity **P**roblem is abbreviated as LCP or $LCP(A,b)$.

This problem has a number of important applications in operations research, economic equilibrium problems and in the engineering sciences. A variety of economic, biological and physical phenomena are most naturally modeled by saying that certain pairs of inequality constraints must be complementary, in the sense that at least one must hold with equality. This problem has attracted much attention due its various applications. For a non exhaustive description of these applications, see for instance Elfoutayeni et al. [7,8,9,10,11,12], Ferris and Pang [13] and Garcia and Gould [14].

Many direct or iterative methods for solving LCP are proposed by several authors Cottle and Dantzig [1], Cottle et al. [2], Eaves and Lemke [3], Eaves and Scarf [4], Kelly and Watson [15], Lemke [17] and Watson [23].

The ideas presented by Lemke [17] were later exploited by Scarf [21] for developing fixed point algorithms. The relationship between the $LCP(A,b)$ and the fixed point problem is well described by Eaves and Lemke [3] and by Eaves and Scarf [4].

Recently, in [5], we have built an interior point method to solve a linear complementarity problem $LCP(A,b)$ for some $n \times n$ matrix A and $b \in \mathrm{IR}^n$; the convergence of this method requires $o(\sqrt{n}L)$ number of iterations where L is the length of a binary coding of the input data of the problem $LCP(A,b)$; the method developed in [5] is globally efficient and has a good iteration complexity but it has the problem of finding a strictly feasible starting point.

In [6], we have developed a globally convergent hybrid method which is based on vector divisions for solving the $LCP(A,b)$ and we have given some numerical simulations to illustrate our theoretical results and to show that this method can solve large-scale linear complementarity problems.

Most studies to date have used the fact that solving linear complementarity problem *LCP(A,b)* requires the existence and uniqueness of the solution (the matrix A is assumed to be symmetric positive definite or, at least, $P-matrix$) see, e.g., Cottle et al. [2], Elfoutayeni and Khaladi [5,6], Kelly and Watson [15], Murty [18,19] and Watson [23].

The aim of this work is to answer the following questions: (a) Under what conditions of A and b the problem *LCP(A,b)* has solutions? (b) How to calculate the solutions when they exist?

The paper is organized as follows. In the next section, we recall some notations and preliminary definitions to be used through the paper. In section 3, we provide the main result of the paper. The numerical examples are shown in Section 4. Section 5 concludes the paper.

2. Preliminaries

We first give some general notations and preliminary definitions to be used through the paper. For any positive integer n, IR^n denotes the $n-$ dimensional real Euclidean space and $\mathrm{IR}^{n\times n}$ be the set of all real $n\times n$ matrices. We will use I to denote the identity matrix. The transpose of a vector x is denoted by x^T (with super script T). Given x, $y\in \mathrm{IR}^n$, $x^T y$ stands for the inner product of x and y. We denote $N=\{1,2,...,n\}$. For $1\le i\le n$, x_i refers to the ith element of x. Let x, $y\in \mathrm{IR}^n$, the expression $x\le y$ (respectively $x<y$) meaning that $x_i\le y_i$ (respectively $x_i<y_i$) for each $i=1,2,...,n$.

If A is a matrix, we will denote by $A_{.j}$, the jth column vector of A.

A is said to be a $Q-matrix$ if (1) has a solution for each $b\in \mathrm{IR}^n$.

A is said to be a $P-matrix$ if all its principal subdeterminants are positive. To test whether a given square matrix A, of order $n\times n$, is a $P-matrix$ directly would require the computation of the determinants of 2^n principal submatrices of A.

A key result in complementarity theory is that LCP has a unique solution for each $b\in \mathrm{IR}^n$ if and only if A is a $P-matrix$ (see Samelson et al. [20]). Refining this characterization, Murty introduced the notion of a finite test set for LCP: if the LCP has a unique solution for each vector b in the collection

$$b\in\Gamma=\left\{\begin{matrix} I_{.1},...,I_{.n},-I_{.1},...,-I_{.n}, \\ A_{.1},...,A_{.n},-A_{.1},...,-A_{.n},e \end{matrix}\right\}$$

where e is any vector in the interior of some complementary cone, then A is a $P-matrix$ and hence the LCP has a unique solution for each $b\in \mathrm{IR}^n$. Note that Γ has (4n+1) select elements.

Tamir [22] studied the set

$$\Gamma_1=\{I_{.1},...,I_{.n},A_{.1},...,A_{.n},-A_{.1},...,-A_{.n},e\}$$

of cardinality (3n +1). He found that Γ_1 was sufficient for testing whether or not A is a *P-matrix*. Kostreva [16] refined the above characterization by proving that A is a $P-matrix$ if and only if LCP has a unique solution for each b in the set Γ_2 where Γ_2 has cardinality (2n+2) and $\Gamma_2=\{I_{.1},...,I_{.n},-A_{.1},...,-A_{.n},e,0\}$.

In what follows, we consider the general case where the matrix A is not necessarily a $P-matrix$. To do so, we are going to need the following definitions:

Definition1 *Given* $A\in \mathrm{IR}^{n\times n}$, *let* X, Y *be a subset of* $\{1,2,...,n\}$. $A[X,Y]$ *denotes the submatrix of* A *having rows indexed by elements of* X *and columns indexed by elements of* Y.

We denote by $A[X]$ the principal submatrix $A[X,X]$.

Definition2 *Given* $b\in \mathrm{IR}^n$, *let* X *be a subset of* $\{1,2,...,n\}$. $b[X]$ *denotes the subvector of* b *with rows indexed by elements of* X.

3. The Characterization of the Solutions

Using the above we now obtain our main result.

Theorem *For a given matrix* $A\in \mathrm{IR}^{n\times n}$ *and a vector* $b\in \mathrm{IR}^n$ *the following are equivalent*

- $LCP(A,b)$ has a solution,
- There is a set $X\subseteq N=\{1,2,...,n\}$ such that the system of linear equations $A[X]x+b[X]=0$ has a nonnegative solution $x^*[X]\ge 0$ and $A[\bar{X},X]x^*[X]+b[\bar{X}]\ge 0$ where $\bar{X}$ denotes $N\setminus X$.

Proof. Let x^* be a solution of the $LCP(A,b)$, i.e.,

$$x^*\ge 0,\ y^*=Ax^*+b\ge 0 \text{ and } x^{*T}y^*=0$$

then we have

$$x_i^*\ge 0,\ y_i^*\ge 0 \text{ and } x_i^*y_i^*=0 \text{ for all } i\in N.$$

Thus, if one of the variables in the pair x_i^*, y_i^* is positive, the other should be zero. Hence $x_i^*=0$ or $y_i^*=0$ for all $i\in N$.

Let

$$X=\{i\in N: x_i^*>0\}$$

and

$$\bar{X}=\{i\in N: x_i^*=0\}$$

$\bar{X}$ is the complement of X in the set N.

There are two possible cases for X:

- $X=\varnothing$, i.e. $x^*=0$, therefore $y^*=Ax^*+b=b$.

  Thus, if $b\ge 0$ then $x^*=0$ is a solution of $LCP(A,b)$.

  Otherwise, there is $i\in N$ such that $b_i<0$

  then $x^*=0$ can not be a solution of $LCP(A,b)$,

  and therefore $X\ne\varnothing$.
- $X\ne\varnothing$, i.e. $y_i^*=0$ for all $i\in X$ that is

$$\sum_{j\in X}a_{ij}x_j^*+b_i=y_i^*=0 \text{ for all } i\in X$$

So, $x^*[X]$ satisfies the system $A[X]x^*[X]+b[X]=0$.

Now for all $i \in \bar{X}$
we have

$$\sum_{j\in X} a_{ij}x_j^* + b_i = y_i^* \geq 0$$

or in matrix form

$$A[\bar{X},X]x^*[X]+b[\bar{X}] = y^*[\bar{X}] \geq 0 .$$

Thus, in both cases, there is a set X such that the system of linear equations $A[X]x+b[X]=0$ has a nonnegative solution:

$$x^*[X] \geq 0$$

and

$$A[\bar{X},X]x^*[X]+b[\bar{X}] = y^*[\bar{X}] \geq 0$$

with conventionally $x[\varnothing]=0$.

Now we assume that there is a set $X \subseteq N$ such that the system of linear equations $A[X]x+b[X]=0$ has a nonnegative solution $x \geq 0$ and $y = A[\bar{X},X]x+b[\bar{X}] \geq 0$.

Let $x^* = (x_i^*)_{i\in N}$ and $y^* = (y_i^*)_{i\in N}$

where $x_i^* = \begin{cases} x_i & \text{if } i \in X \\ 0 & \text{if } i \in \bar{X} \end{cases}$ and $y_i^* = \begin{cases} y_i & \text{if } i \in \bar{X} \\ 0 & \text{if } i \in X \end{cases}$, it can easily be verified that $x^* \geq 0$, $y^* \geq 0$ and $x^{*T}y^* = 0$ and therefore x^* is a solution of the $LCP(A,b)$. This concludes the proof.

As an immediate consequence of previous Theorem we have the following result.

Corollary *For a given matrix* $A \in \mathrm{IR}^{n\times n}$ *and a vector* $b \in \mathrm{IR}^n$. $LCP(A,b)$ has a unique solution if and only if either there is a unique set X such that

- The matrix $A[X]$ is invertible,
- $(A[X])^{-1}b[X] \leq 0$,
- $A[\bar{X},X](A[X])^{-1}b[X]-b[\bar{X}] \leq 0$

or there are many sets X with the same solution associated to all the sets X.

Proof. This is immediate from the above Theorem, it suffices to note that $LCP(A,b)$ has a unique solution if and only if we have one of the two following cases, in the first one, even if there are many sets X there is always only one solution associated with these sets X, in the second one, there is a unique set X such that the system of linear equations $A[X]x+b[X]=0$ has a unique nonnegative solution $x^* \geq 0$ and $A[\bar{X},X]x^*+b[\bar{X}] \geq 0$, this is equivalent to that
on the one hand, the matrix $A[X]$ is invertible
and

$$x^* = -(A[X])^{-1}b[X] \geq 0,$$

therefore

$$(A[X])^{-1}b[X] \leq 0,$$

on the other hand

$$A[\bar{X},X]x^*+b[\bar{X}] \geq 0$$

implies that

$$-A[\bar{X},X](A[X])^{-1}b[X]+b[\bar{X}] \geq 0,$$

therefore

$$A[\bar{X},X](A[X])^{-1}b[X]-b[\bar{X}] \leq 0 .$$

This completes the proof.

In the following section we will give numerical examples where we can find many sets X associated to a unique solution x^* (example 1 and 2). We also give an example where a set X is associated to many solutions (example 3).

4. Numerical Examples

In this section, we provide numerical examples to illustrate the results given below. As announced, the first and the second one illustrate the situation where many subsets X are associated to a unique solution x^*. In the third one we obtain a set X associated to an infinity of solutions. Finally, the LCP given in the fourth example has no solution.

Example1 *Consider the following linear complementarity problem* $LCP(A,b)$

where $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 10 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$.

Let $X_1 = \{1\}$ (resp. $X_2 = \{1,2\}$), the solution of the system of linear equations $A[X]x+b[X]=0$ is given by

$$x^*[X_1] = 1 \text{ (resp. } x^*[X_2] = (1,0)^T \text{),}$$

therefore, we have $y^* = A[\bar{X},X]x^*+b[\bar{X}]$, i.e.,

$$\begin{cases} y_2^* = -4\times 1+4 = 0 \\ y_3^* = 7\times 1-3 = 4 \end{cases}$$

$$\text{(resp. } y_3^* = 7\times 1-8\times 0-3 = 4 \text{).}$$

For the other subsets X of N it is easy to show that there is no solution in each case, to verify that, one need only verified that in each case, whether the solution of $A[X]x+b[X]=0$ is not nonnegative ,

or else ($A[\bar{X},X]x^*+b[\bar{X}]$) is not nonnegative.

Thus, this problem has one and only one solution, this solution is given by $x = (1,0,0)^T$.

Example2 *Let us consider the following linear complementarity problem* $LCP(A,b)$, *find a vector* x *satisfying* $x \geq 0,\ \ Ax+b \geq 0$ *and* $x^T(Ax+b)=0$

where $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 10 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

- Let $X_1 = \{1,2\}$

then we have

$$x^*[X_1] = (1/3, 2/3)^T$$

is the solution of the system of linear equations

$$\begin{cases} x_1 - 2x_2 + 1 = 0 \\ -4x_1 + 5x_2 - 2 = 0 \end{cases}$$

moreover

$$A[\bar{X}_1, X_1]x^* + b[\bar{X}_1] = 7 \times \frac{1}{3} - 8 \times \frac{2}{3} + 3 = 0 .$$

Thus, $x = (1/3, 2/3, 0)^T$ is a solution of this example.

• Let $X_2 = \{2,3\}$

then we have

$$x^*[X_2] = (1, 1/2)^T$$

is the solution of the system of linear equations

$$\begin{cases} 5x_2 - 6x_3 - 2 = 0 \\ -8x_2 + 10x_3 + 3 = 0 \end{cases}$$

moreover

$$A[\bar{X}_1, X_1]x^* + b[\bar{X}_1] = -2 \times 1 + 3 \times \frac{1}{2} + 1 = \frac{1}{2} .$$

Thus, $x = (0, 1, 1/2)^T$ is a solution of this example.

• Let $X_3 = \{1,2,3\}$

then we have

$$x^*[X_3] = (1/3, 2/3, 0)^T$$

is the solution of the system of linear equations

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 1 = 0 \\ -4x_1 + 5x_2 - 6x_3 - 2 = 0 \\ 7x_1 - 8x_2 + 10x_3 + 3 = 0 \end{cases}$$

and of course in this case $Ax^* + b = 0$.

Thus, $x = (1/3, 2/3, 0)^T$ is a solution of this example.

Example3 *Consider the following linear complementarity problem* $LCP(A,b)$ *, find a vector* x *satisfying* $x \geq 0,\ \ Ax + b \geq 0$ *and* $x^T(Ax+b) = 0$

where $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Let $X = \{1\}$ then the system of linear equations associated with this set is $0x_1 + 0 = 0$, it is clear that the solutions of this system is the set IR_+ ,

moreover

$$A[\bar{X}, X]x_1^* + b[\bar{X}] = 2 \times x_1^* + 1 \geq 0$$

where x_1^* is any real number in IR_+ .

Example4 *Let us consider the following linear complementarity problem* $LCP(A,b)$ *, find a vector* x *satisfying* $x \geq 0,\ \ Ax + b \geq 0$ *and* $x^T(Ax+b) = 0$

where $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 10 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$.

It is easy to show that there is no solution of this problem. To verify that, one need only consider the sets

$$X = \varnothing, X = \{1\}, X = \{2\}, X = \{3\}, X = \{1,2\},$$

$$X = \{1,3\}, X = \{2,3\}, X = \{1,2,3\},$$

and verify that in each case, whether the solution of $A[X]x + b[X] = 0$ is not nonnegative or else ($A[\bar{X}, X]x^* + b[\bar{X}]$) is not nonnegative.

5. Conclusion

In this paper we have given a general characterization of a linear complementarity problem. We have demonstrated that $LCP(A,b)$ has a solution if and only if there is a set X such that the system of linear equations $A[X]x + b[X] = 0$ has a nonnegative solution $x^* \geq 0$ and $A[\bar{X}, X]x^* + b[\bar{X}] \geq 0$; this solution is then given by $x = (x_i)_{i \in N}$ where $x_i = x_i^*$ if $i \in X$, and $x_i = 0$ if $i \in \bar{X}$. We then have given numerical examples and we have used our characterization to provide the solutions. We point out that we can find many sets X associated to a unique solution x^* and on the contrary, we can find a set X associated to many solutions.

We showed also that, if there is a unique set X such if the system of linear equations $A[X]x + b[X] = 0$ has a unique nonnegative solution and

$$A[\bar{X}, X](A[X])^{-1} b[X] - b[\bar{X}] \leq 0$$

then the LCP has a unique solution.

References

[1] R. W. Cottle, G. B. Dantzig, *A life in mathematical programming*, Math. Program., 105 (2006) 1-8.

[2] R. W. Cottle, J. S. Pang, R. E. *Stone, The Linear Complementarity Problem*, Academic Press, New York, 1992.

[3] B. C Eaves, C. E. Lemke, *Equivalence of LCP and PLS*, Math. Oper. Res., 6 (1981) 475-484.

[4] B. C Eaves, H. Scarf, *The solution of systems of piecewise linear equations*, Math. Oper. Res., 1 (1976) 1-27.

[5] Y. ELFoutayeni, M. Khaladi, *A New Interior Point Method for Linear Complementarity Problem*, Appl. Math. Sci., 4 (2010) 3289-3306.

[6] Y. ELFoutayeni, M. Khaladi, *Using vector divisions in solving the linear complementarity problem*, J. Comput. Appl. Math., 236 (2012) 1919-1925.

[7] Y. ELFoutayeni, M. Khaladi, A. ZEGZOUTI, *Profit maximization of fishermen exploiting two fish species in competition*, accepted for publication.

[8] Y. ELFoutayeni, M. Khaladi, A. Zegzouti, *A generalized Nash equilibrium for a bioeconomic problem of fishing*, Studia Informatica Universalis-HERMANN, 10 (2012) 186-204.

[9] Y. ELFoutayeni, M. Khaladi, *A bio-economic model of fishery where prices depend harvest*, J. Adv. Model. Optim., 14 (2012) 543-555.

[10] Y. ELFoutayeni, M. Khaladi, *A generalized bio-economic model for competing multiple-fish populations where prices depend on harvest*, J. Adv. Model. Optim., 14 (2012) 531-542.

[11] Y. ELFoutayeni, M. Khaladi, *Prey Switching: How to maximize the species's well being*, J. Adv. Model. Optim., 14 (2012) 577-587.

[12] Y. ELFoutayeni, *Modélisation et étude mathématique et informatique d'un modèle bioéconomique d'exploitation d'espèces marines en compétition*, thèse Université Cadi Ayyad, Marrakech Maroc.

[13] M. C. Ferris, J. S. Pang, *Engineering and economic applications of complementarity problems*, SIAM Rev., 39 (1997) 669-713.

[14] C. B. Garcia, F. J. Gould, *Studies in linear complementarity, Center for Mathematical Studies in Business and Economics*, University of Chicago, Chicago (1980).

[15] L. M. Kelly, L. T. Watson, *Q-Matrices and Spherical Geometry, Linear Geometry and its Applications*, 25 (1979) 175-189.

[16] M. M. Kostreva, *Finite test sets and P-matrices*, Proc. Amer. Math. Soc., 84 (1982) 104-105.

[17] C. E. Lemke, *Bimatrix equilibrium points and mathematical programming*, Manag. Sci., 11 (1965) 681-689.

[18] K. G. Murty, *On a characterization of P-matrices*, SIAM J Appl Math, 20 (1971) 378-384.

[19] K. G. Murty, *On the number of solutions to the complementarity problem and spanning properties of complementary cones*, Linear Algebra and Appl., 5 (1972) 65-108.

[20] H. Samelson, R. M. Thrall, O. Wesler, *A partition theorem for Euclidean n-space*, Proc. Amer. Math. Soc., 9 (1958) 805-807.

[21] H. Scarf, *The approximation of fixed points of a continuous mapping*, SIAM J. Appl. Math., 15 (1967) 1328-1342.

[22] A. Tamir, *on a characterization of P-matrices*, Math. Programming, 4 (1973) 110-112.

[23] L. T. Watson, *A variational Approach to the Linear Complementarity Problem*, Doctoral Dissertation, Dept. of Mathematics, University of Michigan, Ann Arbor, MI, 1974.