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Analysis on Uniform [N, p, q] Summability of Fourier Series and Its Conjugate Series

Suresh Kumar Sahani[1,2,*]

[1]MIT Campus, Janakpurdham, Nepal
[2]Rajarshi Janak Campus, Janakpurdham, Nepal
*Corresponding author: sureshsahani54@gmail.com

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Abstract This paper briefly discusses the uniform (N, p, q) summability of Fourier series and its conjugate series. We prove that if $\lambda(t)$ and $\mu(t)$ be positive (i.e. monotric function of t) and $\{p_m\}$ and $\{q_n\}$ is monotonic sequence of constant with their non-vanishing partial sums p_m and q_n tending to infinity as m, n $\to\infty$ if $\sum_{k=0}^{n}\left|\Delta\left(p_{n-k}q_k\right)\right| = 0\left(\frac{R_n}{n}\right)$, as n $\to\infty, \lambda(n)\cdot R_n = 0\left[\left\{\mu(R_n)\right\}^v\right]$, as n $\to\infty$, Where $0\le v\le 1$ and $\Phi(t)=\int_0^t\left|\phi(u)\right|du - 0\left[\frac{\lambda\left(\frac{1}{t}\right)q_\tau}{\left\{\mu(R_\tau)\right\}^v}\right]$, as t $\to 0$, uniformly in a domain *E* in which f(*x*) is bounded then the **F**ourier series (1.4) is summable (N, p, q) uniformly in E to the sum f(*x*). Also, If (2.4) uniformly in E then (1.5) is summable (N, p,q) uniformly in the domain E to the sum (2.5) whenever the integral exist uniformly in E.

Keywords: *Summability of Fourier Series, Conjugate of Fourier Series etc.*

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1. Introduction

The old hazy notion of convergence of infinite series was placed on sound foundation with the appearance of Cauchy's monumental work course "d Analysis Algebrique" in 1821 and Abel's [1] researches on the binomial series in 1826. However, it was observed that there were certain convergent series which particularly in dynamical Astronomy furnished nearly correct result. A theory of divergent series was formulated explicitly for the first time in 1890, when Cesaro [2] published a paper on the multiplication of series. Since then the theory of series, whose equation of partial sums oscillates, has been the center of attraction and fascination for most of the pioneering mathematical analysis. These process of associating generalized sums known as method of summability Szasz [4,5] and Hardy [3,6] provide a natural generalization of the classical notion of convergence. Hobson [9] and are thus responsible for bringing within the field of applicability a wider class of erstwhile rejected series that used to be tabooed as divergent.

Let $\sum U_n(x)$ be an infinite series with $\{S_n(x)\}$ as the sequence of its nth Partial sums.

1.1. Definition

Let $\{p_n\}$ and $\{q_n\}$ be two non-negative sequence, with

$$\begin{cases} P_n = \sum_{k=0}^{n} p_k \neq 0, \\ Q_n = \sum_{k-0}^{n} qk \neq 0, \\ R_n = \sum_{k=0}^{n} p_{n-k}q_n \neq 0 \end{cases} \quad (1.1)$$

The nth (N,p,q) mean $t_n^{p,q}(x)$ of the sequence $\{s_n(x)\}$ at point x in a domain E is defined by the sequence to sequence transformations.

$$\left\{t_n^{p,q}(x) = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}.q_k.s_k(x) \quad (1.2)$$

If $t_n(x) \to s(x)$ as n $\to\infty$.

Then the series $\sum U_n(x)$ or the sequence $\{s_n(x)\}$ of its its partial sum is said to be summable (N,p,q) to the sum s(*x*) at the point x in a domain E

If

$$t_n^{p,q}(x)-s(x)=0(1) \text{ as } n\to 0 \quad (1.3)$$

uniformly in set E then we say that series $\sum U_n(x)$ is summable (N,p,q) uniformly in E to the sum s (x) where q_n = 1 ∀ n. (N,p,q) summability reduces to the summability (N, p_n) for $p_n = \frac{1}{n+1}$ summability reduces to (N, $\frac{1}{n+1}$) such summability called as harmonic summability.

Let F(t) be a 2π - periodic and Lebesgue integrable function of t in ($-\pi,\pi$). Then the Fourier series [7,8] of the function F(t) is given by

$$F(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin t) = \sum_{n=0}^{\infty}A_n(t) \quad (1.4)$$

The conjugate series of Fourier series is given by

$$\sum_{n=1}^{\infty}(b_n cosnt-a_n sinnt)=\sum_{n=1}^{\infty}B_n(t). \quad (1.5)$$

We write a point at t = x

$$\phi(t)=f(x+t)+f(x-t)-2f(x),$$

$$\varphi(t)=f(x+t)-f(x-t),$$

$$\Phi(t)=\int_0^t|\phi(u)|du,$$

$$\Psi(u)=\int_0^t|\psi(u)|du$$

$$N_n^{p,q}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}.q_k\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

$$\bar{N}_n^{p,q}(t)=\frac{1}{2\pi R_n}.\sum_{k=0}^{n}p_{n-k}.q_k.\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}},$$

$\tau = \left[\frac{1}{t}\right]$, the integral part of $\frac{1}{t}$

$$\bar{F}_n(x)=\frac{\pi}{2}\int_{\frac{1}{n}}^{\pi}\psi(t).\,cos\frac{t}{2}dt$$

and

$$\bar{F}_n(x)=\lim_{n\to\infty}\bar{F}_n(x).$$

2. Preliminaries

Theorem 1:

Let λ(t) and $\mu(t)$ be two positive function of t such that λ(t), $\mu(t)$ and t. $\frac{\lambda(t)}{\mu(t)}$ Increase monotonically with t. Let $\{p_m\}$ and $\{q_n\}$ be two non- negative monotonic non-increasing sequence of constant with there non- vanishing partial sum P_m and Q_n tending to infinity as m, n$\to\infty$ respectively.

If

$$\sum_{k=0}^{n}\left|\Delta(p_{n-k}.q_k)\right|=0\left(\frac{R_n}{n}\right) as\ n\to\infty \quad (2.1)$$

$$\lambda(n).R_n=0\left[\{\mu(Rn)\}^{v}\right], as\ n\to\infty \quad (2.2)$$

Where $0\le v\le 1$, and

$$\Phi(t)=\int_0^t|\phi(u)|du=0\left[\frac{\lambda\left(\frac{1}{t}\right)q\tau}{\{\mu(R_\tau)\}^{v}}\right], \quad (2.3)$$

as t $\to 0$, uniformly in a domain E in which F(x) is bounded then the Fourier series (1.4) is summable (N,p,q) uniformly in E to the sum F(x).

Theorem 2:

Let $\{p_m\}$ and $\{q_n\}$ be non-negative monotonic, non-increasing sequence of constant with their non-varnishing partial sums P_m and Q_n tending to infinity as m, n $\to\infty$ resp if $\sum_{k=0}^{n}\left|\Delta(p_{n-k}.q_n)\right| = 0\left(\frac{R_n}{n}\right)$ as n$\to\infty$ and also λ(t) and $\mu(t)$ be the function of t such that λ(t), $\mu(t)$ and t. $\frac{\lambda(t)}{\mu(t)}$ increase monotonically with t and which satisfy the following condition λ(n). $R_n = 0\left[\{\mu(R_n)\}^{v}\right]$, as n$\to\infty$, $0\le v\le 1$

If

$$\Psi(t)=\int_0^t|\phi(u)|du=0\left[\frac{\lambda(t)q_\tau}{\{\mu R_\tau\}^{v}}\right], \quad (2.4)$$

as t$\to\infty$, $0\le v\le 1$ uniformly in E, then the conjugate series of Fourier series (1.5) is summable (N,p,q) uniformly in the domain E to the sum

$$\bar{F}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cos\frac{t}{2}dt \quad (2.5)$$

Whenever the integral exists uniformly in E.

The following lemmas are required in order to prove our theorem.

3. Lemmas

Lemma (3.1) for $0\le a\le b\le\infty, 0<t\le 2\pi,$

$\left|\sum_{k=a}^{b} p_{n-k} q_k e^{ikt}\right| < cR_\tau$ Where c is an absolute constant

Lemma (3.2) If $R_n \to \infty$, as $n \to \infty$ and the condition (2.1) is satisfy then n. $q_n = 0(R_n)$, as $n \to \infty$.

Proof of Lemma (3.2)

It may be easily noted that if $\sum_{k=1}^{n}\left|\Delta(p_{n-k}q_k)\right| = 0\left(\frac{R_n}{n}\right)$. Then,

$$\sum_{k=1}^{n} k\left|\Delta(p_{n-k}q_k)\right| = 0(R_n)$$

Now,

$$\sum_{k=1}^{n} k\Delta(p_{n-k}.q_k)$$
$$= \sum_{k=1}^{n-2}\Delta k\sum_{v=1}^{k}\Delta(p_{n-v}q_v) + (n+1)\sum_{k=1}^{n-1}\Delta(p_{n-k}.q_k)$$
$$= \sum_{k=1}^{n-2}\Delta k(p_{n-1}.q_1 - p_{n-k-1}q_{k+1}) + (n+1)(p_{n-1}q_1 - p_0 q_n)$$
$$= \sum_{k=1}^{n-1} p_{n-k}q_k - (n-1)\,p_0 q_n$$
$$= \sum_{k=1}^{n} p_{n-k}q_k - p_n q_0 - nq_n p_0.$$

Therefore,

$$nq_n p_0 = R_n - \sum_{k=1}^{n-1} k\Delta(p_{n-k}q_k) - p_n q_0.$$

Implies that n $q_n = 0\ (R_n)$ as $n \to \infty$.

Lemma (3.3) If $0 \le t \le \frac{1}{n}$, then $N_n^{p,q}(t) = 0(n)$

Proof:

We have

$$\left|N_n^{p,q}(t)\right| = \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$
$$= 0\left(\frac{1}{R_n}\right)\left|\sum_{k=0}^{n}(2k+1)\,p_{n-k}q_k\right|$$
$$= 0\left[\frac{2n+1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k\right] = 0(n),\ as\ n \to \infty.$$

Proof of lemma (3.1)

Denoting the n[th] partial sum of the series (1.4) by

$$\sigma_n(x) - f(x) = \int_0^n \frac{\phi(t).Sin\left(n+\frac{1}{2}\right)}{\sin\frac{t}{2}}dt.$$

Therefore following (1.2), the (N,p,q)[th] transform of $\{\sigma_n(x)\}$ is given by

$$t_n^{p,q}(x)\,f(x) = \sum_{k=0}^{n} p_{n-k}q_k\{\sigma_k(x) - f(x)\}$$
$$= \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k \times \int_0^\pi \phi(t)\frac{\sin\left(k+\frac{1}{2}\right)}{\sin\frac{t}{2}}dt$$
$$= \int_0^\pi \phi(t)\left\{\frac{1}{2\pi R_n} \times \sum_{k=0}^{n} p_{n-k}q_k.\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt \quad (3.1)$$
$$= \int_0^\pi \phi(t)\,N_n^{p,q}(t)\,dt = \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\sigma} + \int_\sigma^\pi \phi(t)\,N_n^{p,q}(t)\,dt$$
$$\therefore t_n^{p,q}(x)\,f(x) = I_1 + I_2 + I_3(say).$$

Now in order to prove the **theorem 1** we have to show that under our assumption,

$$I = I_1 + I_2 + I_3 = 0\ (1), \quad (3.2)$$

as $n \to \infty$ uniformly in E.

Now for I_1,

$$|I_1| = 0\left[\int_0^{\frac{1}{n}}|\phi(t)|\left|N_n^{p,q}(t)\right|dt\right]$$

$= 0(n)\ \int_0^{\frac{1}{n}}\phi(t)\,dt$, (using if $0 \le t \le \frac{1}{n}$, then $N_n^{p,q}(t) = 0(n)$)

$= 0\ (n)\ 0\left[\frac{\lambda(n)q_n}{\{\mu(R_n)\}^v}\right]$, (using 2.3)

$= 0\left[\left[\frac{\lambda(n)R_n}{\{\mu(R_n)\}^v}\right]\right]$, (using lemma 3.2)

$$= 0(1)\ as\ n \quad (3.3)$$

Uniformly in E,

Also for I_2,

$$|I_2| = 0\left[\int_{\frac{1}{n}}^{\delta}|\phi(t)|\left|N_n^{p,q}(t)\right|dt\right] = 0\left(\frac{1}{R_n}\right)\int_{\frac{1}{n}}^{\delta}|\phi(t)|\frac{R_\tau}{t}$$
$$= 0\left(\frac{1}{R_n}\right)\left[\left\{\Phi(t)\frac{R_\tau}{t}\right\}_{\frac{1}{n}}^{\delta}\right] + 0\left(\frac{1}{R_n}\right)\int_{\frac{1}{t}}^{\delta}\Phi(t)\frac{R_\tau}{t^2}dt \quad (3.4)$$
$$+\left(\frac{1}{R_n}\right)\int_{\frac{1}{n}}^{\delta}\Phi(t).\frac{1}{t}\mathrm{d}(R_\tau)\mathrm{dt}$$
$$= I_{2,1} + I_{2,2} + I_{2,3}(say)$$

Now

$$I_{2,1}=0\left(\frac{1}{R_n}\right)\left[\left\{\Phi(t)\frac{R_\tau}{t}\right\}_{\frac{1}{n}}^{\delta}\right]$$
$$=0\left(\frac{1}{R_n}\right)+\left[\frac{n\lambda_n q_n}{\{\mu R_n\}^v}\right] \tag{3.5}$$
$$=0(1)+0\left[\frac{\lambda(n)R_n}{\{\mu R_n\}^v}\right]$$
$$=\ 0(1),\ asn\to uniformly\ in\ E.$$

Further

$$I_{2,2}=0\left(\frac{1}{R_n}\right)\left[\int_{\frac{1}{t}}^{\delta}\Phi(t).\frac{R_\tau}{t^2}dt\right]$$
$$=0(1)+0\left(\frac{1}{R_n}\right)\sum_{m=1}^{n-1}\int_{m=1}^{m+1}\Phi\left(\frac{1}{u}\right).\mathrm{R}_{[u]}du.$$

But

$$\int_{m}^{1+m}\Phi\left(\frac{1}{u}\right)R_{[u]}du\le\Phi\left(\frac{1}{m}\right)R_m$$
$$=0\left[\frac{\lambda(m)q_mR_m}{\{\mu(R_m)\}^v}\right]=0(q_m)\ as\to\infty\ uniformly\ in\ E.$$

Thus

$$I_{2,2}=0(1)+0\left(\frac{1}{R_n}\right).0\left(\sum_{m=1}^{q-1}q_m\right)$$
$$=0\ (1)\ +\ 0\ (1) \tag{3.6}$$
$$=0\ (1),\ as\ m\to\infty\ uniformly\ in\ E,$$

Again,

$$I_{2,3}=0\left(\frac{1}{R_n}\right)\int_{\frac{1}{n}}^{\delta}\Phi(t)\times\frac{1}{t}dR_\tau$$
$$=0\left(\frac{1}{R_n}\right)\int_{\frac{1}{\delta}}^{\delta}\phi\left(\frac{1}{v}vdR_{[u]}\right)$$
$$=0(1)+0\left(\frac{1}{R_n}\right)\left\{\sum_{m=1}^{n-1}R_m.\Phi\left(\frac{1}{m}\right)\right\} \tag{3.7}$$
$$=0(1)+0\left[\frac{1}{R_n}\sum_{m=1}^{n-1}\frac{\lambda(m)R_mq_m}{\{\mu R_m\}^v}\right]$$
$$=0(1)+0(1)=0(1),\ as\ n\to\infty\ uniformly\ in\ E.$$

From (3.4) to (3.7) it follows that

$$I_2=0(1)\ as\ n\to\infty\ uniformly\ in\ E. \tag{3.8}$$

Again,

$$|I_3|=0\left[\int_{\delta}^{\pi}|\phi(t)|\left|N_n^{p,q}(t)\right|dt\right]=0(1)\ as\ n\to\infty. \tag{3.9}$$

Uniformly in E by virtue of a known result due to Hardy and Rogosinski and regularity of the method of summation.

Now combing (3.3), (3.8) and (3.9), we get the required result in (3.2). This completes the proof of the **theorem (2).**

Proof of theorem (2)

The n^{th} partial sum $\bar{\sigma}_n(x)$ of the series (1.5) at the point t = *x* in E, is given by

$$\bar{\sigma}_n(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\left\{\cot\frac{t}{2}-\cos\left(n+\frac{1}{2}\right)t\right\}\times\frac{1}{\sin\frac{t}{2}}dt.$$

Denoting by $\bar{t}_n^{p,q}(x)$, the $(N,p,q)^{th}$ transform of $\bar{\sigma}_n(x)$.

We have the following (1.2),

$$\bar{t}_n^{p,q}(x)-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{t}{2}dt$$
$$=\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\bar{\sigma}_k(x)-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{t}{2}dt$$
$$=\int_0^{\pi}\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q_k\times\left\{\frac{\cot\frac{t}{2}-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\psi(t)dt$$
$$=-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{t}{2}dt$$
$$=-\int_0^{\pi}\psi(t).\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q_k\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$
$$=-\int_0^{\pi}\psi(t)\bar{N}_n^{p,q}(t)dt$$
$$=-J\ say,$$

Where, $J=\int_0^{\pi}\psi(t)\bar{N}_n^{p,q}(t)dt$

Now we have to show that

$$J=0(1),\ as\ n\to\infty, uniformaly\ in\ E, \tag{3.10}$$

Let us write with $0<\delta<\pi$,

$$J=\int_0^{\pi}\psi(t)N_n^{-p,q}(t)dt$$
$$=\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}+\int_{\delta}^{\pi}.\right)\psi(t)\times\bar{N}_n^{p,q}(t)dt \tag{3.11}$$
$$=J_1+J_2+J_3\ say.$$

For J_1

$$|J_1| = 0\left[\int_0^{\frac{1}{n}} |\psi(t)|\left|\bar{N}_n^{p,q}(t)\right| dt\right]$$

$$= 0(n)\left[\int_0^{\frac{1}{n}} |\psi(t)| dt\right] \tag{3.12}$$

$$= 0(0).\left[\frac{\lambda(n) q_n}{\{\mu(R_n)\}^v}\right] = 0.\left[\frac{\lambda(n) R_n}{\{\mu(R_n)\}^v}\right]$$

$$= 0(1),\ as\ n \to \infty, uniformly\ in\ E.$$

$$|J_2| = 0\left[\int_{\frac{1}{n}}^{\delta} |\psi(t)|\left|\bar{N}_n^{p,q}(t)\right| dt\right]$$

$$= 0\left(\frac{1}{R_n}\right)\int_{\frac{1}{n}}^{\delta} |\psi(t)| \frac{R_\tau}{t} dt \tag{3.13}$$

$$= 0(1),\ as\ n \to \infty, uniformly\ in\ E.$$

Again,

$$J_3 = 0(1), as\ n \to \infty, uniformly\ in\ E. \tag{3.14}$$

Using the result due to Hardy and Rogosinski and regulation of the method of summation.

Combing (3.11), (3.12), (3.13) and (3.14) we get the required result in (3.10). This completes the proof of the theorem (2).

4. Conclusion

When $\{p_m\}$ and $\{q_n\}$ be non-negative monotonic non-increasing sequence of constant with their non-varnishing partial sums P_m and Q_n tending to infinity as m, n $\to \infty$ resp if $\sum_{k=0}^{n} |\Delta(p_{n-k}.q_n)| = 0 \left(\frac{R_n}{n}\right)$ as n $\to \infty$ and also λ(t) and $\mu(t)$ be the function of t such that λ(t), $\mu(t)$ and $t.\frac{\lambda(t)}{\mu(t)}$. Increase monotonically wit t. and which satisfy the following condition λ(n). R_n= $0\left[\{\mu(R_n)\}^v\right]$, as n $\to \infty$, $0 \le v \le 1$.

If $\Psi(t) = \int_0^t |\phi(u)| du = 0\left[\frac{\lambda(t) q_\tau}{\{\mu R_\tau\}^v}\right]$.

As t $\to \infty$, $0 \le v \le 1$ uniformly in E, then the conjugate series of Fourier series (1.5) is summable (N,p,q) uniformly in the domain E to the sum

$$\bar{F}(x) = \frac{1}{2\pi}\int_0^\pi \psi(t) \cos\frac{t}{2} dt.$$

Whenever the integral exists uniformly in E.

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