# **Theoretical Basis of Modal Analysis**

#### Pavol Lengvarský\*, Jozef Bocko

Department of Applied Mechanics and Mechatronics, Technical University of Košice, Košice, Slovakia \* Corresponding author: pavol.lengvarsky@tuke.sk

Received October 09, 2013; Revised October 18, 2013; Accepted November 13, 2013

**Abstract** There is given the theoretical background of the modal analysis in this paper. In the first part modal analysis, its theoretical and experimental procedure of vibration analysis is defined. In document one degree of freedom system and its behavior, equations and next modification as system of free vibration, harmonic excitation, structural damping is defined. Also, there is defined important property of the modal model - orthogonal property. In the next step is discussed multi degrees of freedom system, its properties, damping and solution.

**Keywords:** modal analysis, vibration, damping, orthogonal properties, excitation

**Cite This Article:** Pavol Lengvarský, and Jozef Bocko, "Theoretical Basis of Modal Analysis." *American Journal of Mechanical Engineering* 1, no. 7 (2013): 173-179. doi: 10.12691/ajme-1-7-4.

#### 1. Introduction

Using of modal analysis we can determine the natural frequencies, the damping at natural frequencies and the mode shapes at natural frequencies [3].

Figure 1 shows the theoretical procedure of vibration analysis. It displays the three phase procedure of the theoretical vibration analysis. We are starting with a description of the physical properties of the structures, usually as their mass, stiffness and damping characteristics [1].

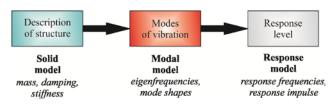


Figure 1. Theoretical procedure of vibration analysis [1]

The theoretical modal analysis of solid model leads to a description of the behavior of the structure as modes of vibration so-called the modal model. This model is defined as a set of natural frequencies with damping factor and natural modes vibration. This solution describes various methods of the structure vibration [1].

The response of model is the next part of the theoretical procedure analysis, which corresponds to the excitation and its amplitude. Model describes a set of frequency response functions [1].



Figure 2. Experimental aim [1]

It is possible to perform the analysis in the opposite direction, from the response properties, which we can deduce the modal model properties. This method is called the experimental procedure of vibration analysis and is shows in Figure 2 [1].

In Figure 3 the general procedure for linking simulation and experimental analysis in modal analysis is shown [6].

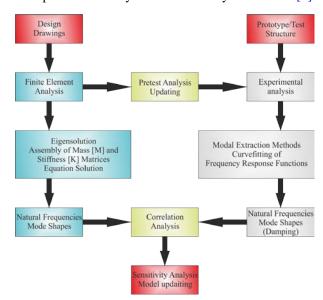


Figure 3. General procedure for linking simulation and experimental analysis in modal analysis [1]

## 2. One Degree of Freedom System

In Figure 4 one degree of freedom system is shown. The system is composed of a perfectly rigid body of mass m, spring with stiffness k and viscous damper with damping ratio d. The force in the spring is directly proportional to the displacement x and the damping force to the velocity  $\dot{x}$ . The spring and damper can be considered as linear [2].

The equation of motion has the form

$$m\ddot{x} + d\dot{x} + kx = F(t). \tag{1}$$

where F(t) is the excitation force and x is the displacement of the body of the equilibrium position.

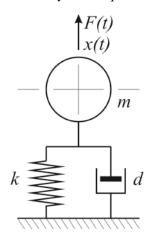


Figure 4. One degree of freedom system [3]

#### 2.1. Free Vibration

Substituting F(t) = 0 to equation (1) we get the equation of motion in the form

$$m\ddot{x} + d\dot{x} + kx = 0. \tag{2}$$

Dividing the equation (2) of mass m, we obtain

$$\ddot{x} + 2\delta \dot{x} + \omega_n^2 x = 0, \tag{3}$$

where  $\delta$  is the damping constant and  $\omega_n$  is the natural angular frequency

$$\delta = \frac{d}{2m},\tag{4}$$

$$\omega_n = \sqrt{\frac{k}{m}}. (5)$$

If we assume the result in the form

$$x(t) = Ce^{\lambda t} \tag{6}$$

we obtain the equation

$$\left(\lambda^2 + 2\delta\lambda + \omega_n^2\right)C = 0. \tag{7}$$

For  $C \neq 0$  has the characteristic equation form

$$\lambda^2 + 2\delta\lambda + \omega_n^2 = 0. (8)$$

The roots of the characteristic equation (8) are

$$\lambda, \lambda^* = -\delta \pm \sqrt{\delta^2 - \omega_n^2}, \tag{9}$$

where  $\delta$  is damping constant. If  $0 < \delta < \omega_n$  we have damping system

$$\lambda, \lambda^* = -\delta \pm \sqrt{\omega_n^2 - \delta^2} = -\delta \pm j\omega_d, \tag{10}$$

where

$$\omega_d = \sqrt{\omega_n^2 - \delta^2}.$$
 (11)

With the general solution of damping system we have equation in the form

$$x(t) = e^{-\delta t} \left[ C_1 e^{j\omega_0 t} + C_2 e^{-j\omega_0 t} \right], \tag{12}$$

where  $C_1$  and  $C_2$  are constants. These ones must have real value to both sides of the equation. The condition is valid if

$$C_1 = \frac{1}{2}(A - jB), \quad C_2 = \frac{1}{2}(A + jB),$$
 (13)

where A and B are real constants.

Substituting the equation (13) into the equation (12) we obtain the displacement in the form

$$x(t) = e^{-\delta t} \left( A \cos \omega_d t + B \sin \omega_d t \right)$$
  
=  $\hat{x}e^{-\delta t} \sin \left( \omega_d t + \phi \right)$  (14)

where amplitude  $\hat{x}$  and phase angle  $\phi$  are

$$\hat{x} = \sqrt{A^2 + B^2},\tag{15}$$

$$\phi = \operatorname{arctg} \frac{A}{B}.$$
(16)

For the case that damping constant is  $\delta = 0$  we get

$$x(t) = A\cos\omega_n t + B\sin\omega_n t = \hat{x}\sin(\omega_n t + \phi). \tag{17}$$

The function in Figure 5 shows the free vibration and consists of the harmonic displacement x. The value  $\omega_n$  is natural angular frequency of the system without damping and with period  $2\pi$  and the time of period  $T_n$  and thus

$$\omega_n T_n = 2\pi, \tag{18}$$

$$T_n = \frac{2\pi}{\omega_n},\tag{19}$$

$$f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi},\tag{20}$$

where  $f_n$  is the natural frequency.

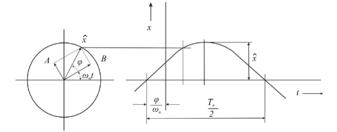


Figure 5. Free vibration without damping [3]

Constants A and B are determined from the initial conditions  $x(t=0) = x_0$  and  $\dot{x}(t=0) = \dot{x}_0$ . Thus

$$A = x_0, \quad B = \frac{\dot{x}_0}{\omega_n}.$$
 (21)

In Figure 6 is the course of function  $e^{-\delta t}$  decreasing in time and the harmonic value corresponding of the damping harmonic vibration is shown [2].

Proportion of damping constant  $\delta$  we can express the value of the damping the coefficient attenuation D, or the logarithmic decrement  $\theta$ .

The damping factor has the form

$$D = \frac{\delta}{\omega_n}.$$
 (22)

Logarithmic decrement has the form

$$\mathcal{G} = \ln \frac{x_n}{x_{n+1}},\tag{23}$$

where  $x_n$  is the value of the displacement from the equation (14) at the time  $t_0$  and  $x_{n+1}$  is the value of the displacement at the time  $t_0 + T_d$ , where

$$T_d = \frac{2\pi}{\omega_d}. (24)$$

 $T_d$  is the time period of damping vibration.

$$\mathcal{G} = \delta T_d = \frac{2\pi D}{\sqrt{1 - D^2}} \tag{25}$$

and for D << 1 can write

$$\mathcal{G} \approx 2\pi D.$$
 (26)

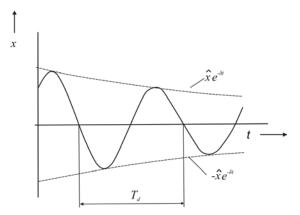


Figure 6. Damping vibration [3]

#### 2.2. Harmonic Excitation

After description of free vibration we consider forced vibration with harmonic excitation. We consider excitation force in the form

$$F(t) = \hat{F}\cos\omega t. \tag{27}$$

The solution of equation of motion (1) is represented by a complex value of the displacement

$$z = x + jy, (28)$$

the excitation force F(t) has the form

$$F(t) = \hat{F}(\cos \omega t + j\sin \omega t) = \hat{F}e^{j\omega t}.$$
 (29)

From the equation (1) we obtain

$$m\ddot{z} + d\dot{z} + kz = \hat{F}e^{j\omega t} \tag{30}$$

and solution

$$z(t) = Ce^{j\omega t}, (31)$$

where constant C is

$$C = \frac{\hat{F}}{k - m\omega^2 + jd\omega}.$$
 (32)

The displacement  $x(t) = \text{Re}\{z(t)\}\$  leads to

$$x(t) = \hat{x}\cos(\omega t - \varepsilon) \tag{33}$$

with amplitude

$$\hat{x} = \frac{\hat{F}}{\sqrt{(k - m\omega^2)^2 + (d\omega)^2}} \tag{34}$$

and phase angle

$$\varepsilon = arctg \frac{d\omega}{k - m\omega^2}.$$
 (35)

The mass performs harmonic motion during harmonic excitation with frequency  $\omega$ . The displacement pursute the force with phase angle  $\varepsilon$ . For static force is  $\omega = 0$ . The displacement is then

$$x_S = \frac{\hat{F}}{k}.\tag{36}$$

Amplitude  $\hat{x}$  is divided by magnification factor  $x_S$  with a constant ratio of frequencies, thus coefficient of tuning is

$$\eta = \frac{\omega}{\omega_n} \tag{37}$$

and damping factor D from equation (22) is substituted into (34), thus function of magnification is obtained

$$V = \frac{\hat{x}}{x_s} = \frac{1}{\sqrt{\left(1 - \eta^2\right)^2 + \left(2D\eta\right)^2}}$$
 (38)

and from the equation (35) we obtain

$$\varepsilon = arctg \ \frac{2D\eta}{1 - n^2}. (39)$$

The values V and  $\varepsilon$  are shown in Figure 7 and Figure 8 as a ratio of the function force relative frequency and damping factor. The function of magnification is also known as the resonance curve. Each of the resonance curves starts at point 1 and decreases to 0 for large values  $\eta$ .

For  $D \le 1/\sqrt{2} = 0.707$  curve has maximum

$$V_{\text{max}} = \frac{1}{2D\sqrt{1 - 2D^2}} \tag{40}$$

in relative frequency

$$\eta_{\text{max}} = \sqrt{1 - 2D^2} \,. \tag{41}$$

For small damping D << 1 is obtained

$$V_{\text{max}} \approx \frac{1}{2D}$$
 (42)

$$\eta_{\text{max}} \approx 1.$$
(43)

The form 1/2D is called resonance sharpness or Q factor

$$Q = \frac{1}{2D} \approx V_{\text{max}} = \frac{\hat{x}_{\text{max}}}{x_{\text{s}}}.$$
 (44)

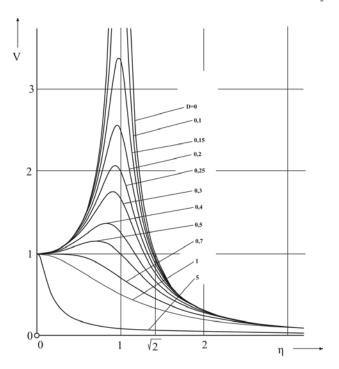


Figure 7. Function of magnification V [3]

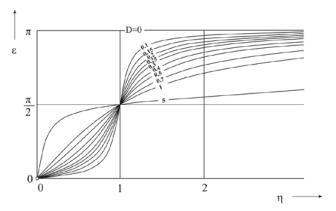


Figure 8. The phase angle [3]

From Figure 8 is clear that phase angle lies between the values 0 and  $\pi$ . For  $\eta=1$  is  $\omega=\omega_n$  and is equal to  $\pi/2$ , independent of the damping coefficient. Figure 7 indicates that a resonance curve is wider with growth of damping. For D<<1 is

$$D \approx \frac{\omega_2 - \omega_1}{2\omega_{\text{max}}} = \frac{\Delta\omega}{2\omega_{\text{max}}},\tag{45}$$

where  $\omega_1$  and  $\omega_2$  are angular frequencies at  $\hat{x} = \hat{x}_{\text{max}} / \sqrt{2} \approx 0.707 \hat{x}_{\text{max}}$  (obr.9).

The system from Figure 4 is supplemented of the source excitation with imbalance (Figure 10). The result is the excitation force in the form

$$F(t) = m_0 e \omega^2 \cos \omega t. \tag{46}$$

With a total mass  $m = m_1 + m_0$  we obtain the equation of motion

$$m\ddot{x} + d\dot{x} + kx = m_0 e\omega^2 \cos \omega t. \tag{47}$$

The partial results follow directly from the results of amplitude with constant excitation  $\hat{F}$ , which will be

replaced by  $m_0e\omega^2$ . Using the equations (33) and (34) the solution is in the form

$$x(t) = \hat{x}'\cos(\omega t - \varepsilon), \tag{48}$$

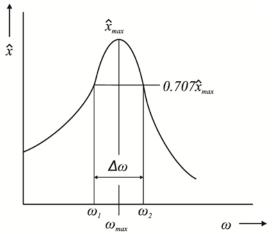


Figure 9. Curve of resonance [3]

where amplitude is

$$\hat{x}' = \frac{m_0 e \omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(d\omega\right)^2}} \tag{49}$$

or

$$\hat{x}' = \frac{m_0}{m_1 + m_0} V'e, \tag{50}$$

where the value magnification is

$$V' = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + (2d\eta)^2}}.$$
 (51)

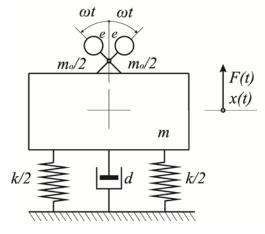


Figure 10. Excited system [3]

The amplitude of constant force is equal to the phase angle  $\varepsilon$ . From equation (40) is obvious, that  $V_{\rm max}$  is identical with the maximum value V' while the function of magnification V' starts at 0 and ends at 1. The maximum value V' is for the case

$$\eta'_{\text{max}} = \frac{1}{\sqrt{1 - 2D^2}}.$$
 (52)

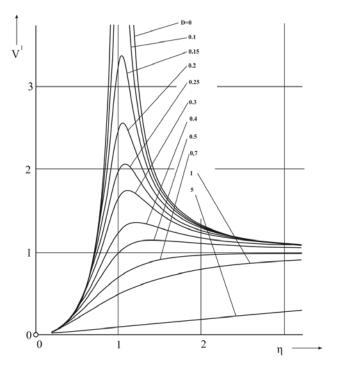


Figure 11. Function of magnification V [3]

In Figure 12 is shown the system with excitation from the ground. The equation of motion has the form

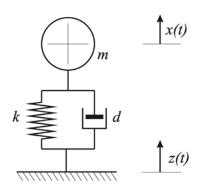
$$m \ddot{y} + d \dot{y} + k y = k z + d \dot{z}.$$
 (53)

Substituting

$$y = x - z \tag{54}$$

we obtain the equation of motion in the form

$$m \ddot{y} + d \dot{y} + k y = -m \ddot{z}(t).$$
 (55)



**Figure 12.** System with excitation from the ground [3]

For the harmonic motion  $\hat{z}(t) = z \cos \omega t$  has the equation of motion the form

$$m \ddot{y} + d \dot{y} + k y = m \hat{z} \omega^2 \cos \omega t, \tag{56}$$

where right-hand sites of the equations (56), (47) are similar, thus

$$y(t) = V'\hat{z}\cos(\omega t - \varepsilon), \tag{57}$$

where the function of magnification V' is obtained from the equation (51) [2].

#### 2.2. Structural Damping

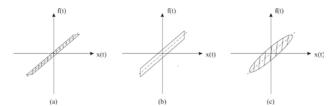
Structural damping causes faster damping of free vibration, reducing of amplitude vibration in the resonance

field, reducing the value of the growth of vibration and thus also the noise when passing through the resonance area. Structural damping is generally smaller than other types of damping [1].

All materials exhibit some degree of damping due to their hysteresis properties. The typical example of this effect is the graph of the time dependence of the displacement on force shown in Figure 13a, in which the area contained by loop represents lost of energy in one cycle vibration between illustrated boundaries [1].

Another common source of energy absorption in real structures and also the damping is the friction, which exists in the connections between the components of the structure. If these effects are macro slip between adjacent parts, or rather micro slip on connection between them can be described by a simple model of dry friction shown in Figure 13b with its corresponding dependence displacement on force [1].

Viscous damper is shown in Figure 13c, while it is necessary to assume the harmonic motion at the frequency  $\omega$ , for the purpose of constructing a dependence displacement on force. The problem that arises with a model viscous damping is that the frequency dependence of the energy loss per cycle, while the dry friction device is unaffected frequency of load [1].



**Figure 13.** The time dependence of the displacement on force [1] (a) Hysteresis material, (b) Dry friction, (c) Viscous damper

# 3. Undamped Multidegrees of Freedom Systems

#### 3.1. Free Vibration - Modal Properties

For undamped system MDOF with N degrees of freedom, we can write the equations of motion in the matrix notation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t), \tag{58}$$

where **M** and **K** are NxN matrices of mass and stiffness,  $\mathbf{x}(t)$  and  $\mathbf{f}(t)$  are Nx1 vectors of displacement and forces dependent on time [1].

We will consider the first solution of free damping (to determine the normal and modal properties) that

$$\mathbf{f}\left(t\right) = \mathbf{0}.\tag{59}$$

We assume the solution in the form

$$\mathbf{x}(t) = \mathbf{X} e^{i\omega t},\tag{60}$$

where  $\mathbf{X}$  is Nx1 vector of time-independent amplitudes. For this case it is clear that

$$\ddot{\mathbf{x}} = -\omega^2 \mathbf{X} e^{i\omega t}. \tag{61}$$

Substituting the equations (59), (60) and (61) in the equation of motion (58), leads to the equation

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{X} e^{i\omega t} = \mathbf{0},\tag{62}$$

for which the only non-trivial solution satisfies

$$\det \left| \mathbf{K} - \omega^2 \mathbf{M} \right| = 0, \tag{63}$$

or

$$\alpha_{2N}\omega^{2N} + \alpha_{2N-2}\omega^{2N-2} + \alpha_0 = 0, (64)$$

from which we can find N values of  $\omega^2 = (\overline{\omega}_1^2, \overline{\omega}_2^2, ..., \overline{\omega}_r^2, ..., \overline{\omega}_N^2)$ , undamped natural angular frequencies of the system.

### 3.2. Orthogonal Property of Modal Model

The most important property of the modal model is orthogonal property. The equation of motion for free vibration we can written in the form

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{X} e^{i\omega t} = \mathbf{0}. \tag{65}$$

The equation (65) for r-th mode shape we multiply by the transposed vector  $\mathbf{\psi}_{s}^{T}$  and we get

$$(\mathbf{\psi}_s^T \mathbf{K} \mathbf{\psi}_r) - \omega_r^2 (\mathbf{\psi}_s^T \mathbf{M} \mathbf{\psi}_r) = 0.$$
 (66)

Then for the s-th mode shape we multiply by the transposed vector  $\boldsymbol{\psi}_r^T$ , we get the equation in the form

$$(\mathbf{\psi}_r^T \mathbf{K} \mathbf{\psi}_s) - \omega_s^2 (\mathbf{\psi}_r^T \mathbf{M} \mathbf{\psi}_s) = 0.$$
 (67)

Since the stiffness matrix  $\mathbf{K}$  and mass matrix  $\mathbf{M}$  are symmetrical matrices, we can the equation (67) transpose and written in the form

$$(\mathbf{\psi}_{s}^{T}\mathbf{K}\mathbf{\psi}_{r}) - \omega_{s}^{2}(\mathbf{\psi}_{s}^{T}\mathbf{M}\mathbf{\psi}_{r}) = 0.$$
 (68)

Subtracting the equation (68) from the equation (66) we get expression

$$\left(\overline{\omega}_s^2 - \overline{\omega}_r^2\right) \mathbf{\psi}_s^T \mathbf{M} \mathbf{\psi}_r = 0, \tag{69}$$

if it is satisfied  $\omega_r \neq \omega_s$ , that

$$\mathbf{\psi}_{s}^{T}\mathbf{M}\,\mathbf{\psi}_{r} = 0; \quad r \neq s. \tag{70}$$

From the equation (70) follows that mode shapes r and s are orthogonal with respect to the mass matrix  $\mathbf{M}$ .

Substituting the equation (70) to the equation (66) we get expression

$$\mathbf{\psi}_{s}^{T}\mathbf{K}\,\mathbf{\psi}_{r}=0;\quad r\neq s. \tag{71}$$

From the equation (71) follows that mode shapes r and s are orthogonal with respect to the stiffness matrix  $\mathbf{K}$ .

In the special case when r = s or  $\omega_r = \omega_s$ , equations (70) and (71) are not valid.

For this special case we can write the equation (68) in the form

$$(\mathbf{\psi}_r^T \mathbf{K} \mathbf{\psi}_r) = \overline{\omega}_r^2 (\mathbf{\psi}_r^T \mathbf{K} \mathbf{\psi}_r), \tag{72}$$

if 
$$\overline{\omega}_r^2 = k_r / m_r$$
, then

$$\mathbf{\psi}_r^T \mathbf{K} \mathbf{\psi}_r = m_r \quad a \quad \mathbf{\psi}_r^T \mathbf{K} \mathbf{\psi}_r = k_r. \tag{73}$$

# 4. Multidegrees of Freedom Systems with Structural Damping

# **4.1. Solution of Free Vibration- Complex Modal Properties**

In this section we will examine properties of the general elements of hysteric damping. The general equation of motion for the system with multi degrees of freedom with hysteric damping and harmonic excitation has the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\dot{\mathbf{x}} + i\mathbf{D}\mathbf{x} = \mathbf{F}e^{i\omega t}.$$
 (74)

We will considered the case without excitation and we choose a solution in the form

$$\mathbf{x} = \mathbf{X} e^{i\lambda t}. (75)$$

Substituting to the equation (74), the solution (75) leads to eigen problem, which solution is in the form of two matrices containing eigenvalues and eigenvectors. In this case each natural frequency and each mode shape is described by complex quantities such as

$$\lambda_r^2 = \omega_r^2 \left( 1 + i \eta_r \right), \tag{76}$$

where  $\omega_r$  is natural angular frequency and  $\eta_r$  is the damping loss factor.

It is not necessary that natural angular frequency  $\omega_r$  equal to natural angular frequency of the undamped system  $\omega_r$ .

This solution can be considered as the same type of orthogonal properties which have been demonstrated for undamped system and can be defined by equations

$$\mathbf{\Psi}^{T} \mathbf{M} \mathbf{\Psi} = \mathbf{m}_{r}; \quad \mathbf{\Psi}^{T} (\mathbf{K} + i\mathbf{D}) \mathbf{\Psi} = \mathbf{k}_{r}. \tag{77}$$

### 5. Conclusion

This paper is only general entry to the modal analysis. It define the theoretical background of modal analysis while other models of modal analysis exist, for example models with different damping, excitation and other examples of calculation of modal analysis.

## Acknowledgement

This article was created with support of VEGA grant project VEGA 1/1205/12 Numerical modeling of mechatronic systems.

#### References

- EWINS, D.J., Modal Testing: Theory, Practice and Application, England: Wiley, 2000. Second Edition.
- [2] TREBUŇA, F. ŠIMČÁK, F.: Príručka experimentálnej mechaniky, SjF TU, Košice, 2007.
- [3] MEAD, D.: Passive Vibration Control, University of Southampton, UK: Wiley, 2000.

- [4] KRÄMER, E.: Dynamics of Rotors and Foundations: Springer Verlag, 1993.
- [5] ŽIARAN, S., Kmitanie a akustika. Znižovanie kmitania a hluku v priemysle, STU Bratislava, 2006.
- [6] DASCOTTE, E.: Linking FEA with Test. In: Sound and Vibration, April 2004, p. 12-17.
- [7] LENGVARSKÝ. P., Štrukturálna a modálna analýza súčiastok z polymérov použivaných v domácich spotrebičoch, Košice, 2013.
- [8] ZÁHOREC, O. CABAN, S., Dynamika, Olymp, Košice, 2002.
- [9] LMS a Siemens Business [online], http://www.lmsintl.com.