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Large Strain Elasto Plasticity

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Abstract In this paper we present a continuum theory for large strain Elasto-Plasticity based on formulations: Eulerian Formulation and Multiplicative Elasto-Plasticity. The theories includes Cauchy and Kirchhoff stress tensor as well as Truesdell rate and Jauman rate for the Cauchy stress and the multiplicative elasto-plastic decomposition. We show detailed derivative for mentioned formulations.

Keywords: *Elasto-Plasticity, Kirchhoff stress tensor, Cauchy stress tensor, elasto-plastic decomposition, Jauman rate, Truesdell rate*

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1. Introduction

In recent decades improved numerical solutions of continua subjected to large inelastic strains have been obtained by changing the initial hypo-elastic-plastic algorithms to procedures based on hyperplastic formulation. Typically, a hyperelastic constitutive relation governs the elastic deformations, but the main purpose of this approach is to by-pass the integration of the stress rates in the plasticity part, rather than considering large elastic strains. Indeed, most approaches that have been presented assume that the elastic strains remain small, at least compared with the plastic In this article we will first outline the Eulerian approach, and then directly proceed with the multiplicative decomposition.

2. Eulerian Formulation

An Eulerian finite element formulation is most convenient developed starting from the virtual work expression in the current configuration :

$$\int_V \delta\boldsymbol{\varepsilon}:\boldsymbol{\sigma}dV = \int_V \rho\delta\mathbf{u}\cdot\mathbf{g}dV + \int_S \delta\mathbf{u}\cdot\mathbf{t}dS. \tag{1}$$

Using Equations:

$$\rho dV = \rho_0 dV_0 \tag{2}$$

which is mathematically expression for conservation of mass of an elementary volume and other one

$$\mathbf{k} = \frac{\rho_0}{\rho} = \det\mathbf{F}\boldsymbol{\sigma}. \tag{3}$$

This identity can be also expressed in terms of Kirchhoff stress tenzor [5]:

$$\int_{V_0} \delta\boldsymbol{\varepsilon}:\mathbf{k} = \int_{V_O} \rho_0\delta\mathbf{u}\cdot\mathbf{g}dV + \int_{S_0} \delta\mathbf{u}\cdot t_0 dS \tag{4}$$

which has advantage that the integration can now be carried out for the known volume V_0 . As with the Lagrange formulation we use the kinematic relation between the virtual strains and the virtual displacement, Equation

$$\delta\boldsymbol{\varepsilon} = \mathbf{L}\delta\mathbf{u} \tag{5}$$

relation between the variation of the strain tensor and that of the continuous displacement field and with Equation

$$\mathbf{u} = \mathbf{H}\mathbf{a}_e \tag{6}$$

which is a continuous displacement field for field for all points within an element, to arrive at:

$$\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\mathbf{a} \tag{7}$$

where for notational simplicity, the element index e has been omitted. For general three-dimensional conditions we have [2]

$$\mathbf{B} = \mathbf{J}^{-1}\cdot\begin{bmatrix} \frac{\partial h_1}{\partial\xi} & 0 & 0 & \dots & \dots & \dots & \frac{\partial h_n}{\partial\xi} & 0 & 0 \\ 0 & \frac{\partial h_1}{\partial\eta} & 0 & \dots & \dots & \dots & 0 & \frac{\partial h_n}{\partial\eta} & 0 \\ 0 & 0 & \frac{\partial h_1}{\partial\zeta} & \dots & \dots & \dots & 0 & 0 & \frac{\partial h_n}{\partial\zeta} \\ \frac{\partial h_1}{\partial\eta} & \frac{\partial h_1}{\partial\xi} & 0 & \dots & \dots & \dots & \frac{\partial h_1}{\partial\eta} & \frac{\partial h_1}{\partial\xi} & 0 \\ 0 & \frac{\partial h_1}{\partial\zeta} & \frac{\partial h_1}{\partial\eta} & \dots & \dots & \dots & 0 & \frac{\partial h_1}{\partial\zeta} & \frac{\partial h_1}{\partial\eta} \\ \frac{\partial h_1}{\partial\zeta} & 0 & \frac{\partial h_1}{\partial\xi} & \dots & \dots & \dots & \frac{\partial h_1}{\partial\zeta} & 0 & \frac{\partial h_1}{\partial\xi} \end{bmatrix} \tag{8}$$

with $\mathbf{J}^{-1}$ the 6x6 inverse of the Jacobian matrix. Substitution of the discrete kinematics relation, and requiring that the resulting discrete equations hold for any virtual nodal displacement $\delta\mathbf{a}$, yields the discrete equilibrium equations:

$$\mathbf{f}_{\text{int}} = \mathbf{f}_{ext}$$

end, for the Cauchy stress tensor, the internal force vector reads:

$$\mathbf{f}_{\text{int}} = \int_V \mathbf{B}^T \boldsymbol{\sigma} dV.$$

Whereas for the Kirchhoff stress tensor we have:

$$\mathbf{f}_{\text{int}} = \int_{V_0} \mathbf{B}^T \mathbf{k} dV \tag{9}$$

The tangential stiffness matrix can be obtained in a standard manner, namely by differentiating the internal virtual work. For the Kirchhoff stress this results in [5]:

$$\begin{aligned} \int_{V_0} \dot{\overline{\delta\boldsymbol{\varepsilon}:\boldsymbol{\kappa}}}\, dV &= \int_{V_0} \delta\dot{\boldsymbol{\varepsilon}}:\boldsymbol{\kappa} dV + \int_{V_0} \delta\boldsymbol{\varepsilon}:\dot{\boldsymbol{\kappa}} dV \\ &= \int_{V_0} \delta\mathbf{l}:\boldsymbol{\kappa} dV + \int_{V_0} \delta\boldsymbol{\varepsilon}:\dot{\boldsymbol{\kappa}} dV \end{aligned} \tag{10}$$

where the latter equality sign holds because of the symmetry of the Kirchhoff stress tensor. We now elaborate $\delta\mathbf{l}$ and first note because $\boldsymbol{\xi}$ is fixed,

$$\delta\left(\frac{\partial\dot{\mathbf{x}}}{\partial\boldsymbol{\xi}}\right) = \delta\left(\frac{\partial\dot{\mathbf{x}}}{\partial\mathbf{x}}\right)\cdot\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}} + \frac{\partial\dot{\mathbf{x}}}{\partial\mathbf{x}}\cdot\delta\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}}\right) = 0 \tag{11}$$

So that, using the velocity gradient $\mathbf{l}$ as:

$$\mathbf{l} = \nabla\dot{\mathbf{x}}$$

we get

$$\begin{aligned} \delta\mathbf{l} &= -\mathbf{l}\cdot\delta\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}}\right)\cdot\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}}\right)^{-1} \\ &= -\mathbf{l}\cdot\frac{\partial\delta\mathbf{x}}{\partial\boldsymbol{\xi}}\cdot\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}}\right)^{-1} = -\mathbf{l}\cdot\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}. \end{aligned} \tag{12}$$

Next, we choose the Truesdell rate of the Kirchhoff stress as the objective stress rate [1].

We can $\dot{\boldsymbol{\kappa}}$ to express

$$\dot{\boldsymbol{\kappa}} = \mathbf{D}^{TK}:\dot{\boldsymbol{\varepsilon}} + \mathbf{l}\cdot\boldsymbol{\kappa} + \boldsymbol{\kappa}\cdot\mathbf{l}^T \tag{13}$$

where

$\mathbf{D}^{TK}$ - is material tangential stiffness tensor,

Note that the superscript ´*T*´ continues to denote the transpose of a quantity. We next substitute Equations (12) into (10) to obtain:

$$\begin{aligned} \int_{V_0} \dot{\overline{\delta\boldsymbol{\varepsilon}:\boldsymbol{\kappa}}}\, dV &= -\int_{V_0}\left(\mathbf{l}\cdot\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right):\boldsymbol{\kappa} dV \\ &+ \int_{V_0} \delta\boldsymbol{\varepsilon}:\left(\mathbf{D}^{TK}:\dot{\boldsymbol{\varepsilon}} + \mathbf{l}\cdot\boldsymbol{\kappa} + \boldsymbol{\kappa}\cdot\mathbf{l}^T\right) dV \end{aligned} \tag{14}$$

Exploiting the symmetry of the Kirchhoff stress tensor this identity can be reworked to give:

$$\int_{V_0} \dot{\overline{\delta\boldsymbol{\varepsilon}:\boldsymbol{\kappa}}}\, dV = \int_{V_0} \delta\boldsymbol{\varepsilon}:\mathbf{D}^{TK}:\dot{\boldsymbol{\varepsilon}} dV + \int_{V_0}\left(\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right)^T\cdot\dot{\boldsymbol{\kappa}}\cdot\mathbf{l} dV \tag{15}$$

This equation can be discretised in a straightforward manner, yielding:

$$\int_{V_0} \delta\boldsymbol{\varepsilon}:\mathbf{D}^{TK}:\dot{\boldsymbol{\varepsilon}} dV + \int_{V_0}\left(\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right)^T\cdot\boldsymbol{\kappa}\cdot\mathbf{l} dV = \delta\mathbf{a}^T\mathbf{K}\dot{\mathbf{a}} \tag{16}$$

with

$$\mathbf{K} = \int_{V_0} \mathbf{B}^T\mathbf{D}^{TK}\mathbf{B} dV + \int_{V_0} \mathbf{G}^T\mathbf{Z}\mathbf{G} dV \tag{17}$$

the tangential stiffness matrix, and $\mathbf{B}$ given in Equation (8), while $\mathbf{Z}$ and $\mathbf{G}$ are given by [5]

$$\mathbf{Z} = \begin{bmatrix} z_{xx} & z_{xy} & z_{xz} & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{yx} & z_{yy} & z_{yz} & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{zx} & z_{zy} & z_{zz} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{xx} & z_{xy} & z_{xz} & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{yx} & z_{yy} & z_{yz} & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{zx} & z_{zy} & z_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_{xx} & z_{xz} & z_{xy} \\ 0 & 0 & 0 & 0 & 0 & 0 & z_{yx} & z_{yy} & z_{yz} \\ 0 & 0 & 0 & 0 & 0 & 0 & z_{zx} & z_{zy} & z_{zz} \end{bmatrix} \tag{18}$$

and

$$\mathbf{G} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & 0 & 0 & \frac{\partial h_2}{\partial x_1} & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{\partial h_1}{\partial x_2} & 0 & 0 & \frac{\partial h_2}{\partial x_2} & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{\partial h_1}{\partial x_3} & 0 & 0 & \frac{\partial h_2}{\partial x_3} & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & \frac{\partial h_1}{\partial x_1} & 0 & 0 & \frac{\partial h_2}{\partial x_1} & 0 & \cdots & \cdots & \cdots \\ 0 & \frac{\partial h_1}{\partial x_2} & 0 & 0 & \frac{\partial h_2}{\partial x_2} & 0 & \cdots & \cdots & \cdots \\ 0 & \frac{\partial h_1}{\partial x_3} & 0 & 0 & \frac{\partial h_2}{\partial x_3} & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \frac{\partial h_1}{\partial x_1} & 0 & 0 & \frac{\partial h_2}{\partial x_1} & \cdots & \cdots & \cdots \\ 0 & 0 & \frac{\partial h_1}{\partial x_2} & 0 & 0 & \frac{\partial h_2}{\partial x_2} & \cdots & \cdots & \cdots \\ 0 & 0 & \frac{\partial h_1}{\partial x_3} & 0 & 0 & \frac{\partial h_2}{\partial x_3} & \cdots & \cdots & \cdots \end{bmatrix} \tag{19}$$

Evidently, the choice of the Truesdell rate of the Kirchhoff stress yields a particularly simple expression for the tangential stiffness matrix. This is not so for other choice of the objective stress rate and the stress tensor. For instance, if we take the Jaumann rate in conjunction with Kirchhoff stress tensor, the resulting expression for the tangential stiffness matrix is [3]:

$$\mathbf{K} = \int_{V_0} \mathbf{B}^T\left(\mathbf{D}^{JK} - \mathbf{Z}^{JK}\right)\mathbf{B} dV + \int_{V_0} \mathbf{G}^T\mathbf{Z}\mathbf{G} dV \tag{20}$$

Where the superscript “JK” at the constitutive matrix $\mathbf{D}^{JK}$ denotes the Jaumann derivative of the Kirchhoff stress. Using of the term [4]

$$\mathbf{D}^{JK}:\dot{\boldsymbol{\varepsilon}}=\mathbf{D}^{JK}:\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\kappa}-\boldsymbol{\kappa}\cdot\dot{\boldsymbol{\varepsilon}}. \tag{21}$$

So that, in index notation, both constitutive matrices are related through:

$$D_{ijkl}^{TK}=D_{ijkl}^{JK}-\frac{1}{2}\left(\kappa_{il}\delta_{jk}+\kappa_{jl}\delta_{ik}+\kappa_{ik}\delta_{jl}+\kappa_{jk}\delta_{il}\right) \tag{22}$$

where it is noted that expressing one objective stress rate into another through a modification of the constitutive matrix is always possible. Rewriting in matrix-vector format yields [5]:

$$\mathbf{D}^{TK}=\mathbf{D}^{JK}-\mathbf{Z}^{JK} \tag{23}$$

which explains (12.16), with

$$\mathbf{Z}^{JK}=\begin{bmatrix} 2z_{xx} & 0 & 0 & 0 & z_{zx} & z_{xy} \\ 0 & 2z_{yy} & 0 & z_{yz} & 0 & z_{xy} \\ 0 & 0 & 2z_{xx} & z_{yz} & z_{zx} & 0 \\ 0 & z_{yz} & z_{yz} & \frac{1}{2}\left(z_{yy}+z_{zz}\right) & z_{xy} & z_{zx} \\ z_{zx} & 0 & z_{zx} & z_{xy} & \frac{1}{2}\left(z_{zz}+z_{xx}\right) & z_{yz} \\ z_{xy} & z_{xy} & 0 & z_{zx} & z_{yz} & \frac{1}{2}\left(z_{yy}+z_{zz}\right) \end{bmatrix} \tag{24}$$

Using a similar procedure the tangential stiffness matrices for other objective rates in conjuction with different stress measures can be derived. For instance, when using the Jaumann rate for the Cauchy stress, one arrives at

$$\mathbf{K}=\int_V \mathbf{B}^T\left(\mathbf{D}^{JC}-\mathbf{S}^{JK}+\mathbf{S}^{JC}\right)\mathbf{B}dV+\int_V \mathbf{G}^T\mathbf{S}\mathbf{G}dV \tag{25}$$

with

$$\mathbf{S}=\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{xx} & \sigma_{xy} & \sigma_{\sigma_{xz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{26}$$

and $\mathbf{S}^{JK}$ as $\mathbf{Z}^{JK}$,but with Cauchy stresses in lieu of Kirchhoff stresses. The matrix $\mathbf{S}^{JK}$ is derived from the relation between the Jaumann rates of the Cauchy and the Kirchhoff stress tensors, resulting in [1]:

$$\mathbf{D}^{JK}:\dot{\boldsymbol{\varepsilon}}=\det\mathbf{F}\left(\mathbf{D}^{JC}:\dot{\boldsymbol{\varepsilon}}+\boldsymbol{\sigma}tr\left(\dot{\boldsymbol{\varepsilon}}\right)\right) \tag{27}$$

In index notation, this equation can be reworked to give:

$$D_{ijkl}^{JK}=\det\mathbf{F}\left(D_{ijkl}^{JC}+\sigma_{ij}\delta_{kl}\right) \tag{28}$$

so that we finally obtain the matrix:

$$\mathbf{S}^{JC}=\begin{bmatrix} \sigma_{xx} & \sigma_{xx} & \sigma_{xx} & 0 & 0 & 0 \\ \sigma_{yy} & \sigma_{yy} & \sigma_{yy} & 0 & 0 & 0 \\ \sigma_{zz} & \sigma_{zz} & \sigma_{zz} & 0 & 0 & 0 \\ \sigma_{xy} & \sigma_{xy} & \sigma_{xy} & 0 & 0 & 0 \\ \sigma_{yz} & \sigma_{yz} & \sigma_{yz} & 0 & 0 & 0 \\ \sigma_{zx} & \sigma_{zx} & \sigma_{zx} & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

Indeed, the non-symmetry in $\mathbf{S}^{JC}$ can be balanced by a non-symmetry in $\mathbf{D}^{JC}$, which, for instance, is the case when a hyperelastic constitutive relation is adopted [3].

By contrast, the use of the Truesdell rate of the Cauchy stress does not result in a non-symmetric tangential stiffness matrix. Substituting expression

$$\mathbf{D}^{TK}:\dot{\boldsymbol{\varepsilon}}=\det\mathbf{F}\mathbf{D}^{TC}:\dot{\boldsymbol{\varepsilon}} \tag{30}$$

into (a) and using the relation between the Cauchy and the Kirchhoff stress tensor, we yields the tangential stiffness matrix [4]:

$$\mathbf{K}=\int_V \mathbf{B}^T\mathbf{D}^{TC}\mathbf{B}dV+\int_V \mathbf{G}^T\mathbf{S}\mathbf{G}dV. \tag{31}$$

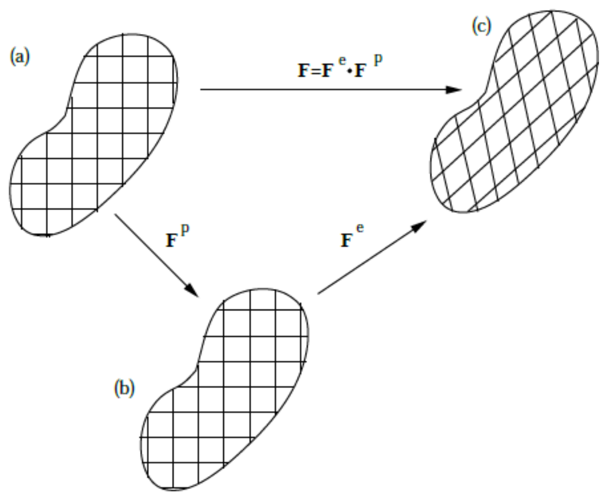

Figure 1. Multiplicative elasto-plastic decomposition: (a) initial state, (b) intermediate state,(c)final state [5]

3. Multiplicative Elasto – Plasticity

The multiplicative elasto-plastic decomposition assumes the existence of three configurations: the initial, undeformed configuration, with a line segments $d\boldsymbol{\xi}$, which is first moved, by a purely plastic deformation, into an intermediate configuration $d\hat{\mathbf{x}}$, and subsequently, into the final configuration $d\mathbf{x}$ through a pure elastic deformation (Figure 1). For each configuration we have [4]

$$d\hat{\mathbf{x}}=\frac{\partial\hat{\mathbf{x}}}{\partial\boldsymbol{\xi}}\cdot d\boldsymbol{\xi} \quad\rightarrow\quad \mathbf{F}^P=\frac{\partial\hat{\mathbf{x}}}{\partial\boldsymbol{\xi}} \tag{32}$$

for the mapping from the initial state to the intermediate configuration, with $\mathbf{F}^P$ the plastic part of the deformation gradient, and

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \hat{\mathbf{x}}} \cdot d\hat{\mathbf{x}} \quad \rightarrow \quad \mathbf{F}^e = \frac{\partial \mathbf{x}}{\partial \hat{\mathbf{x}}} \tag{33}$$

the mapping from the intermediate state to the final configuration, with the elastic part of the deformation gradient, which, in standard manner, can be decomposed into a rotational part [3].

$\mathbf{R}^e$ and a contribution that stems from a pure deformation, $\mathbf{U}^e$,as follows:

$$\mathbf{F}^e = \mathbf{R}^e \cdot \mathbf{U}^e \tag{34}$$

Subsequently, definition (33) can be used to define the right Cauchy – Green deformation tensor referred to the intermediate, elastic reference state $\hat{\mathbf{x}}$,

$$\mathbf{C}^e = \left(\mathbf{F}^e\right)^T \cdot \mathbf{F}^e \tag{35}$$

the "elastic" Green – Lagrange strain tensor based upon $\mathbf{C}^e$,

$$\gamma^e = \frac{1}{2}\left(\mathbf{C}^e - \mathbf{I}\right) \tag{36}$$

and the left Cauchy – Green deformation tensor referred to the intermediate, elastic reference state:

$$\mathbf{B}^e = \mathbf{F}^e \cdot \left(\mathbf{F}^e\right)^T. \tag{37}$$

Considering the definition of the deformation gradient tensor and combining Equations (32) and (33) give the multiplicative decomposition of the deformation gradient for elasto – plastic deformations [1]:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \xi} = \frac{\partial \mathbf{x}}{\partial \hat{\mathbf{x}}} \cdot \frac{\partial \hat{\mathbf{x}}}{\partial \xi} = \mathbf{F}^e \cdot \mathbf{F}^P. \tag{38}$$

For the one – dimensional case, Equation (38) particularizes as:

$$\lambda = \frac{l}{l_0} = \frac{l}{l^P} \cdot \frac{l^P}{l_0} = \lambda^e \cdot \lambda^P \tag{39}$$

Whit λ the stretch ratio, which is multiplicative decomposed into an elastic and plastic stretch ratio, λ^e and λ^P, respectively [2].

The multiplicative decomposition for elasto – plasticity is not unique. For instance, it would be equally possible to rotate the intermediate configuration by $\mathbf{R}$, such that:

$$d\bar{\mathbf{x}} = \frac{\partial \bar{\mathbf{x}}}{\partial \hat{\mathbf{x}}} \cdot d\hat{\mathbf{x}} \quad \rightarrow \quad \mathbf{R} = \frac{\partial \bar{\mathbf{x}}}{\partial \hat{\mathbf{x}}} \tag{40}$$

whence, using Equation (32) and (33) we straightforwardly arrive at the following, alternative elasto – plastic multiplicative decomposition:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}} \cdot \frac{\partial \bar{\mathbf{x}}}{\partial \xi} = \bar{\mathbf{F}}_e \cdot \bar{\mathbf{F}}_P \tag{41}$$

which is equivalent to the decomposition of Equation (38), since [4]

$$\bar{\mathbf{F}}_e \cdot \bar{\mathbf{F}}_P = \mathbf{F}^e \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{F}^P = \mathbf{F}^e \mathbf{F}^P$$
$$\bar{\mathbf{F}}^P = \mathbf{R} \cdot \mathbf{F}^P \tag{42}$$
$$\bar{\mathbf{F}}^e = \mathbf{F}^e \cdot \mathbf{R}^T.$$

For crystalline materials, it is physically reasonable to consider that the plastic deformation gradient $\mathbf{F}^P$ purely represents the plastic sliding between crystals lattice and its rotation. This is represented by the decomposition of Equation (38) and is shown graphically in Figure 1.

Unfortunately, the Second Piola – Kirchhoff stress tensor τ cannot be related generally to the Green – Lagrange strain tensor γ^e than can be constructed on the basis of the elastic deformation gradient:

$$\gamma^e = \frac{1}{2}\left(\left(\mathbf{F}^e\right)^T \cdot \mathbf{F}^e - \mathbf{I}\right) \tag{43}$$

Since it is not invariant with respect to a rotation of the intermediate configuration. In particular, using Equation (42), one obtains:

$$\bar{\gamma}^e = \mathbf{R} \cdot \gamma^e \cdot \mathbf{R}^T$$
$$\bar{\boldsymbol{\tau}} = \mathbf{R} \cdot \boldsymbol{\tau} \cdot \mathbf{R}^T \tag{44}$$

and the stress is not affected by the frame in which the intermediate configuration is represented. The important consequence of the decomposition of Equation (38) is that, although the elastic and plastic deformations are decomposed in a multiplicative sense, this is not so for the strains rates [2].

Using the following relations

$$\mathbf{l} = \dot{\mathbf{F}}^e \cdot \mathbf{F}^P \cdot \mathbf{F}^{-1} + \mathbf{F}^e \cdot \dot{\mathbf{F}}^P \cdot \mathbf{F}^{-1}$$
$$= \dot{\mathbf{F}}^e \cdot \left(\mathbf{F}^e\right)^{-1} + \mathbf{F}^e \cdot \dot{\mathbf{F}}^P \cdot \mathbf{F}^{-1}$$
$$\mathbf{l}^e = \dot{\mathbf{F}}^e \cdot \left(\mathbf{F}^e\right)^{-1} \tag{45}$$
$$\mathbf{l}^P = \mathbf{F}^e \cdot \dot{\mathbf{F}}^P \cdot \mathbf{F}^{-1}$$

where, in view of the definition of the velocity gradient, the additively decomposed elastic and plastic velocity gradients, $\mathbf{l}^e$ and $\mathbf{l}^P$, refer to the current configuration. It is emphasized that this additive decomposition of the velocity gradient depends crucially on the definition for $\mathbf{l}^P$ as given in Equation (45). for instance, when [1]

$$\mathbf{L}^P = \dot{\mathbf{F}}^P \left(\mathbf{F}^P\right)^{-1} \tag{46}$$

is substituted for $\mathbf{l}^P$, which would then be similar to the definition of $\mathbf{l}^e$, an additive decomposition is not obtained. Nevertheless, the symmetric port of $\mathbf{L}$,

$$\mathbf{D}^P = \frac{1}{2}\left(\mathbf{L}^P + \left(\mathbf{L}^P\right)^T\right) \tag{47}$$

Is a measure for the plastic stretching, as the eigenvalues D_i^P of the spectral decomposition,

$$D^P = \sum_{i=1}^{3} D_i^P \mathbf{e}_i \otimes \mathbf{e}_i. \tag{48}$$

The anti – symmetrical part of $\mathbf{L}^P$ is named the plastic spin tensor [3],

$$\mathbf{W}^P = \frac{1}{2}\left(\mathbf{L}^P - \left(\mathbf{L}^P\right)^T\right). \tag{49}$$

And represents the instantaneous rate of plastic spin of the intermediate configuration. In principle, a constitutive equation must be postulated for the plastic spin tensor, but in this treatment the hypothesis is made that the plastic spin vanishes:

$$\mathbf{W}^P = 0. \tag{50}$$

This hypothesis holds rigorously for plastic isotropy, but not necessarily for plastic anisotropy.

Using the hypothesis of Equating (50) it directly follows that:

$$\mathbf{D}^P = \mathbf{L}^P.$$

Using the definition of Equation (46), the plastic strain rate in the currently configuration can be written as:

$$\dot{\boldsymbol{\varepsilon}}^P = \left(\mathbf{l}^P\right)^{sym} = \frac{1}{2}\begin{pmatrix} \mathbf{F}^e \cdot \mathbf{L}^P \cdot \left(\mathbf{F}^e\right)^{-1} \\ +\left(\mathbf{F}^e\right)^{-T} \cdot \left(\mathbf{L}^P\right)^T \cdot \left(\mathbf{F}_e\right)^T \end{pmatrix} \tag{51}$$

and, with expression can be obtained in the intermediate configuration [5]:

$$\dot{\boldsymbol{\gamma}}^P = \left(\mathbf{F}^e\right)^T \cdot \dot{\boldsymbol{\varepsilon}}^P \cdot \mathbf{F}^e = \frac{1}{2}\left(\left(\mathbf{C}^e\right)^T \cdot \mathbf{L}^P + \left(\mathbf{L}^P\right)^T \cdot \mathbf{C}^e\right) \tag{52}$$

$$\dot{\boldsymbol{\gamma}}^P = \left(\mathbf{F}^e\right)^T \cdot \dot{\boldsymbol{\varepsilon}}^P \cdot \mathbf{F}^e = \frac{1}{2}\left(\left(\mathbf{C}^e\right)^T \cdot \mathbf{L}^P + \left(\mathbf{L}^P\right)^T \cdot \mathbf{C}^e\right) \tag{53}$$

with $\mathbf{C}^e$ as in Equation (53). Clearly, there is no unequivocal definition for the plastic strain rate, neither in the current configuration, nor in the intermediate configuration. For instance, under the assumption that the elastic strains remain small, so that $\mathbf{C}^e \approx \mathbf{I}$, Equation (53) can be approximated as [1]:

$$\dot{\boldsymbol{\gamma}}^P = \frac{1}{2}\left(\mathbf{L}^P + \left(\mathbf{L}^P\right)^T\right) \equiv \mathbf{D} \tag{54}$$

with using relations

$$\dot{\boldsymbol{\varepsilon}}^P = \left(\mathbf{F}^e\right)^{-T} \cdot \mathbf{D}^P \cdot \left(\mathbf{F}^e\right)^{-1} \tag{55}$$

$$\dot{\boldsymbol{\varepsilon}}^P = \mathbf{F}^e \cdot \mathbf{D}^P \cdot \left(\mathbf{F}^e\right)^{-1} \tag{56}$$

we get:

$$\dot{\boldsymbol{\varepsilon}}^P = \mathbf{R}^e \cdot \mathbf{D}^P \cdot \left(\mathbf{R}^e\right)^T \tag{57}$$

When the elastic strains remain small, $\mathbf{U}^e \approx \mathbf{I}$, and using Equation (34), $\mathbf{F}^e \approx \mathbf{R}^e$ and $\left(\mathbf{F}^e\right)^{-T} \approx \left(\mathbf{R}^e\right)^{-T} = \mathbf{R}^e$.

$\mathbf{D}^P$ - plastic stretching, which is transformed to the deformed configuration through the elastic rotation [4].

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