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Vibration Analysis of Flexible Structural Components

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Abstract This article deals with the vibration analysis of flexible structural or mechanical components. The purpose of this article is to derive the general nonlinear differential equations of motion regarding the free vibration of flexible members of uniform and/or variable cross section along their length. Unique solution methodologies are developed that simplify the solution of very complex problems.

Keywords: *flexible, structural, mechanical components, vibration, nonlinear differential equations*

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1. Introduction

The main purpose in this article is to provide an introduction to the very complex problem regarding the free vibration of flexible members. Free vibrations, in general, take place from the static equilibrium position of the member. For a flexible member, the static equilibrium position is associated with large static amplitudes, and the differential equation of motion that expresses the free vibration of the member becomes nonlinear. If the vibrational amplitudes, measured from the static equilibrium position, are small, then the free frequencies of vibration are independent of the amplitude of vibration, but they do depend upon the static amplitude that defines the static equilibrium position. If, however, the frequency amplitudes are also large, then the free frequencies of vibration will be dependent on both static and vibrational amplitudes. In this article the objective is to derive the general nonlinear differential equations of motion regarding the vibration analysis of flexible members of uniform and/or variable cross section along the length of the member. The analysis in this chapter is dealing primarily with small oscillation vibration superimposed on large static displacements which define the static equilibrium position of the flexible member

2. The General Nonlinear Differential Equation of Motion

We start with the Euler-Bernoulli Law. This nonlinear differential equations is as follows:

$$\frac{y''}{\left[1+(y')^2\right]^{3/2}} = -\frac{M_x}{E_x I_x} \tag{1}$$

By multiplying both sides of Eq. (1) by $E_x I_x$

$$E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}} = -M_x \tag{2}$$

Differentiating both sides of Eq. (2) with respect to x we get

$$\frac{d}{dx}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -\frac{dM_x}{dx} \tag{3}$$

or

$$\frac{d}{dx}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -V(x) \tag{4}$$

The shear force $V(x)$ at any cross section must be the same whether it is defined in the reference configuration, or deformed configuration. That is $V(x_0) = V(x(x_0))$.

That Eq. (4) may be written as

$$\frac{d}{dx}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -\frac{V(x_0)}{\cos\theta} \tag{5}$$

By differentiating Eq. (5) with respect to x, we obtain the following expression:

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -\frac{d}{dx}\left\{\frac{V(x_0)}{\cos\theta}\right\} \tag{6}$$

Since the expression for the transverse weight is well defined in the undeformed configuration we can rewrite Eq. (6) as follows:

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -\frac{1}{\cos\theta}\frac{d}{dx}\left\{\frac{V(x_0)}{\cos\theta}\right\} \quad (7)$$

For example, for distributed weight, the relation of the shear force $V(x_0)$ to the load $w(x_0)$ is a following:

$$\frac{d}{dx_0}V(x_0) = -w(x_0)\cos\theta \quad (8)$$

Performing the indicated differentiation on the right-hand side of Eq. (7) we obtain

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -\frac{1}{\cos\theta}w(x_0) \quad (9)$$

Whit using the following equations

$$x_0(x) = \int_0^x \sqrt{1+(y')^2}\,dx \quad (10)$$

and

$$\cos\theta = \frac{1}{\sqrt{1+(y')^2}} \quad (11)$$

we obtain for uniformly distributed loading equation where $w(x_0) = w_0$ [2]

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} = -\left[1+(y')^2\right]^{1/2} w(x_0) \quad (12)$$

For transverse free vibration, the weight w_0 is replaced by the inertia force w_{in}.

The inertia force is express by equation

$$w_{in} = m\frac{d^2y}{dt^2} \quad (13)$$

where m is the uniform mass density, and d^2y/dt^2 is the relative acceleration.

There for the transverse free vibration of flexible member with a uniform distributed weight, or a distributed mass m, is given by the equation

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} + \left[1+(y')^2\right]^{1/2} m\frac{d^2y}{dt^2} = 0 \quad (14)$$

In general, for arbitrarily distributed weight, the mass m is replaced with an appropriate expression $m(x_0)$.

Note that m is a function of. Thus, the nonlinear differential equation of motion becomes

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y''}{\left[1+(y')^2\right]^{3/2}}\right\} + \left[1+(y')^2\right]^{1/2} m(x_0)\frac{d^2y}{dt^2} = 0 \quad (15)$$

Equation (15) is the one, as stated earlier, that cannot be solved be using separation of variables, and where the natural frequencies of vibration are amplitude dependent. A procedure that could be used for the solution of such problems, would be to assume the naturals of vibration and to determine the natural frequencies of vibration and the corresponding of the fundamental frequency of vibration this approach is realistic, but it becomes extremely difficult for the computation of higher frequencies of vibration [5].

2.1. Small Amplitude Vibration of Flexible Members

We start the derivation of the appropriate nonlinear equation of motion with small amplitude vibrations by considering the flexible cantilever beam in Fig.1a, where w_0 is the uniform weight of the member per unit of length. However it should be realize that w_0 may include other weights witch are attached to the beam and participating in its vibration motion. The deformation configuration of the of the member is show in Figure 1b, where $y_S(x)$ is the large deformation that defines the static equilibrium position. The dynamic amplitude which represent the small vibration of the member from the static equilibrium position is denoted by $y_d(x,t)$. From the undeformed straight configuration of the member, the amplitude is defined by

$$y(x,t) = \left[y_S(x) \pm y_d(x,t)\right] \quad (16)$$

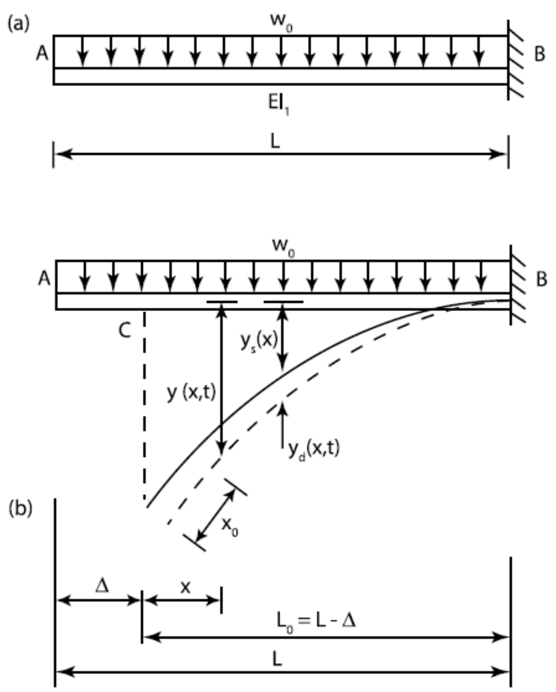

Figure 1. (a) Original undeformed flexible cantilever member (b) Deformed configuration of uniform flaxible cantilever member [3]

In this equation $y_S(x)$, as stated earlier, is the large static deflection which defines the static equilibrium position of the member, and $y_d(x,t)$ is the dynamic amplitude of its free vibration which is considered to be small. The slope of the dynamic amplitude curve is also small when it is compared to the large slope of the static configuration curve. This means that $y_S' >> y_d(x,t)$. Thus, if we differentiate Eq.(16) once with respect to x, we find [1]

$$y'(x,t) = y_S'(x) \tag{17}$$

By differentiate Eq. (16) twice with respect to x, we obtain

$$y''(x,t) = y_S''(x) \pm y_d(x,t) \tag{18}$$

Also, if we differentiate Eq.(16) twice with respect to time, we obtain

$$\frac{d^2 y(x,t)}{dt^2} = \frac{d^2 y_d(x,t)}{dt^2} \tag{19}$$

The substitution of Eqs. (18) and (19), into Eq.(15) yields

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_S''(x) \pm y_d''(x,t)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} + \\ +\sqrt{1+(y')^2}\, m(x_0)\frac{d^2 y_d}{dt^2} = 0 \tag{20}$$

From the large static equilibrium configuration, we have the expression

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_S''(x)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} = -\frac{w(x_0)}{\cos\theta} \tag{21}$$

and from the dynamic equilibrium configuration, we have

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_d''(x,t)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} \\ -\sqrt{1+(y_S')^2}\, m(x_0)\frac{d^2 y_d(x,t)}{dt^2} \tag{22}$$

Since the time function is harmonic, we can write M_x

$$\frac{d^2 y_d(x,t)}{dt^2} = -\omega^2 y_d(x,t) \tag{23}$$

where ω is the free vibration of the flexible member in radians per second. By substituting Eq.(23) into Eq.(22), we obtain the following differential equation:

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_d''(x,t)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} \\ -\left\{\sqrt{1+(y_S')^2}\, m(x_0)\right\}\omega^2 y_d(x,t) = 0 \tag{24}$$

From the separation of variables, Eq.(24) may also be written as

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_d''(x,t)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} \\ -\left\{\sqrt{1+(y_S')^2}\, m(x_0)\right\}\omega^2 y_d(x) = 0 \tag{25}$$

We can also write Eq.(25) in terms of an equivalent variable stiffness $I_e(x)$ and an equivalent variable mass density $m_e(x)$ as follows:

$$\frac{d^2}{dx^2}\left\{E_x I_e(x) y_d''(x)\right\} - m_e(x)\omega^2 y_d(x) = 0 \tag{26}$$

where

$$I_e(x) = \frac{I_x}{\left[1+(y_S')^2\right]^{3/2}} \tag{27}$$

$$m_e(x) = \sqrt{1+\left(y_S'\right)^2}\, m(x_0) \tag{28}$$

and I_x is the moment of inertia of the original member at any $0 \le x \le L_0$. The equivalent quantities $I_e(x)$ and $m_e(x)$, given by Eqs. (27) and (28), respectively, take into account the change in moment of inertia and mass caused by the large static deformation. Therefore the differential equation given by Eq.(26) may by thought of as representing a straight beam of length L_0, which vibrates with the same frequencies of vibration as the original member from its statics equilibrium position $y_S(x)$. The variation of its equivalent moment of inertia $I_e(x)$ and equivalent mass $m_e(x)$, along its length L_0, are given by Eqs.(27) and (28), respectively. In summary, the transverse vibration of a flexible member subjected to large static deformation $y_S(x)$ with small amplitudes of vibration, is completely defined by the following two differential equations:

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_S''(x)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} = -w(x_0)\sqrt{1+(y_S')^2} \tag{29}$$

$$\frac{d^2}{dx^2}\left\{E_x I_x y_d''(x)\right\} - m_e(x)\omega^2 y_d(x) = 0 \quad (30)$$

Equations (29) and (30) must be solved simultaneously in order to obtain the frequencies of vibration of a flexible member. Equation (29), however, is equivalent to the Euler- Bernoulli equation [3]

$$\frac{y_S''}{\left[1+\left(y_S''\right)^2\right]^{3/2}} = -\frac{M_x}{E_x I_x} \quad (31)$$

where M_x is the banding moment of the member, and $E_x I_x$ is its bending stiffness. We have to realize that both M_x and I_x in Eq. (31) are integral equation which depend on the large static deformation of the member. We can think of Eq. (31) as representing the transverse bending vibration of a dynamically equivalent pseudo variable stiffness member of length L_0, and having an equivalent mass density M_e and an equivalent moment of inertia I_e. The depth $h_e(x)$ of the dynamically equivalent straight member is given by the following equation:

$$h_e(x) = \frac{h(x)}{\left[1+(y_S')^2\right]^{1/2}} \quad (32)$$

where $h(x)$ represents the variation in depth of the original member. The depth $h(x)$ of the origin member may have any arbitrary variation along its length. The solution of the differential equation of motion given by Eq. (30) may be obtained by using know linear methods of analysis for free vibration of beams. The application of the finite difference method, or utilization of the Galerkins finite element method, should yield reasonable results [5].

3. Free Vibration of Flexible Simply supported Beams

The methodology is general and it applies equally well to members of both uniform and variable cross section and stiffness EI along their length. For illustration purpose, the analysis in this section deals primarily with the free vibration of uniform flexible simply supported beams, such as the one shown in Fig. 2 In this figure, the distributed load w acting on the member, represents the distribution of its weight along its length. However, it may include the weight of other possible object which are securely attached to the member and participate in its vibration motion. We assume here that the static deflection of the member caused by such weight distribution is large, and from its large static equilibrium position the member behaves like a curved beam during its vibration configuration. We have here a small amplitude free vibration taking place about the large static equilibrium position. The problem is very nonlinear, and its nonlinearity is characterized by the heavily curved large static equilibrium position. In order to proceed with the free vibration analysis of the uniform simply supported flexible member, we need first to derive the equivalent pseudo variable straight simply supported beam of length L_0, as we have done it previously for the cantilever beam. This task requires the computation of the rotation y_S' of the static equilibrium position curve y_S. If we assume that w in Fig.2 is the weight distribution of the simply supported member, then the deflection curve in the figure is the static equilibrium position of the member produced by the weight w [4].

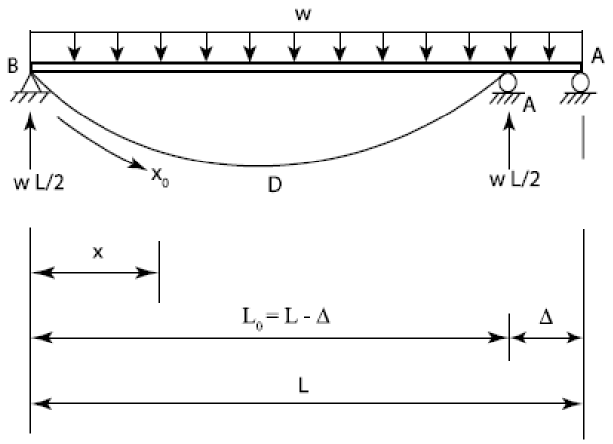

Figure 2. Straight and deflected configuration of a uniform simply supported beam loaded by a uniform distributed load w [5].

The large deformation configuration, and consequently y_S' can be determined by using the Euler-Bernoulli nonlinear differential equation. We rewrite this equation below.

$$\frac{y_S''}{\left[1+\left(y_S'\right)^2\right]^{3/2}} = -\frac{M_x}{E_x I_x} \quad (33)$$

assuming that

$$\Delta(x) = \Delta\frac{x}{L_0} \quad (34)$$

we find

$$y_S' = \frac{G(x)}{\left\{1-\left[G(x)\right]^2\right\}^{1/2}} \quad (35)$$

where

$$G(x) = \frac{wL}{24EI(L-\Delta)}\left[-6(L-\Delta)x^2 + 4x^3 + (L-\Delta)^3\right] \quad (36)$$

In this equation, the unknown horizontal displacement Δ of the right support A, can be determined by using, as in preceding section of this text, the equation

$$L = \int_0^{L_0}\left[1+(y')^2\right]^{1/2} dx \quad (37)$$

and applying a trial-and-error procedure. After having determined Δ, we can find the rotation y_S' of the static equilibrium position y_S, at any $0 \le x \le L_0$, by using Eqs. (35) and (36). With known y_S', we can proceed with the solution of the differential equation of motion which is given by Eq.(30). We rewrite this equation for convenience

$$\frac{d^2}{dx^2}\left\{E_x I_e(x) y_d''(x)\right\} - m_e(x)\omega^2 y_d(x) = 0 \quad (38)$$

In Eq.(38) $I_e(x)$ and $m_e(x)$ are given by the following equations:

$$I_e(x) = \frac{I_x}{\left\{1+\left(y_S'\right)^2\right\}^{3/2}} \quad (39)$$

$$m_e(x) = \sqrt{1+(y_S')^2}\, m(x_0) \quad (40)$$

where I_x is the moment of inertia of the original member at any $0 \le x \le L_0$. Note that Eqs. (38) - (40) are completely defined since y_S' can be obtained from Eq.(35). The following numerical examples illustrate application of the methodology.

4. The Effect of Mass Position Change During the Vibration of Flexible Members

It is already established that the free vibration a flexible member takes place with respect to the static equilibrium position y_S. At this position, the geometric distribution of the mass $m_0(x)$ of the flexible member is changed substantially when it is compared to its straight configuration. In order to clarify this point further, we recall Eq.(24) and we rewrite it as follows:

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_d''(x,t)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} - \left(\sqrt{1+\left(y_S'\right)^2}\right) m(x_0)\omega^2 y_d(x,t) = 0 \quad (41)$$

In the second term of Eq.(41), the quantity $\left[1+\left(y_S'\right)^2\right]^{1/2}$ is the one that takes into consideration the geometric configuration of the mass $m(x_0)$ at the large static equilibrium position y_S. If the static deformation y_S is small, or moderately large, the quantity $\left[1+\left(y_S'\right)^2\right]^{1/2}$ could be assumed to be equal to unity, and Eq.(41) becomes

$$\frac{d^2}{dx^2}\left\{E_x I_x \frac{y_d''(x,t)}{\left[1+\left(y_S'\right)^2\right]^{3/2}}\right\} = m(x_0)\omega^2 y_d(x,t) \quad (42)$$

Equations (41) and (42) will be used here to demonstrate the error introduced in the vibration analysis of flexible members if the changes of the mass geometry during large deformation are not taken into consideration [4].

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